

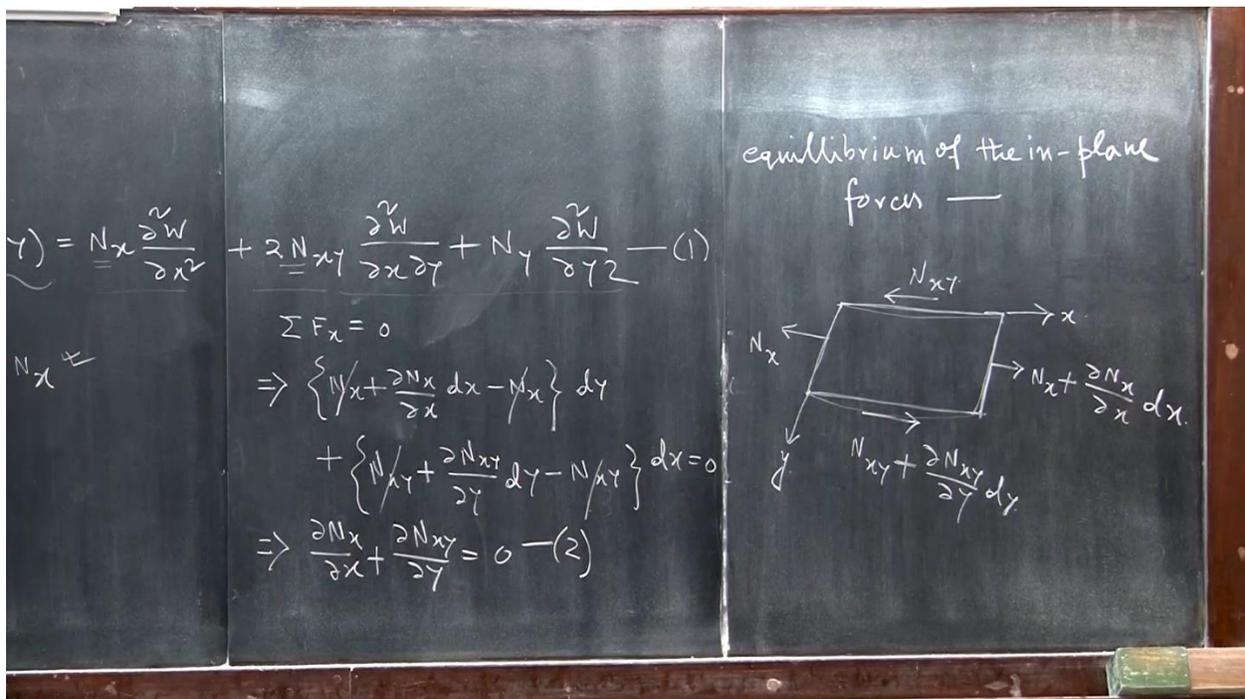
Stability of structure
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WEEK-08
Lecture 15: Post Critical Analysis of Plate

Okay, welcome to the 15th lecture on the stability of structures. So, let us briefly recapitulate what we are discussing. We have completed studying the buckling phenomena in plates subjected to uni-axial compression and We briefly touched on the bi-axial compression and studied the shear buckling as well. And we have seen that we can obtain the expression for the critical load, critical compression that will trigger buckling, as well as the respective mode, and then we have seen two distinct approaches. One is based on directly solving the equilibrium equation in the deformed configuration, which we have derived, and the second approach is, of course, the energy approach. We have also seen that the depending aspect ratio is a very important parameter and how the buckling behavior changes with the changing aspect ratio. So, on changing the aspect ratio, the plate tries to shift from one mode to another. So, mode shifting occurs, and then when the mode shifting occurs, the point in the stability domain. Basically, they form tongue-like structures, which are called Arnold tongue, and this phenomenon can also be seen in axial compressed plates as well as under shear. So, then we started with the, you know, next level of information. That's what the postbuckling behavior is. But then there is a problem; we have to extend our theory a little bit. When going from buckling to post-buckling and from critical load to post-critical regime, if you want to study the behavior, whatever was the governing equation for plate buckling needs to be changed a little. So, what we have seen is that the buckling equation, the governing equation for periodic buckling, was

$$\nabla^2 \nabla^2 W(x, y) + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} = 0 \quad (1)$$

So, this is coming, you know, this is in terms of displacement; there is also, this is for, you know, the force equilibrium in the vertical direction, right, and augmented with the force equilibrium in x and y as well. The moment equilibrium is right, and this one is coming from the component of

in-plane forces because of the deformed configuration, right? But in buckling, our equation was essentially a linear equation because we have considered N_x , N_y , and N_{xy} to be constant over the plate. Right? There was no variation assumed. So, our only unknown for buckling, as far as buckling is concerned when we are studying the buckling analysis, is that we have assumed this n_x is known and constant. There is no variation in N_x , and the critical value of N_x for triggering buckling is what we find out, so since basically that is the linearized theory, right? Okay. And that was giving an eigenvalue buckling problem, right? Trigonometric eigenvalue problem assumed double Fourier series solution, okay? Involving trigonometric functions that satisfy the essential boundary condition correctly, and then we did some minimization; I think in some places we have used the direct Galerkin approach.



However, when we go beyond buckling, then what happens? The theory is that there will be little change. The main change that will occur is that N_x , N_y , and N_{xy} will no longer remain constant. Okay. The variation of in-plane forces along the plate will vary. So, when N_x , N_y , and N_{xy} are functions of x and y , you can clearly understand that the nonlinearity will come because these are no longer constants; these are variable coefficients. And then, they are appearing in this equation as a product. So that becomes a nonlinear differential equation. Understand that? And that becomes much more complicated to solve. So, when we take N_x and N_y to vary along the plate, then only

this equation is not sufficient. Okay. Because this equation only indicates how many unknowns will be there, we will have four unknowns: N_x , N_y , and N_{xy} . Therefore, we also need to include more equations to solve the problem and make it determinate. So, what are the additional equations? Look, this equation is nothing but the equilibrium force in the Z direction, okay, the vertical direction, but we have to consider equilibrium for the in-plane forces. When we consider the equilibrium of forces in-plane, let us focus on the equilibrium of the in-plane forces. So, if we consider, you know, one second you go, and then you know, you consider the deformed configuration, that's not a problem. I'm still considering the deformed configuration. Okay, that's not a problem. Huh? So here you see that I have N_x here. This is x and this is y, and here it is $N_x + \frac{\partial N_x}{\partial x} dx$. And here you will have N_{xy} . And here $N_{xy} + \frac{\partial N_{xy}}{\partial y} dy$, right? So, now if we consider the summation of forces along x, then what you will get is

$$(N_x + \frac{\partial N_x}{\partial x} dx - N_x)dy + (N_{xy} + \frac{\partial N_{xy}}{\partial y} dy - N_{xy})dx = 0$$

Right? So, then this cancels out and what it gives is

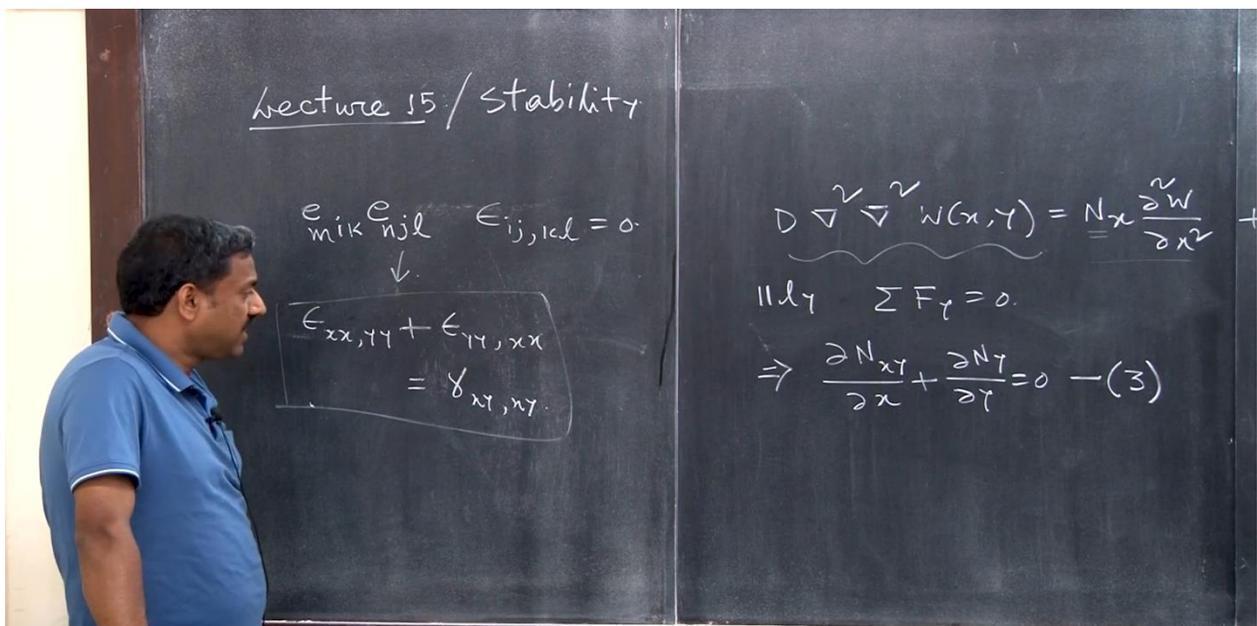
$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad (2)$$

Another equation. So that basically satisfies the equilibrium equation along the X direction. Okay. Now please note that when writing here, N_x , you know, we consider the in-plane equal along the x direction, okay? Because of the deflected configuration, there might be little inclination towards those things, but those we are not considering as if this is what we are writing in the undeformed configuration. Please note that. Similarly, if you want to enforce the force equilibrium along the y direction, then we get

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \quad (3)$$

which is equation three. So, there are three equations: 1, 2, and 3. You see that. Now there are still four unknowns, but three equations are correct. Four unknowns, three equations. So, the problem is not determinate. Okay, so what else can you do? So, right now the system of equations is four, you know, or three, but there are four unknowns, right? So, we will adopt an approach you know that is similar to the approach that you have adopted in elasticity. In elasticity, there are two

different formalisms for the formulation of the governing equation. One is, of course, displacement-based; another is stress-based. The displacement-based approach considers the equilibrium equation first, which is the Navier equation right $\sigma_{ij,j} = 0$ in the absence of body force and inertia body force, and then you know inertial forces. There, static, then you substitute the stress-strain relationship in that equation. Then you substitute the strain-displacement relationship in that equation. And that's what you get: the Navier equation where the equation looks like $U_{i,ijj} = 0$. Okay. So, your displacement field needs to be Y harmonic. Okay, that was the governing equation, right? So, that's the displacement formulation. So, when you do the elasticity displacement formulation, you will have things that are basically nothing but the Navier equation, right? But now, if you use the stress approach, what do you do? You choose a stress function. So, the stress function, the derivative of the stress function, gives the stress component; the gradient of the stress function gives the stress component. So, σ_{xx} is nothing but $\frac{\partial \phi}{\partial y}$, σ_{yy} is nothing but $\frac{\partial \phi}{\partial x}$, and σ_{xy} is nothing but $-\frac{\partial^2 \phi}{\partial x \partial y}$, okay? Then you just substitute the same thing in the equilibrium equation. You see the equilibrium equations are identically satisfied. But then if the equations are identically satisfied, which equation will give you the governing equation?



Yes. You cannot just say that because you are satisfying the equilibrium, it doesn't mean that there will be a legitimate displacement field or stress field. Right? So, you have to satisfy the

compatibility equation of strain compatibility. Why? Because otherwise, your strength field will not be unique. If there is discontinuity, you have to have a displacement field that needs to be continuous. Right? That's why you need to satisfy the compatibility equation for strain compatibility. The strain compatibility is expressed as the curl of the curl of the strain tensor being equal to zero. That means you know Levi's beta symbol ϵ . You know what Levi's beta symbol is, right? Alternating symbol, right? Huh? So, this was the compatibility equation in 3D, but in 1D. In 2D, it said it is very simple. $\epsilon_{xx,yy} + \epsilon_{yy,xx}$ is nothing but $2\epsilon_{xy,xy}$. That was the, or rather, here you write $\gamma_{xy,xy}$. That was the compatibility equation. So, only one equation, right? So, you substitute there, and then you will see the equation where we give you the Laplacian equation, right? That equation you get, so in three dimensions, if you write that equation, it is called the Beltrami equation, okay? So, in stress, if you formulate everything, treating the displacement as the primary variable for the formulation, then it is the displacement formulation that gives the Navier equation. If you formulate everything in terms of the stress function, then you get the Beltrami equation.

The image shows a chalkboard with handwritten mathematical derivations. On the left side, there are two equations for stress components in terms of a stress function \tilde{w} and a parameter h :

$$\sigma_{xx} = N_x \frac{\partial^2 \tilde{w}}{\partial x^2} + 2N_{xy} \frac{\partial^2 \tilde{w}}{\partial x \partial y} + N_y \frac{\partial^2 \tilde{w}}{\partial y^2} \quad (1)$$

$$\sigma_{yy} = h \frac{\partial^2 \phi}{\partial y^2} \frac{\partial^2 \tilde{w}}{\partial x^2} - 2h \frac{\partial^2 \phi}{\partial x \partial y} \frac{\partial^2 \tilde{w}}{\partial x \partial y} + h \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 \tilde{w}}{\partial y^2}$$

Below these, the compatibility equation is written as:

$$\epsilon_{xx,yy} + \epsilon_{yy,xx} - \gamma_{xy,xy} = \frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

On the right side of the chalkboard, the strain components are defined in terms of displacements u and v :

$$\epsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \left(\frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right)$$

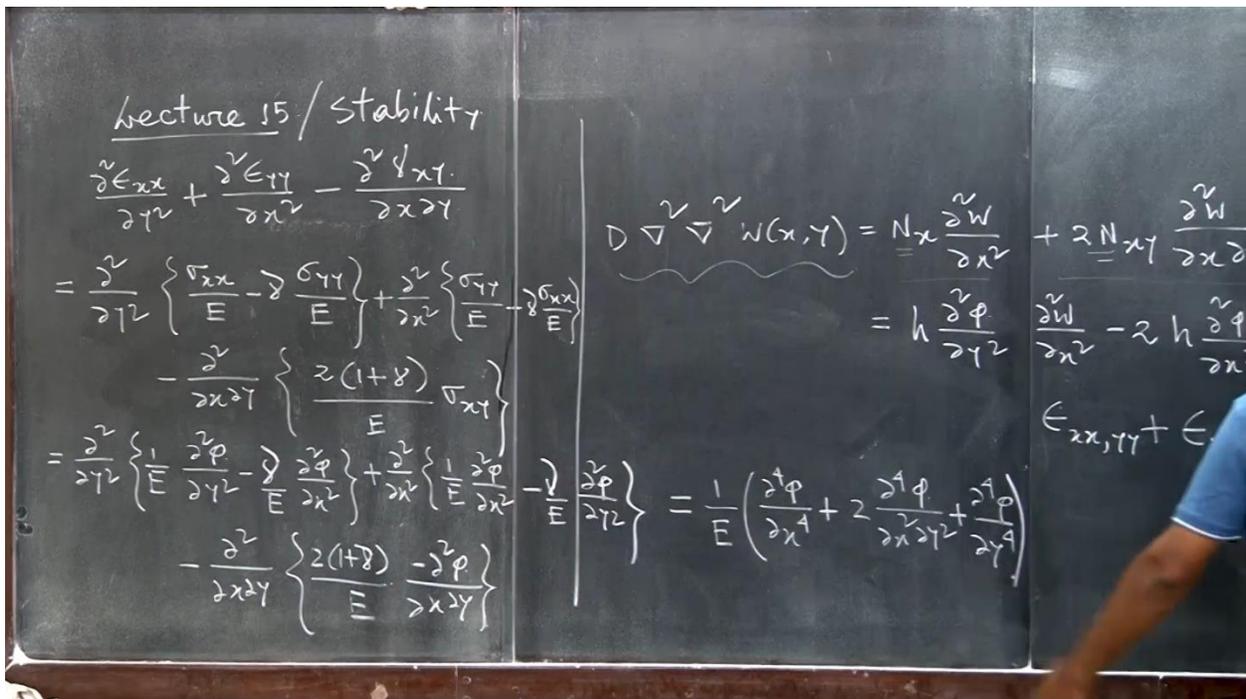
At the bottom right, there is a boxed expression:

$$= \left\{ \left(\frac{\partial^2 \tilde{w}}{\partial x \partial y} \right)^2 - \left(\frac{\partial^2 \tilde{w}}{\partial x^2} \frac{\partial^2 \tilde{w}}{\partial y^2} \right) \right\}$$

The Michell-Beltrami equation is the correct stress formulation. You will see what you have learned in elasticity; all your solved problems that can have an analytical solution are based on stress formulation. Displacement formulation is difficult to approach for a solution except in a few

cases. That's why the displacement formulation is often used in finite element analysis and numerical methods. Whereas the stress function-based approach gives you the Beltrami equation that is used for the analytical solution, that's what a bunch of boundary value problems you have solved, right? You know Bosonic equations, the stress concentration problem, Euler's equation, and other things. Here is what we are going to do. We are going to follow a similar approach as in the stress function, but of course, we have to have our displacement field here, okay. So, what are we going to do about that? Let us assume, let us not assume but let us define the stress function. You know stress. So, the stress function is nothing but a potential function, right? The stress function. So, N_x , N_y , and N_{xy} . So N_x is well thickness, $h \frac{\partial^2 \phi}{\partial y^2}$, $h \frac{\partial^2 \phi}{\partial x^2}$, $-h \frac{\partial^2 \phi}{\partial x \partial y}$. Could you please see that this equation directly satisfies this? With this kind of solution, then what are you going to get? $N_{xy,x}$ is $-h \frac{\partial^3 \phi}{\partial x^2 \partial y}$, and $\frac{\partial}{\partial y}$ is $+h \frac{\partial^2 \phi}{\partial x^2 \partial y}$, which is equal to zero and is identically satisfied. So, with this definition of the stress function, we can see our in-plane equilibrium equations are identically satisfied, right? So, essentially our three in-plane forces can be condensed into a single variable that is a scalar function ϕ , which is the stress function. Clear? Okay. So, since the equilibrium equations are identically satisfied, let us forget about that. Forget about in-plane equation. So now what will this equation be? N_x will be $h \frac{\partial^2 \phi}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - 2h \frac{\partial^2 \phi}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y}$. N_y is $h \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 w}{\partial y^2}$. You see how the nonlinearity is coming into the picture: the product of these two second-order derivatives. You know that's why this equation becomes nonlinear. It's a nonlinear differential equation; it's complicated to solve. We'll see how to solve it, and we'll see that right. But now, is that okay? No, what additional equations are required? We require a compatibility equation. So, compatibility, what is the compatibility equation? Of course, the compatibility equation here is to derive these things. Okay. So, $\epsilon_{xx,yy} + \epsilon_{yy,xx} - \gamma_{xy,xy}$. So, the same unit is $\frac{\partial^2 \epsilon_{xx}}{\partial y^2}$; it is nothing but $\frac{\partial^2}{\partial y^2}$ plus $\frac{\partial^2 \epsilon_{yy}}{\partial x^2}$ minus, see, you have to recall one thing: look. We are considering moderate deformation, okay? So, whatever we have assumed in elasticity, you know, okay, it's not infinitely smaller. We are considering moderate deformation. Okay, let's take a look at that. Okay. $\frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$. So, what we are going to do? we will now substitute, we'll not substitute what is ϵ_{xx} Though ϵ_{xx} , ϵ_{yy} , and γ_{xy} , what are those expressions? What is ϵ_{xx} ? ϵ_{xx} , we have the in-plane forces, $\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2$

von Karman nonlinear term ϵ_{yy} is $\frac{\partial v}{\partial y}$. Why is $\frac{\partial}{\partial x \partial y}$ coming? Because we are considering in-plane deformation here, do you understand that earlier in buckling we didn't consider in-plane deformation? But here there is a variation in the in-plane stress. That's why in-plane deformation will occur; please keep this in mind, okay? In your mind. So, $\frac{1}{2} \left(\frac{\partial w}{\partial y}\right)^2$ and here it is $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}$. Okay. So, now, if you substitute all these things, you can clearly substitute here and then. It's not difficult to substitute, right? You can directly substitute; you can differentiate it, right? Okay. So, and then if you simplify then final expression I'm writing. the final expression will be $\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2}$. You see that you have all these things that have gone. So, the reason why it is gone? Because with this definition of $\frac{\partial u}{\partial y}$, this last component already satisfies the compatibility. Okay, so only the out-of-plane deflection is what was remaining, and that's what is clear. Now this is basically expressing this expression in terms of displacement out of plane. We can express the same thing in terms of the stresses. Because we can have a stress relationship, we can directly substitute the stress function there. So, what we will do here is please note to follow what we are doing, you see? So, the same expression you just take huh.



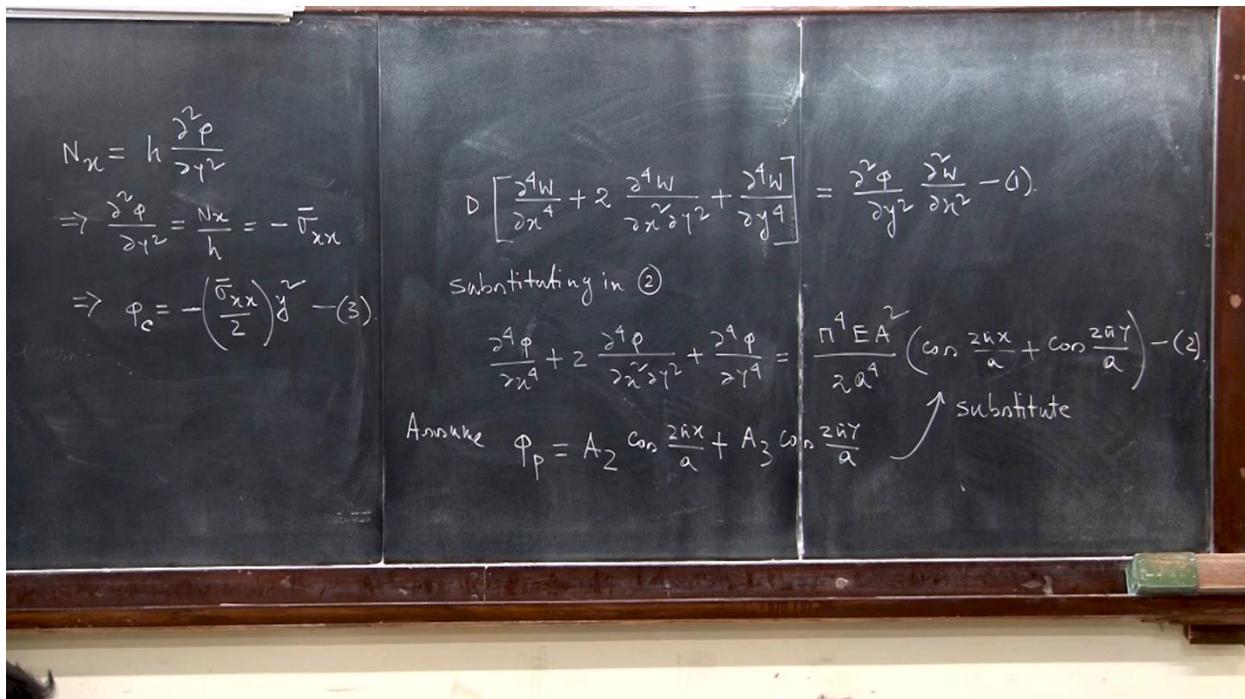
so, you know $\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$. So, here $\frac{\partial^2}{\partial y^2}$, what is it? See the plate bending here; we are considering this with generalized plane stress. So, there is generalized plane stress. So σ_{xx} will be what? ϵ_{xx} is $\left\{ \frac{\sigma_{xx} - \nu \sigma_{yy}}{E} \right\}$ right fine. Then plus here you put then $\frac{\partial^2}{\partial x^2} \left\{ \frac{\sigma_{yy} - \nu \sigma_{xx}}{E} \right\}$ right. Then minus γ_{xy} is nothing but $\frac{\sigma_{xy}}{G}$, which is $-\frac{\partial^2}{\partial x \partial y} \frac{2(1+\nu)}{E} \sigma_{xy}$, and we can clearly see, you know, $2(1 + \nu)$. Yeah, here will be, you know, γ_{xy} . So, $2(1 + \nu)$ times $2(1 + \nu)$ divided by E times σ_{xy} , right? This is equal to $2G(1 + \nu)$, right? This by G, and now you just substitute $\frac{\partial^2}{\partial y^2} \frac{1}{E} \left(\frac{\partial^2 \phi}{\partial y^2} - \frac{\nu}{E} \frac{\partial^2 \phi}{\partial x^2} \right) + \frac{\partial^2}{\partial x^2} \frac{1}{E} \left(\frac{\partial^2 \phi}{\partial x^2} - \nu \frac{\partial^2 \phi}{\partial y^2} \right)$ here $\frac{\partial^2}{\partial x \partial y} \left\{ \frac{2(1+\nu)}{E} \right\}$ then σ_{xy} is nothing but. Here it is $-\frac{\partial^2 \phi}{\partial x \partial y}$. So, if you simplify it further, then you'll see this; you will get on simplifying it further. So, this you will get; let me write it down. This will be $\frac{1}{E} \left(\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} \right)$. Okay. So, look at the right-hand side; the left-hand side in both cases is the same. May I equate this one with this one? Yes. Equating these two, this one and this one, these are the same thing. Okay. So, what here I'm writing. Please see. So,

$$\frac{1}{E} \left(\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} \right) = \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2}$$

This is equation two. Now, you see that all three in-plane equations, all three you know in-plane forces N_x , N_y , and N_{xy} , essentially come into play when we find a stress function. which can express these things N_x , N_y , and N_{xy} by automatically satisfying the equilibrium equation then You know we essentially reduce the dimension of the problem, but then we want to enforce another equation through compatibility. And that's what we have derived. So now you say that we have two unknowns, W and ϕ , and we have two equations. So, we can solve it now. We have two unknowns, W and ϕ ; W is the out-of-plane deflection, and ϕ is the stress function. And then the gradient of ϕ , of course, is the second-order gradient. The second-order derivative of ϕ . Will give the all in-plane forces right. Of course, this equation is also nonlinear, because you see it appears in the product here, okay? And this is clear; everything is clear to everybody. All of you have noted it down, right? Okay, good. So, I'm now removing this. Two important things I would like to point out here. One thing is that, see, when we are doing post-critical analysis for the plate, we have the governing equation. We have to solve it; the equations are nonlinear. However, what is the

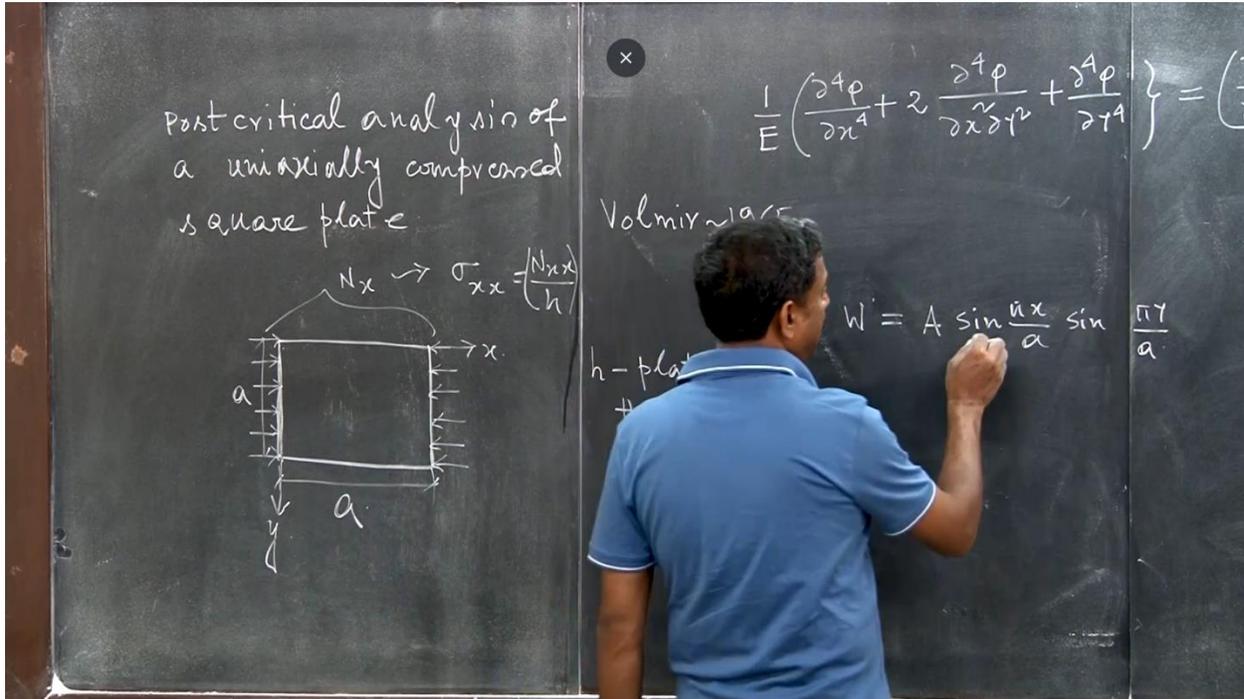
difference from the plate post-buckling analysis of a plate when you consider the elastica? In elastica, when you consider the large deformation of a column, right? Elastica recalls elastica. Elastica, we are considering the curvature, right? And although we didn't express it specifically, what led to an elliptical integral is something you can all recall, right? From there, we have approximated it and then integrated it, and we have shown that. The column indeed shows very little postcritical strength, even when the end rotation is 30° , which is a very large deformation, right? And it shows only a 3 to 4% increase in the load carrying capacity P by P . Here, although we didn't write it specifically, what is this expression? Can you recall that the curvature relation was $\frac{1}{\rho} = \frac{d\theta}{ds}$? It's this. So, see, we literally have considered this expression without explicitly mentioning it. But it involves higher-order derivatives; this nonlinearity is coming from $(\frac{d^2w}{dx^2})^{3/2}$. So, it involves not only second-order terms but higher-order ones as well, right? So, when we are discussing the postcritical analysis of the column, we are considering the higher-order terms in curvature. Are we considering the same thing here? No. Here curvatures are still linear curvature. I mean $\frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial x \partial y}$ cross curvature. Here is curvature. These are still you know not nonlinear. But here it is non-linear. So, then, from where? This is a basic difference between the post-critical analysis of columns and plates. The curvature we are considering is not really what we need, so we are not considering large deformation. In elastic, we have considered finite deformation. Here we are not considering finite deformation. We are considering moderate deformation, which is represented by the linearized expression for curvature, but then when it comes and where it contributes. Look here, you may wonder that, well, if you substitute the expression for N_x , these appear in square terms. I mean these are, you know, accurate until square, but expressing it as appearing in a quadratic term doesn't mean that it is nonlinear. Because when potential energy can only be expressed in quadratic terms, the actual things, which are the actual reasons to capture the post-criticality, are. The fact that N_x is varying; and please note that N_x is $\frac{\partial^2 \phi}{\partial x^2}$, right? So, N_x is not constant, unlike previously; see in the column that the load was essentially the same; you know there was no way to redistribute the load for equilibrium; the load needed to be the same, you see that. "But here, out-of-plane loads are not necessary because it is a two-dimensional plate that allows for the redistribution of in-plane stresses." which was prevented in column, that's why, when do post critical analysis for column, we need to invite higher order terms in curvature here

it is not but then Because N_x is nonlinear and we are taking the product of this term, this term itself is second order accurate, and then N_x , N_y , and N_{xy} are also second order accurate. So, in product, they will be accurate until the fourth, I mean quartic, terms. You see that, and that's why it will capture the post-critical redistribution of stresses and subsequent post-critical behavior. So, do you see the qualitative difference in the post-critical analysis between column and plate? Here, the post-critical analysis does not necessarily require us to consider the higher term in curvature. We are still working in the linearized way. Clear? Okay. Now we will demonstrate one, well I mean one small important thing I would like to point out here, which is that the origin of this term, if you can consider it in terms of the second Piola-Kirchhoff stress tensor. So, here, if you see that when you consider, you know this is the second Piola-Kirchhoff stress tensor. So, this part was essentially contained in this, and this part was giving rise to this, right? Now this S_{ik} second Piola-Kirchhoff stress tensor happens to be constant; that's what is critical in the buckling analysis; these are all constants. But now for post-critical, they are all functions of x and y . So here you cannot take it out; rather, it should be inside. So, now we will present a very simplified solution for the post-critical analysis of a uniaxially loaded, uniaxially compressed square plate.



Okay, so you're going to consider this plate. Huh? And why are we assuming square? Because it's just to simplify the expression. Okay. Instead of assuming it to be rectangular, it is to be square.

Okay. So, this is a, this is a, and then what about this? This is x, and this is y, and all other things. It is subjected to compression. You know this is n_x compressed.

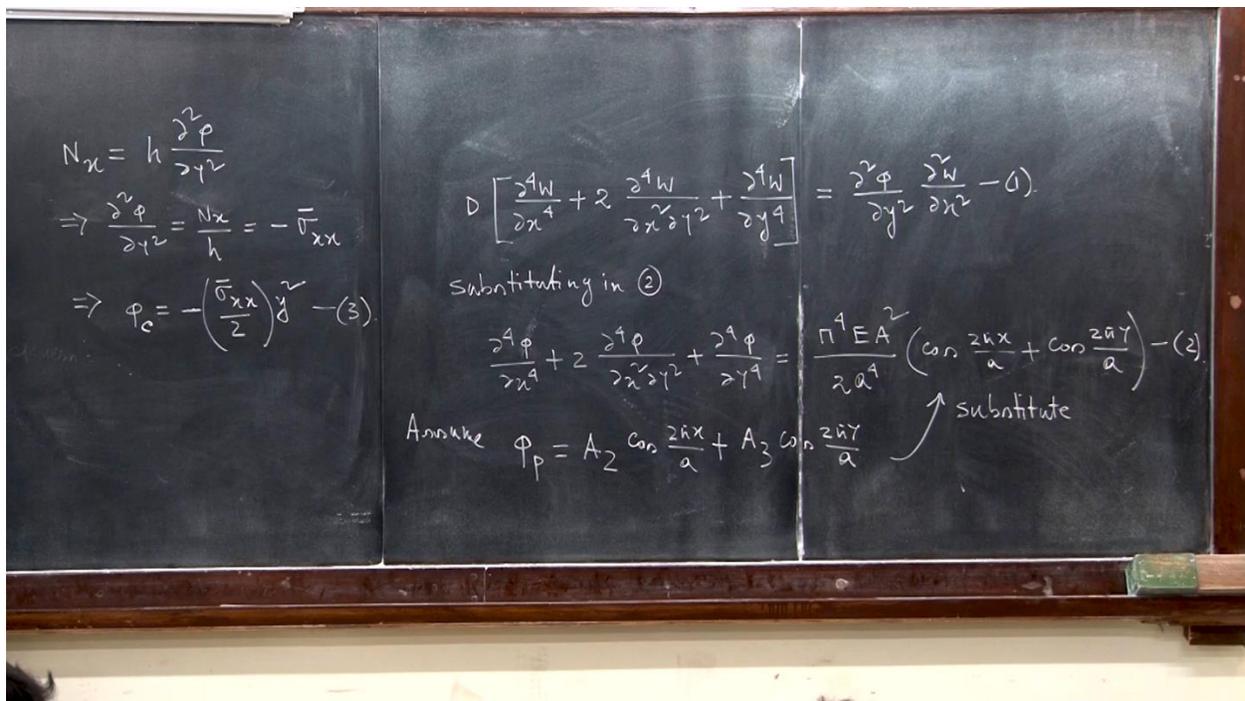


Huh? So, if it is under compression then it will be only N_x . Okay. So, the equation here will be

$$D \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) = \frac{\partial^2 \phi}{\partial y^2} \frac{\partial^2 w}{\partial x^2}. \text{ So, this is equation one. This is equation two. Right? Huh?}$$

These are the two equations we have to satisfy. Right? Now it is uniaxial N_x , and rather maybe the respective stress σ_{xx} will be nothing but n_x over the thickness divided by h. H is the thickness of the plate. H is the plate thickness. Huh? The thickness of the plate. Now we can assume that deflection W can be expressed in terms of sine functions, right? We can assume that right as $\sin(\pi x/a)$, by assuming that these are all simply supported, you know. simply supported simply supported uh and then in plane it is free in the in-plane direction. So, we allow it to compress right. So, w is $A \sin(\pi x/a) \sin(\pi y/a)$. Why we are not considering additional term? Because things are complicated, let us use this solution. We can essentially capture the physics, so we are considering only a single-term approximation of the displacement field. A is the unknown constant. Okay. Now, whatever I am going to present is basically a solution that is very simple and very ad hoc; it is not a very accurate solution that I will present. But this is that it will involve the solution of a nonlinear equation that's analytically cumbersome, and it is, you know, beyond classroom

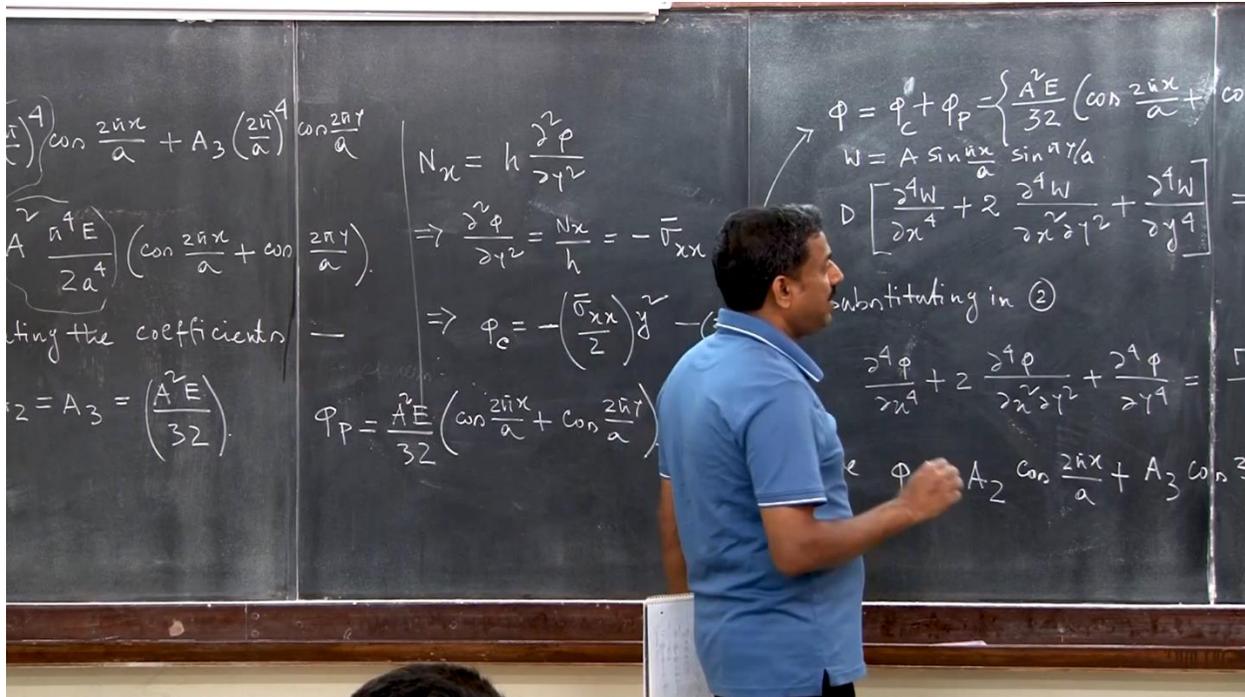
presentation. Because you have to use "but," nevertheless, we have done that. In fact, one of our PhD students has worked in this area. You have to when you want to solve these nonlinear equations. No, analytically, then you have to use some kind of asymptotic expansion perturbation technique, you know. I will present that using PowerPoint, but this involves horrendous algebra, okay? It is very complicated algebra and lots of it is very tedious and boring; that's why I'm not presenting, but what I'm presenting is a working solution given by Volmir, a Russian scientist. Way back in 1965, the 1970s volume is okay. So, this solution was written by Volmir, and I have adopted it from the book of Alexander Chazes. Okay, this is around 1960, 1965, something like that, the Volmir solution. Okay, and this solution is very simple and a little ad hoc. Okay. So, when we assume w to be this, then what we can do is that we cannot predict ϕ as a priority, right? But if we assume it to be this, you know what we can do. We see out of these two questions, this is a little simple, right? So, we can substitute it over here; you can substitute it over here, okay? Okay.



Substituting into two, the equation will be here. This is the simplified formula because the right-hand side is known. So, you just take this solution, differentiate it, substitute it, and then express it using, you know. Sum and difference of the product of sine and cosine functions, and like this. Okay, fine. Okay. Now, what will we do? You see, what we see is that initially the σ_{xx} is given. So, σ_{xx} here is analytically very easy to solve. Because analytically, I mean numerically we can

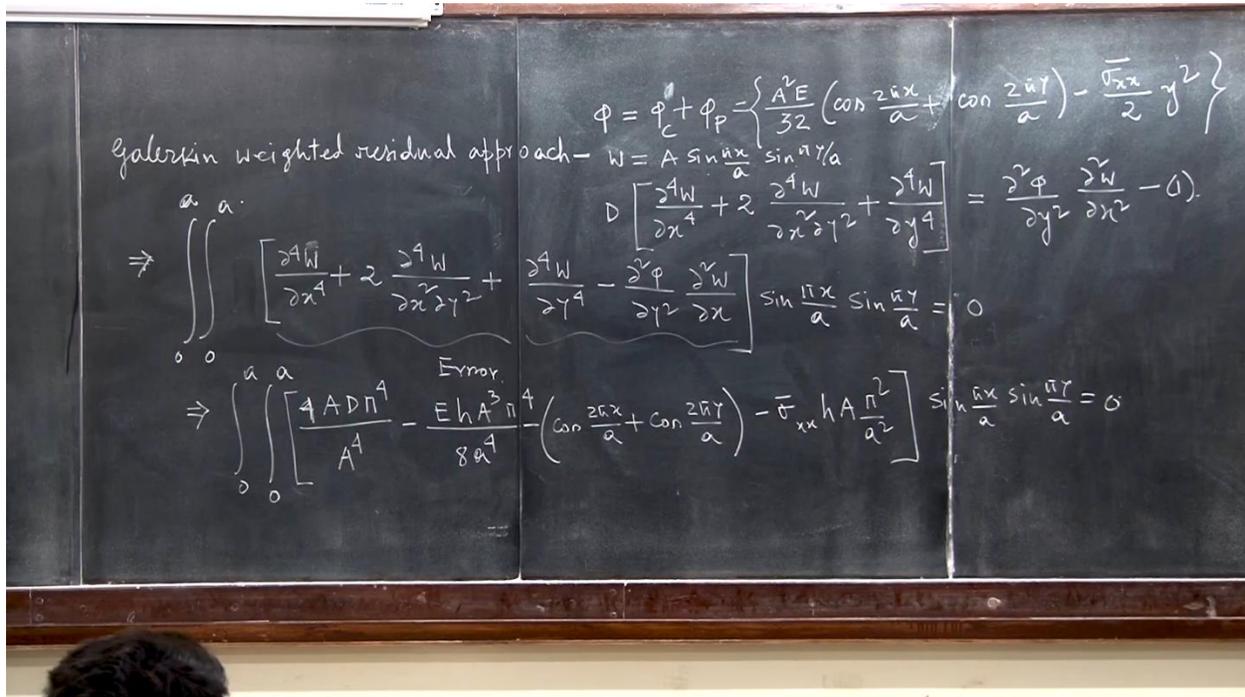
do Bubnov-Galerkin, right? But so, then what we will do is see what we have assumed that, you know, N_x to be. You know, what we know N_x is $h \frac{\partial^2 \phi}{\partial y^2}$, right? So, $\frac{\partial^2 \phi}{\partial y^2}$ is N_x/h , and n_x/h means what? There's some compressive force, right? I'm assuming it is σ_{xx}^- , the applied compressive force. Okay, this one. What you are applying is the sigma compressive force, σ_{xx}^- , applied compressive force, right? So, if we differentiate, you know what you will get? ϕ is $-\left(\frac{\sigma_{xx}^-}{2}\right) y^2$, right? There can be some squares, etc., that we are not counting, you know, considering now, okay? At least, once this kind of term comes into play concerning the expression of the ϕ stress function, we will use some equations. Right now, what we will do is... Then, so this is one of the solutions, okay? One of the components of the stress function, right? Now what we see on the right-hand side is that these kinds of terms are coming, okay? So can we assume that, well, it can have a particular solution which involves this kind of term, so we can assume, you know, another solution, so this is maybe ϕ_1 . You have also learned the particular integral and you know the complementary function, which can be denoted as C, the complementary solution, and another particular solution. That we can assume $A_2 \cos(2\pi x/a) + A_3 \cos(2\pi y/a)$, and then substitute this in this expression. Substitute if you substitute in this expression. So, when you are substituting both side then you will see this kind of things you will get. Okay. $-A_2(2\pi/a)^4 \cos(2\pi x/a)$ sorry, there is no minus plus. Okay. Plus $A_3(2\pi/a)^4 \cos(2\pi y/a)$ is equal to $\frac{A^2 \pi^4 E}{2a^4} [\cos(2\pi x/a) + \cos(2\pi y/a)]$. Now we will equate the coefficients. So, we'll equate these coefficients. We'll equate these coefficients with these coefficients, and we are equating these coefficients to equations. So, then you will see the equating of the coefficients. Okay. Then you will see that $A_2 = A_3 = \frac{A^2 E}{32}$. Okay, so if that is so, then here you can see we may write this ϕ_P , that whatever we have assumed, $A_2 = A_3$. So, $\phi_P = \frac{A^2 E}{32} [\cos(2\pi x/a) + \cos(2\pi y/a)]$, right? Okay. So now this is ϕ_P ; we treat this as one solution. This is one solution, then we can sum up this. Look, the thing is that. So, by combining these two contributions, we can assume the stress function to be these two components complementing each other; we just write it as follows. $A^2 E$, A is an unknown constant that we have initially introduced while approximating the displacement field.

$\frac{A^2 E}{32} [\cos(2\pi x/a) + \cos(2\pi y/a)]$ and then the one which we obtain from the $-\frac{1}{2} (\sigma_{xx}^-) y^2$, let this be the f function ϕ .



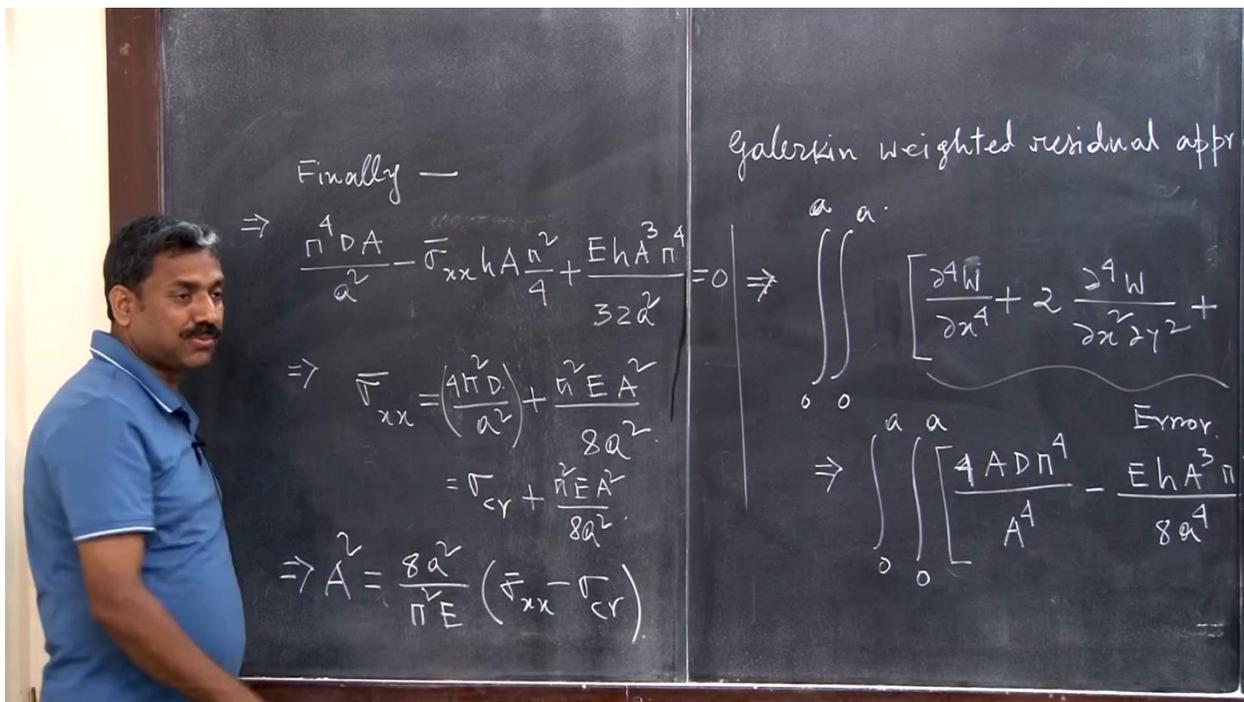
So, you see how we have obtained the displacement function; it was kind of okay to assume that it has to satisfy the boundary condition. So, that's what we were; it was easy to assume the displacement field. You can recall it was a sine once again: $\sin(\pi x/a)\sin(\pi y/a)$, right? But for this stress function, how we have done that is by substituting the compatibility equation. We have assumed that N_x can be expressed as a stress function with this kind of, you know, gradient formula, right? Integrated, this kind of term should be there. So, we sum it up, and then this. So fine. Now we have made use of the definition of the stress function. We have made use of the compatibility equation. The only thing that needs to be satisfied is the equilibrium equation. This one, and we have a ϕ expression for ϕ expression w , and we have only one unknown, a . So now, what will you do? Now we cannot do it; we have to apply the Bubnov-Galerkin method. Okay, so let us use this ϕ , let us use this W , and let us substitute into this equation. Whatever error is in this equation, you use that error. This is to be the weight function, and then you integrate over the thickness to make that error zero, following the Galerkin weighted residual approach. And from there, you will get another condition that will give you A . Then let us see what happens. So, the approach is not great. I mean, this is a very simple approach, a little ad hoc. But the beauty is that even with this kind of simple stuff, you know we can obtain a working solution that beautifully explains the post-critical phenomena and what is going on in the physics. That's why we are doing it. It should have been a very elegant and very accurate solution. A quite accurate solution could

have been obtained using perturbation asymptotic expansion. But then those involve tremendous algebra, you know. So, we want to avoid those kinds of dangerous algebra. Okay. We want to make it simple. That's what we want to do. Okay. Okay, so let us do this. Now let us do the Galerkin thing. Huh. So, in Galerkin, what will we get? Same as what we have to do: integration from 0 to a, integration from 0 to a.



So, the Galerkin weighted or residual approach that you have learned in finite element is also correct. What is Galerkin? Bubnov Galerkin. What is the difference between Bubnov and Petrov? Put in the finite element. Huh? What is the difference between Bubnov and Petrov? Huh? I didn't teach you, right? So, you go and review. This is the error in the equation, right? This is the error in the equation, and I don't care about D, which is anyway a constant. So, this is the error term. If you are having an approximate, sorry, you assume that W is not exact when you are assuming that W is not exact. It's some kind of approximation, and we want to minimize the error to find out a; this one is the error, and then what is the weight function? Which weight function should you use, the same one that has been used to approximate the displacement field? That's your trial function. This is your trial function and test function; both are the same. Trial and test functions are the same for Bubnov-Galerkin, $\int_0^a \int_0^b \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} - \frac{\partial^2 \phi}{\partial y^2} \frac{\partial^2 w}{\partial x^2} \right) (\sin(\pi x/a) + \sin(\pi y/a)) = 0$. We substitute w for ϕ , you know, ϕ you substitute w, and you integrate it. Can you do that? Okay. It's

not very complicated. I will write down the final expression. If you do, then, well, I mean I can do it if you want. So, Oh my God. Do you want me to write the expression? I will write it partially. If you substitute, then you will get this expression: $\int_0^a \int_0^b \frac{4\pi^4 DA}{A^4} - \frac{Eh\pi^4 A^3}{8a^2} - (\cos(2\pi x/a) + \cos(2\pi y/a))\sigma_{xx}^- h \frac{\pi^2}{a^2} A = 0$. Please note that capital A is the unknown coefficient. So, whether small a is the width or length of the plate, I will simplify it from here. You can do the integral step by step; whatever I am not doing from here directly, whatever we get finally, if you do this integration and the same integrating trigonometric function, it's not a big deal, right? So, finally, you will get this kind of equation. Okay. So, you have obtained W. You see that σ_{xx}^- is the extra compression that is applied externally. What is this $(4\pi^2 D/a^2)$?

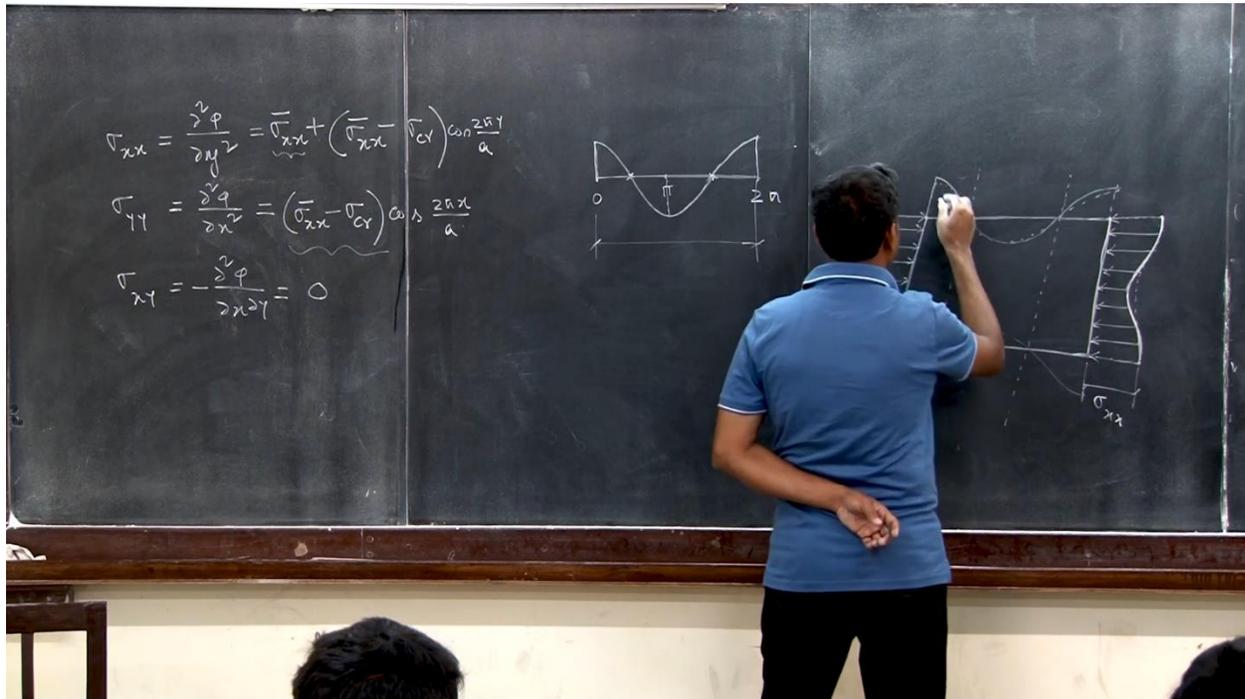


This is nothing but σ_{cr} that we derived in the previous class $(4\pi^2 D/a^2) + \frac{\pi^2 EA^2}{8a^2}$. Okay. So, from here, we'll further see that a is nothing, σ_{xx}^- , σ_{xx} is nothing but $\sigma_{cr} + \frac{\pi^2 EA^2}{8a^2}$. So, from there we can find a. So, from here you will see that A^2 is nothing but $\frac{8a^2}{\pi^2 E} (\sigma_{xx}^- - \sigma_{cr})$. This is so, it is nothing but, you know, you can take the square root of a square; let it be A^2 , okay?

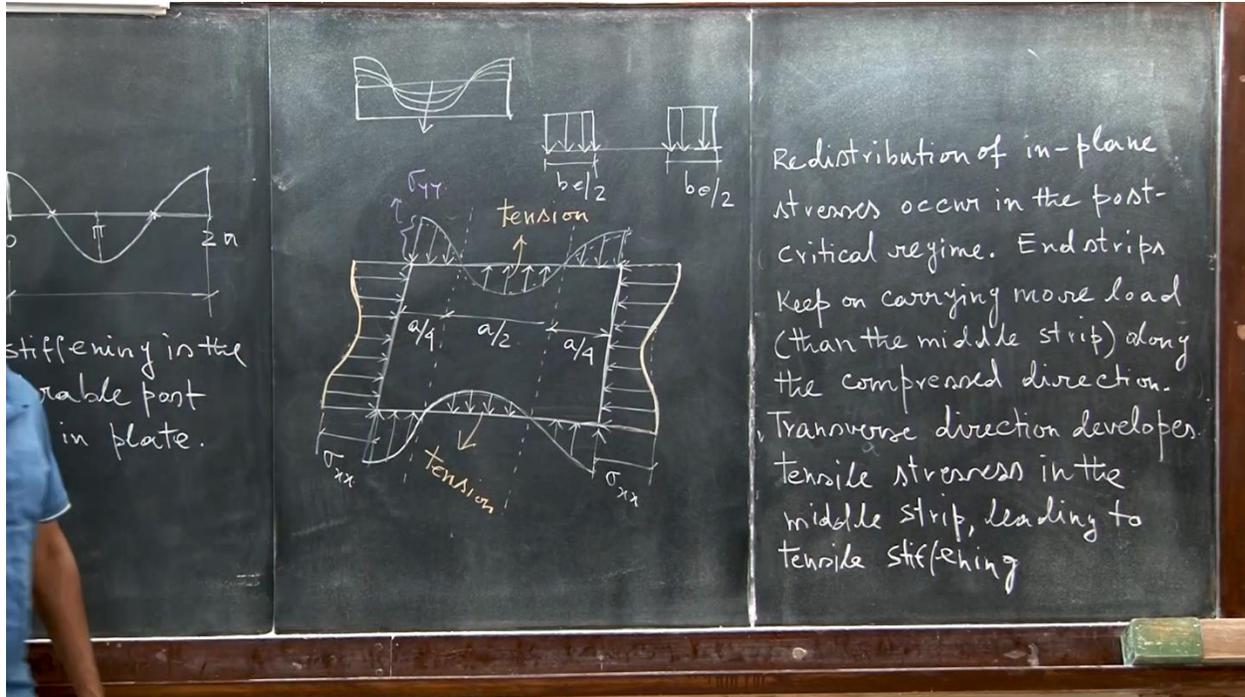
$$\begin{aligned}
 \phi &= \frac{A^2 E}{32} \left(\cos \frac{2\pi x}{a} + \cos \frac{2\pi y}{a} \right) - \left(\frac{\bar{\sigma}_{xx}}{2} \right) y^2 \\
 &= \frac{8a^2}{\pi^2 E} (\bar{\sigma}_{xx} - \sigma_{cr}) \left(\frac{E}{32} \right) \left(\cos \frac{2\pi x}{a} + \cos \frac{2\pi y}{a} \right) - \left(\frac{\bar{\sigma}_{xx}}{2} \right) y^2 \\
 \Rightarrow \phi &= \frac{a^2 (\bar{\sigma}_{xx} - \sigma_{cr})}{4\pi^2} \left(\cos \frac{2\pi x}{a} + \cos \frac{2\pi y}{a} \right) - \left(\frac{\bar{\sigma}_{xx}}{2} \right) y^2
 \end{aligned}$$

So, from here we find out which was the unknown coefficient that all of you have noted down, removing this extra, right? Huh? So, we don't require all other things, but we require ϕ , the stress function, which also includes a huh. So here is ϕ . If you can recall, it was $\frac{A^2 E}{32} [\cos(2\pi x/a) + \cos(2\pi y/a)] - \left(\frac{\bar{\sigma}_{xx}}{2}\right) y^2$ and A^2 ; you just substitute this A^2 over there. this square was nothing but $\frac{8a^2}{\pi^2 E} (\bar{\sigma}_{xx} - \sigma_{cr})$, $\frac{E}{32} [\cos(2\pi x/a) + \cos(2\pi y/a)]$ Whatever is coming, you know square $\bar{\sigma}_{xx} - \sigma_{cr}$ here divided by $4\pi^2$ into $\left\{ \cos\left(\frac{2\pi x}{a}\right) + \cos(2\pi y/a) \right\}$. Okay, this is the stress function expression. Now, once you obtain the stress function, note that there is a difference because earlier, all the axial forces were constant, but here they are not. So σ_{xx} is no longer prevalent all the way up the plate. Let us find out σ_{xx} , σ_{yy} , and σ_{xy} . Okay. σ_{xy} , anyway, will be zero. But what is σ_{xx} ? σ_{xx} is $\frac{\partial^2 \phi}{\partial y^2}$ and $\frac{\partial^2 \phi}{\partial x^2}$ minus $\frac{\partial^2 \phi}{\partial x \partial y}$, and $\frac{\partial}{\partial x} \frac{\partial}{\partial y}$. Of course, it will be zero. You can clearly see that σ_{xx} ; you will see from there if I'm directly writing the expression. So here it will be $\bar{\sigma}_{xx}$, and then plus $(\bar{\sigma}_{xx} - \sigma_{cr}) \cos(2\pi y/a)$, that one. So, you finally obtain this expression right; please let's see that. Now see when you, what is the variation of cosine. Cosine, see 0 to 2π . Now from 0 to 2π , you see that π is here, $\pi/2$ is here, and $3\pi/2$ is here. So, here it is: Sin 0 cosine 0 is 1 and π is basically -1. So, this varies, right? This is positive, +1, +1, -1, and there it is 0, right? Huh. So, if you see

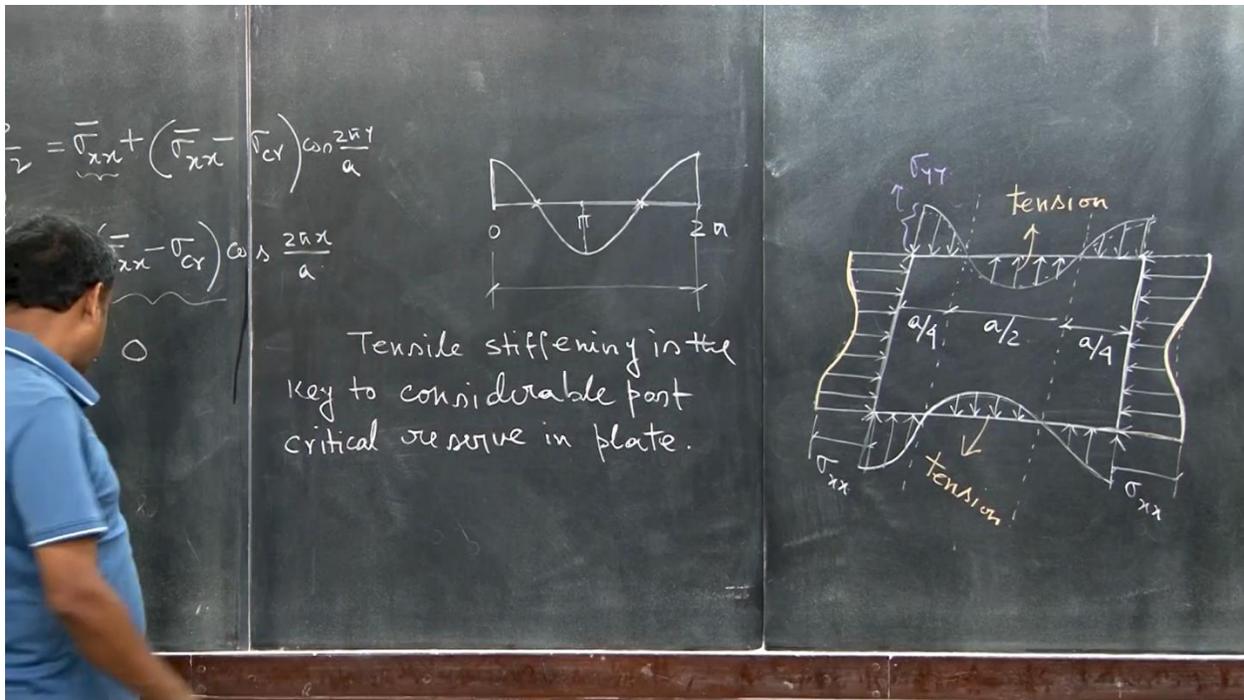
that σ_{xx} , this compression was there, but this is associated with $\cos(2\pi y/a)$. So, this will be modified with the cosine function. In the other direction, this σ_{xx} is minus $\sigma_{cr} \cos(2\pi x/a)$.



So, it depends on whatever the sign is; depending on this, it can be negative or positive, right? So let us plot this. So, this is σ_{xx} plus whatever this is. So σ_{xx} , see this is σ_{xx} , okay. σ_{xx} and what is σ_{yy} ? Look at what is happening. This is all σ_{yy} , and this is σ , huh? What is happening? You, see? Maybe at the end, σ_{xx} will be, you know, prevalent. But as you gradually move, in fact, at the end also, once it is in postbuckling, there is a redistribution of stresses that occurs. So, the end portions will remain the same, but in the middle, you know there will be fewer forces. Okay. And why this is happening is because of this variation. For the other, you see, there is some nonuniform redistribution. Okay. Is, and then if you know, keep on increasing the extent of postcritical, you know, postcritical mobilization of the stresses in this. What will happen eventually? This will deplete. This will keep depleting, and more and more forces will be shared by the end strip because it will reduce. The other direction, what is happening, is still under compression. Earlier, it was in the pre-buckling regime or even just on the verge of buckling; in the other direction, there was no load. There were no stresses, but here there are stresses developed. So, at the end, there is still compression, compression, compression. But do you see what is happening in the middle? These are under tension. Do you see that? This is tension.



So, if there is tension, then what will happen? There will be some tensile stiffening because, as we all know, there is erosion of stiffness under compression. But under tension, there is a stiffening effect. So, it will further stiffen; that means it can still carry load beyond buckling. And so, unlike a column, a plate doesn't lose its strength upon reaching buckling; rather, it has significant post-critical reserve, and more out-of-plane deflection will mobilize more such influence stresses. At the end, what will happen is that the end strip will essentially carry the whole load. So, please note down here that the redistribution of influences of in-plane stresses occurs in the postcritical regime. Hmm, the end strip keeps carrying more load than the middle shape along the along or in the compressed direction. Huh? Compressed direction. Transverse direction. The transverse direction develops tensile forces and tensile stresses in the middle strip. The strip leading to tensile stiffening, and all this you know, if you see this one, is basically a 4 by 4; this strip is a 4 by 4, and this middle one is a 2 by 2, okay? So, this tensile stiffening is the reason for post-buckling significant tensile stiffening; it is the key to considerable post-critical reserve in the plate. At the end, as you keep on increasing the out-of-plane deflection, the redistribution keeps on increasing, and in the end, you will see what will happen. Then this is a dynamic process.



So maybe here this is happening, you know this is the, so initially it will be something like this, okay, and gradually it will go, and then things will be like this. I mean, of course, this will increase. Rather, sorry, you know. So initially, you know it was constant, okay? It was all the same, and then what happened was this fellow would increase while this would be decreasing, something like that, okay? Then this will further increase, and this will further decrease something like this. This will further increase, and this will further decrease something. So, that is basically the dynamic process of redistribution. In the end, what will happen is that this two-end strip will take the whole load. The end strip will take the whole load, you know, and then you can, I mean, this is extreme; this is some kind of idealization, be effective by two. So, there is something called the effective width concept that has also been adopted in the AISC standard and the Canadian standard. In 1933, von Karman gave some expressions for the effective width. And that you know, of course, there have been, you know, statistical analyses done by Lind and others to, you know, make it accountable for, you know, score stipulation. But you see, what will happen essentially is how you will find out the effective width. The effective width you can find out; you see that the way you will do it is to satisfy the equilibrium, right? So, I'm assuming that σ_{max} is the maximum stress, okay? σ_{max} will be equal to σ_{xx} , and along dy, this is from 0 to b. This is okay because this is varying, right? σ_{xx} is, so from there you can find out what the effective W is. So, this is some kind of ideology for the development of code stipulation. Okay, Both Canadian code and AISC

American code still adopt that for the design of plates, and in plate design, you know they make use of this post-critical reserve of plates. After buckling, it can still take some load, okay, because of inlendary distribution. So, this is the difference in plate buckling compared to post-buckling behavior of columns. Anyway, in the next class, we will continue. Thank you very much.