**Stability of structure**

Prof: Sudib Kumar Mishra

Department of Civil Engineering

IIT KANPUR

WEEK-07

Lecture 13: Buckling Plate

So, okay, welcome to lecture 13 on the stability of structure. So, let us briefly recapitulate what we are discussing. We were discussing the buckling of plates, plate buckling in general, and before buckling, we briefly reviewed the theory of plate bending. We have reviewed the theory of plate bending, and then we have derived the equilibrium equation for plate bending. The theory of plate bending, you know, involves some assumptions about plate bending, and the plate bending we are considering is Kirchhoff's plate bending theory.
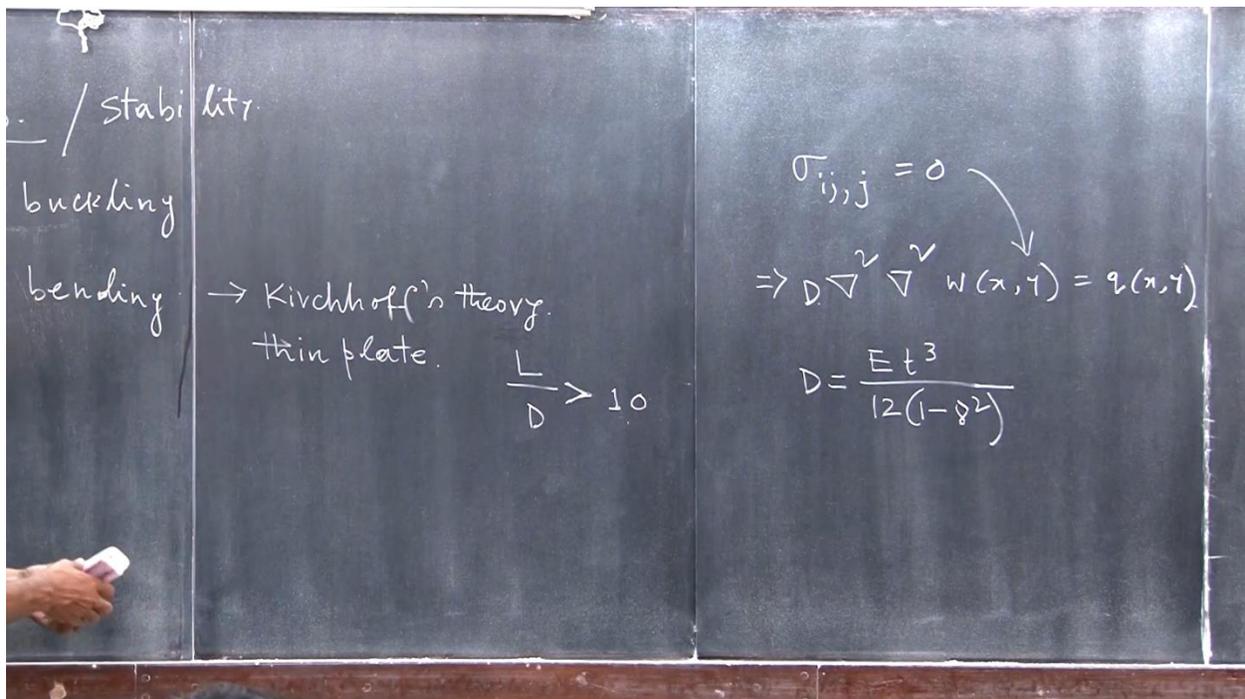

So, here it is: thin plate bending. We are assuming that it is thin plate bending, and then there are certain assumptions, and the main assumption is that you know. Of course, thin plate means, you know, its depth over the length over this ratio is much greater than 10, you know, okay. And height to 10, whatever, okay. It's not very, you know, sec number though, and then the plain section remains plain after bending, right? All these assumptions, and then we have, you know, introduced.

We have reviewed that there are two approaches: one we can start with directly, using the equilibrium equation, right? From what we have learned from continuum mechanics. Or we can, and then, the first approach that we consider is that $\sigma_{ij,j} = 0$, of course in the absence of body force and then in the absence of inertia. And then, this will give three equilibrium equations. So now, if you consider the Cartesian coordinate system, we integrate this; you know, these are the equilibrium equations in terms of the stress tensor, right? So, if you integrate this over thickness, okay, along the Z direction, you know, over thickness, directly integrate the equilibrium equation over thickness. Then you will get the equilibrium equation; you know where you will relate the shear forces to the externally applied load, right? And then you multiply the equation by Z, which means the thickness coordinate, and then you integrate for equilibrium along x and z, okay? So, there are three equilibrium equations: one equilibrium equation along the z direction, another equilibrium equation along the x direction, and another equilibrium equation along the y direction. So, along the z direction, that's what you integrate over the thickness; that will give you the equilibrium in terms of shear, I mean externally. Now the other two equilibrium equations in the X and Y directions, you have to multiply those things by the thickness direction Z and then integrate over the thickness. That will give you two equations where the bending moment and the twisting moment are related to shear forces, right? Then, you combine three equilibrium equations, everything expressed in terms of bending moment and twisting moment. Then you consider the kinematics of deformation, or you relate the bending moment in terms of the texture rigidity of the plate and then the curvature of the plate. So, of course, there will be direct curvature, $\frac{\partial^2 w}{\partial x^2}$, but cross curvature $\frac{\partial^2 w}{\partial x \, \partial y}$, because the twisting moment is there, right? So, there will be three, you know, moment-curvature relationships. Two are for direct bending moments, and another one is for the twisting moment. Okay. And then everything, if you combine, everything you are getting the governing equation: $D\nabla^2\nabla^2 w(x,y) = q(x,y)$, okay? And D is the flexural rigidity for the plate. So, $D = \frac{Et^3}{12(1-v^2)}$, v is the Poisson ratio, w is the vertical deflection right and $q(x,y)$ is the external loading right, and the choice of coordinate that's what we have made is x, y, z right It is now this: we can directly derive from there, or what we have considered. I have also shown in the previous lecture that you can directly consider a differential element, you know, dx. And then you can write down the bending moment, and then, you know, sorry, the twisting moment, and then the bending moment, okay, and then the shear force. And then you directly, of course, these are the bending

moment and twisting moment per unit length of the plate, right? And then you can directly write down the equilibrium equation and moment equation. And then you can derive the same governing equation, right? So basically, we have reviewed the theory of plate bending. Okay. Kirchhoff thin plate bending, right? So now, what? This must have been, and then it will be accompanied by the boundary conditions. Right? You can see this is the fourth-order governing equation, right?
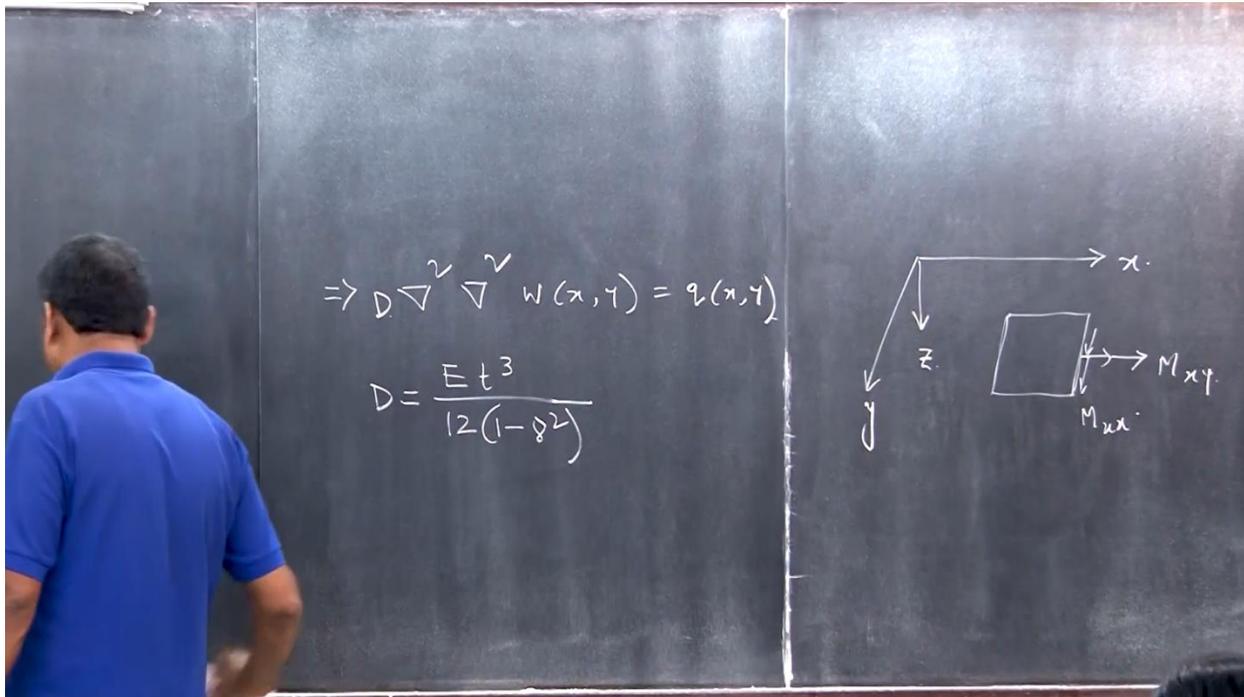

A partial differential equation is linear, and you require four boundary conditions, right? So, on each side, there are two boundary conditions. Okay. Now, if you want to do plate bending, then what additional thing do you require? The additional thing that is required for the buckling of the plate is, you know, two ways you can do it. You know about plate buckling. The first thing you have to consider is the equilibrium in the deformed configuration. Here we didn't consider the equilibrium; in the deformed state, we have considered the equilibrium in the configuration. Okay, without deflection, right? So, the first assumption that we are going to make in the theory of plate buckling is that this equation, and then buffer buckling, we don't care about the external loading $q(x, y)$, right? So, we have considered, let us consider this part, the first part; this is the flexural rigidity of the plate per unit length, right? Now, you may wonder why this equation is not changing because we have to consider the equilibrium in the deformed configuration, right?

Lecture 13. / Stability

Plate buckling.

$$D \nabla^2 \nabla^2 W(x,y) = 0$$
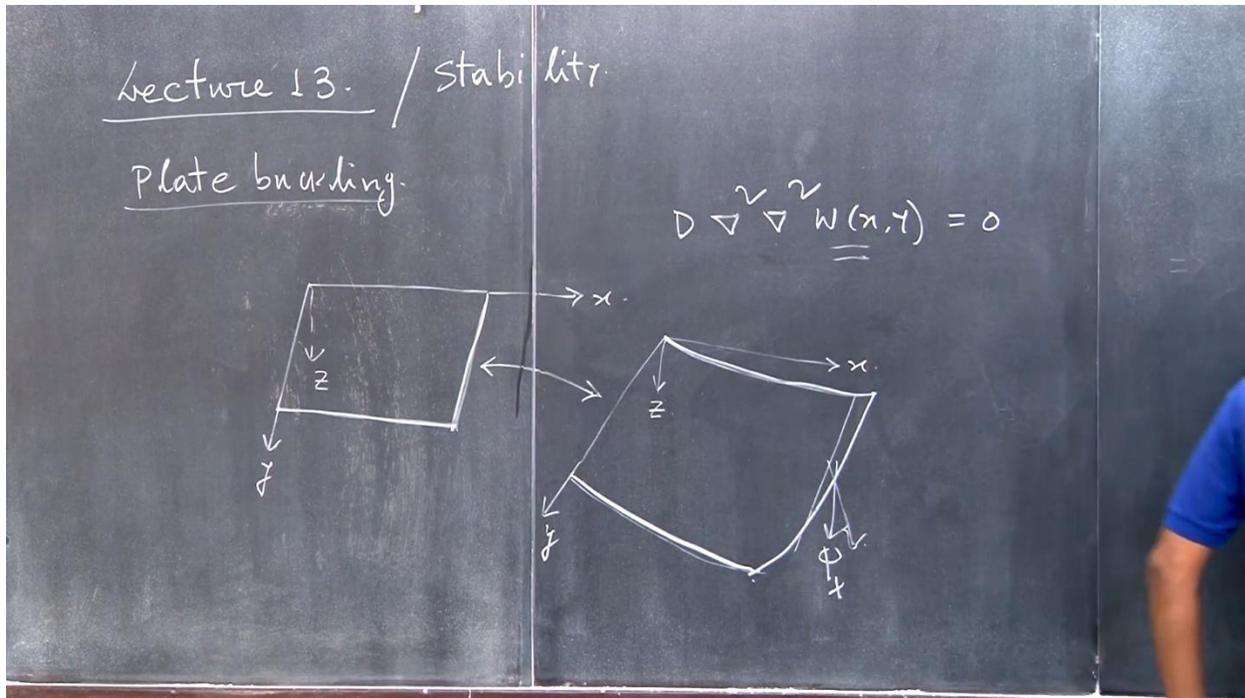
So, what we are considering is the equilibrium, you know, of the plate. It is something like this: X, Y, Z, something like that, right? But what we are supposed to do, we must consider the equilibrium in this configuration, right? You know, the deformed configuration, right? Something like this, you see that. Maybe here is the X, here is Y, and here is Z. Okay. Of course, here you do not understand that you know something like this, and then here it will be something like this. It will be something like this, something like this, right? So, there will be some deformation, right? What assumption we are going to make is that, in the deformed undeformed configuration, from undeformed to deformed configuration, okay? What will happen? Because it is deformed, we should consider the equilibrium in terms of shear force or bending moment. It might be that there is an extra component that might come due to additional curvature that is coming. But we are considering that, as far as these equations, this part of the equation is concerned. This part of the equation is basically in terms of shear forces, vertical shear, not out-of-plane shear, right? Shear forces, whatever the bending moment you know, don't really change significantly because of this deformed configuration. So, as far as this equation is concerned, we don't distinguish between the deformed and undeformed configuration. You understand what I'm trying to say; this equation may change a little bit by considering this different configuration of the plate. Because there is little slope here, you know, okay, and so earlier this was the shear force q_x, right? Because there is little slope, there might be some, you know, vertical component when you are considering. There

will be some component q_x into something like $\frac{\partial w}{\partial x}$ into something like that, okay? But we are not considering that; we are considering those contributions to be negligible. Do you understand that? Now, however, we cannot do it. So, because you know, buckling also essentially require that you need to have in plain forces, right? So, what we are considering here is basically the equilibrium of the out-of-plane forces. This basically considers the equilibrium of forces in the z direction and moment equilibrium along the X and Y axes. That is the consequence of these three equations. But we don't consider the plane forces here, right? We have to consider in-plane or compressive forces here, right? So, let us keep it. And then what we are considering, let us consider now the influent forces, and you see the component of the influent forces that is contributing to this equation. because of this deformed configuration. only that part we are going to consider. So, for that let us see what we are going to do. Okay. We are assuming that we are considering the plate; you know this is the plate, but it's in a deformed configuration. You know, and what you can see, the in-plane forces you can consider, you know, in-plane forces, please draw it, huh, are N_x, N_y, and N_{xy}.
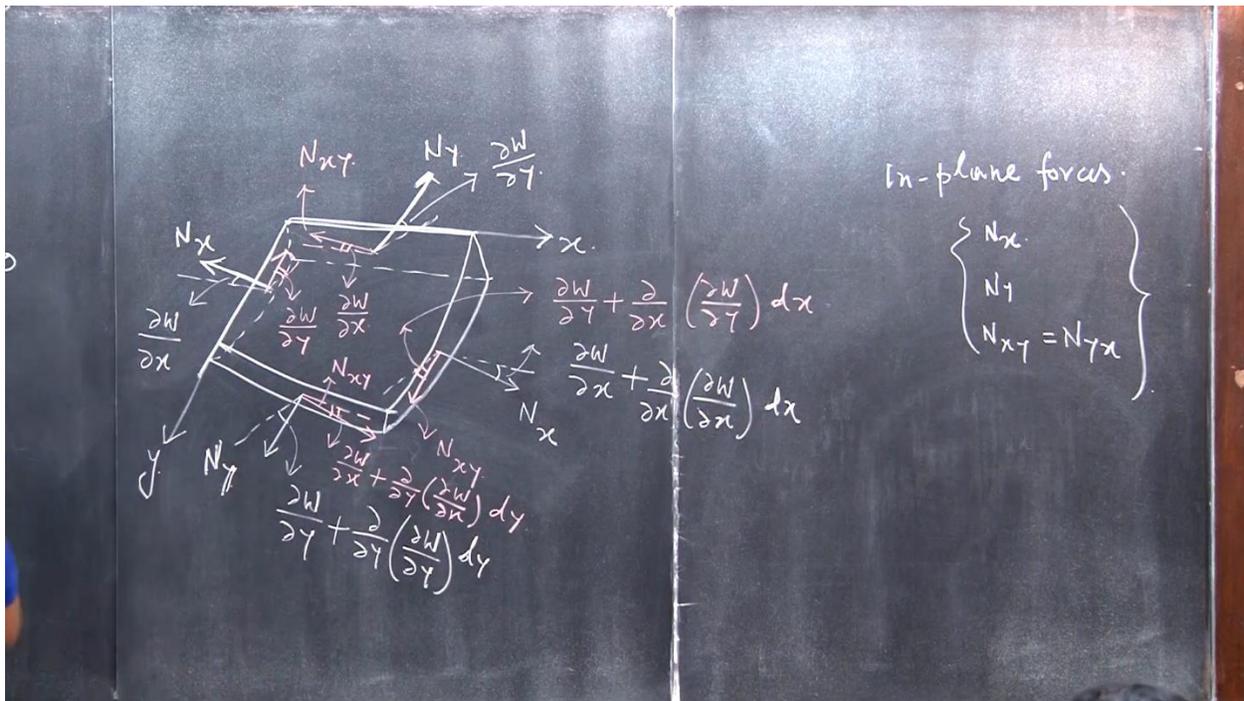

Please note that N_{xy} and N_{yx} are the same. As of now, we have assumed that these are tensile forces, but later the auto sign will automatically come. So now this is N_x, this is N_y, and this is N_{xy}. Okay, N_{xy} and N_{yx} are complementary; that's why we are considering all of the in-plane

forces and deformed configurations. So, what we really want to do is see how, because of the deformation and the perturb configuration, you know, as required in stability analysis. We want to see how these in-plane forces are going to contribute to the vertical equilibrium. Essentially, this equation is nothing but a vertical equilibrium equation in the Z direction. Right? What is the origin of this equation? Equilibrium along the Z direction. Right. Equilibrium along Z, that is the first equation right, and then of course we have. So, we want to see how these in-plane forces are affecting the equilibrium along Z. Okay. So, the component of in-plane forces along the Z direction. Okay. So, what are the components N_x and N_y, and if you see that, we are assuming N_x, N_y, and N_{xy} to be constant over the plate. There is no variation, you see. So, N_{xy}, N_x is the same. So, it's subjected to constant axial force, membrane force. Okay, so what is this? This will come downward, right? $N_x \frac{\partial w}{\partial x} + dx \frac{\partial}{\partial x}(\frac{\partial w}{\partial x})$ right and this one will come downward; this will come upward. So, this is $-\frac{\partial w}{\partial X}$, right? Similarly, for N_y, what are we going to get? N_y, look, this is N_y, right? So, this vertical component downward, this one, it is $\frac{\partial w}{\partial y} + \frac{\partial}{\partial y}(\frac{\partial w}{\partial y})dy - \frac{\partial w}{\partial y}$, right? And of course, if you want to consider n_x, we are considering a very small element of length dx and dy. So, this distance is dy. I want to multiply by dy, right? The dy component in-plane forces is right. So, dx this similarly here, we should multiply it with dx, right? Then what are the additional stuff, plus here, plus n_{xy}? What about n_{xy}? This is n_{xy}. It is downward. What is that? $\frac{\partial w}{\partial y} + \frac{\partial}{\partial x}(\frac{\partial w}{\partial y})dy$ right, and then minus this one, this one is what? $\frac{\partial w}{\partial y}$ right, $N_x N_{xy} \frac{\partial w}{\partial y}$, and this is into dx plus N_{xy}. Now you consider this one: $\frac{\partial w}{\partial x} + \frac{\partial}{\partial y}(\frac{\partial w}{\partial x})dy - N_{xy}$, and then $\frac{\partial w}{\partial x}$ and dx. So, these are all plus signs. You see that this is for N_x, this is for N_y, and this is n_{xy}, right? All of you understood how this expression is coming from, right? Okay, then we are considering this force and its component along the Z direction, right? And then from both sides, that's what within one expression, we are basically subtracting, for the component for negative x negative y and this. So, if we simplify this, this will go; this and this will go; this and this will go; this and this will go. So, ultimately what we are going to get is here; here you see that

$N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \, \partial y} dxdy$ here, right. So, over this small segment, you know, the small plate segment in the deformed configuration, because of this deformed configuration, if we

consider the contribution of the in-plane forces along with the vertical equilibrium, then we get these forces right. So now, for per unit length, you know the component of in-plane forces along the positive Z direction per unit length. This is for the $dxdy$ element. So, per unit length, what will it be? Now you just divide $dxdy$, right? So, $N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \, \partial y} + N_y \frac{\partial^2 w}{\partial y^2}$ right now. So, you understand that. This must contribute to this equation.
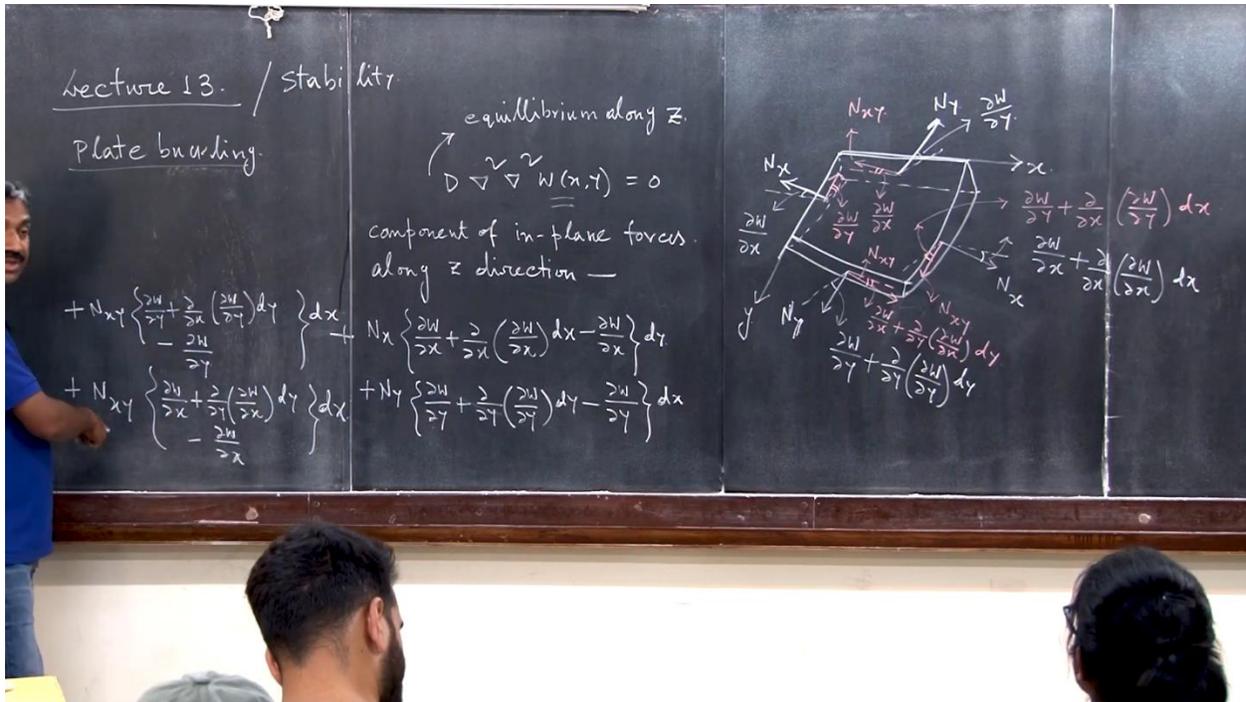
So, on the right-hand side, these are nothing but vertical equilibrium. On the right-hand side, we will add this term. So, this basically is going to be $N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \, \partial y}$, right? So, I'm removing all other things. Right? All other things. Huh? But please note that this is very important because of this deformed configuration; if we had considered this, shear forces q_x and q_y would be okay. They will also have little inclination, and that component will contribute vertically; however, they were neglecting the higher-order contribution. But for the in-plane forces, for the out-of-plane forces, say, because of the deformed configuration, the contribution of the in-plane forces is in the vertical equilibrium. So, that is basically the governing equation to understand. So, of course, this is one way all of you have noted down, all right. See, the thing is that, well, you know, here once again, we did it in an ad hoc way, where we are considering a small element, you know. And we are trying to find out the component; well, I mean, this is one way to derive the

equation. But if you want to do it smarter than that, what will you do? Is there a smarter way to derive the equation? Yes, why not? How will you do that? Do you have a clue? How. Now the way we deal with it is by considering, you know, these equations the way we have derived them here. We always consider a small element and then we enforce this, right? But we never, the way I mentioned to you, just by integrating this, can get this part of the equation right. How will you get this equation?
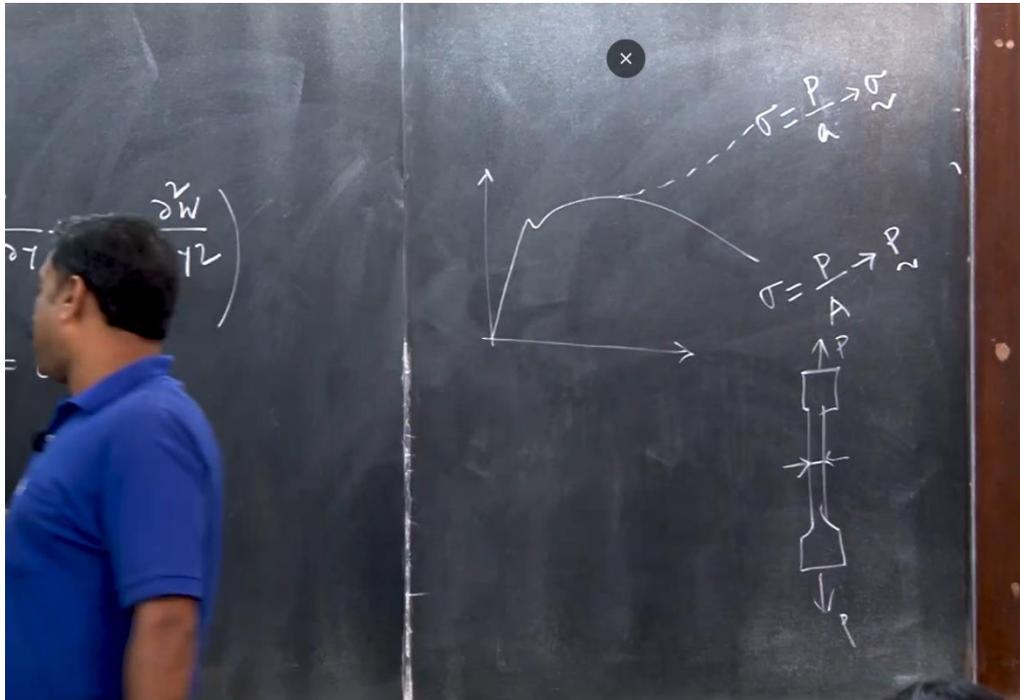
This is a smarter way to do it. Directly from continuum mechanics, you will get this equation right. At least from this equation, you will get this part right. But how will you get this part? This is a much smarter derivation, and you don't have to care about anything; just start from a very basic Cauchy equilibrium equation, then you integrate. This is sufficient for this, but how will you do it? See, the Cauchy stress tensor doesn't distinguish between the deformed and undeformed configurations. Right? From this equation, you'll get this part. This is what we'll never get. Because of how the σ_{ij} is defined, which is the Cauchy stress tensor, σ is defined in the context where both sigmas represent force per unit length, right? Both the force and the area reference are in the deformed configuration. you have learned Right Eulerian and Lagrange. Right. is in the undeformed configuration?

Because then it will distinguish between the two. Of course we will see; this equation is nothing but what? The divergence of the stress tensor is zero. So, we'll instead you know, divergence of first Piola–Kirchhoff stress tensors to be zero. First Piola–Kirchhoff stress tensors is what? First Piola–Kirchhoff stress tensors are defined; it is a three-dimensional generalization of the uniaxial tensile testing you did in your second-year strength of materials class, right? And then can you recall that you have plotted the plot? You have plotted two graphs, right? One was like this and then coming down, but another one, you know, this plot—can you recall? So, the first one was, σ was what? σ this one P/A. This one is P/A_0. That means you can recall that this was your tensile specimen, right? Huh? Here, if force P is always there, right? But if you consider the area, look, when you are considering the area, when it is expanding, the area is also contracting because of Poisson's ratio, right? So, the area is also changing. If you consider that, the change, you know, the reduction in the diameter, that means the area is changing. So, if you consider the changing area, then you will get that rising curve. But if you don't consider that nominal, then you will always get something like this. Now this is one-dimensional. If you generalize it in three dimensions, this is the Cauchy stress tensor. If you generalize it, this is the first Piola–Kirchhoff stress tensor. Okay. So, the problem with Piola–Kirchhoff stress tensors is that, I don't know, this is the equilibrium equation, but in the momentum equation, there is a problem. Because what happened is that the Piola–Kirchhoff stress tensors are not symmetric, that means p is not the first Piola–Kirchhoff stress tensor; it's not symmetric. So, what happens is that, although this is a linear momentum balance, the linear momentum balance is linear, but then the angular momentum balance is a nonlinear equation. Okay. Because it is not symmetric. Okay. So, that's what we define the second Piola–Kirchhoff stress tensor as: $P = F \cdot S$. So, this is the first Piola–Kirchhoff stress tensor, and this is the second Piola–Kirchhoff stress tensor, and F is the deformation gradient. So, then, it will be $\nabla \cdot (F \cdot S) = 0$. F is the deformation gradient. Okay. So now that, earlier, this is nothing but a linear momentum balance. That's the beauty of the second Piola-Kirchhoff stress tensor; that's why the second Piola-Kirchhoff stress tensor was introduced. You see that I don't care about this. But yes, I care about this; I will show you from here how it is coming. So let us work with the second Piola-Kirchhoff stress tensor, and the beauty is that, unlike the Cauchy stress tensor, the second Piola-Kirchhoff stress tensor will allow you to write it in terms of the initial notation. In this notation, write that this repeated index, now please note that capital represents the undeformed configuration and small represents the deformed configuration. Okay, you must

understand that the deformation gradient is a two-point tensor because the deformation gradient is what? So, one leg is in the undeformed configuration and the other leg is in the deformed configuration. So, it's a two-point tensor, right? And what is f_{jk}? What is F? F_{jk} is nothing but what? F_{jk} is nothing but, you know, deformation gradient is nothing but what? $\frac{\partial x_j}{\partial X_k}$ and $\frac{\partial (x_j + U_j)}{\partial X_k}$,

and this is $\frac{\partial x_j}{\partial X_k}$. You express the displacement gradient in the deformation gradient, U_j, by $\frac{\partial}{\partial X_k}$; this is nothing but the Kronecker delta δ_{jk}, and this is nothing but $\frac{\partial U_j}{\partial X_k}$. So, then you substitute it here.

When you substitute here, S_{ik} and F_{jk} are nothing but $(\delta_{jk} + \frac{\partial U_j}{\partial X_k})$. Simplified $S_{ik}\delta_{jk}$ will be K as a repeated index, so K will be dropped, right? So, it will be $\{S_{ij} + S_{ik}\frac{\partial U_j}{\partial X_k}\}$. All of you are convergent with convergent, what initial notation, right? All of you, right? We're assuming that the plate is subjected to uniform forces, right? So, I can neglect its variation over X_k. So, I can take this thing out of I_k. And how can I write this? $\frac{\partial^2 U_j}{\partial X_i \partial X_k} = 0$.
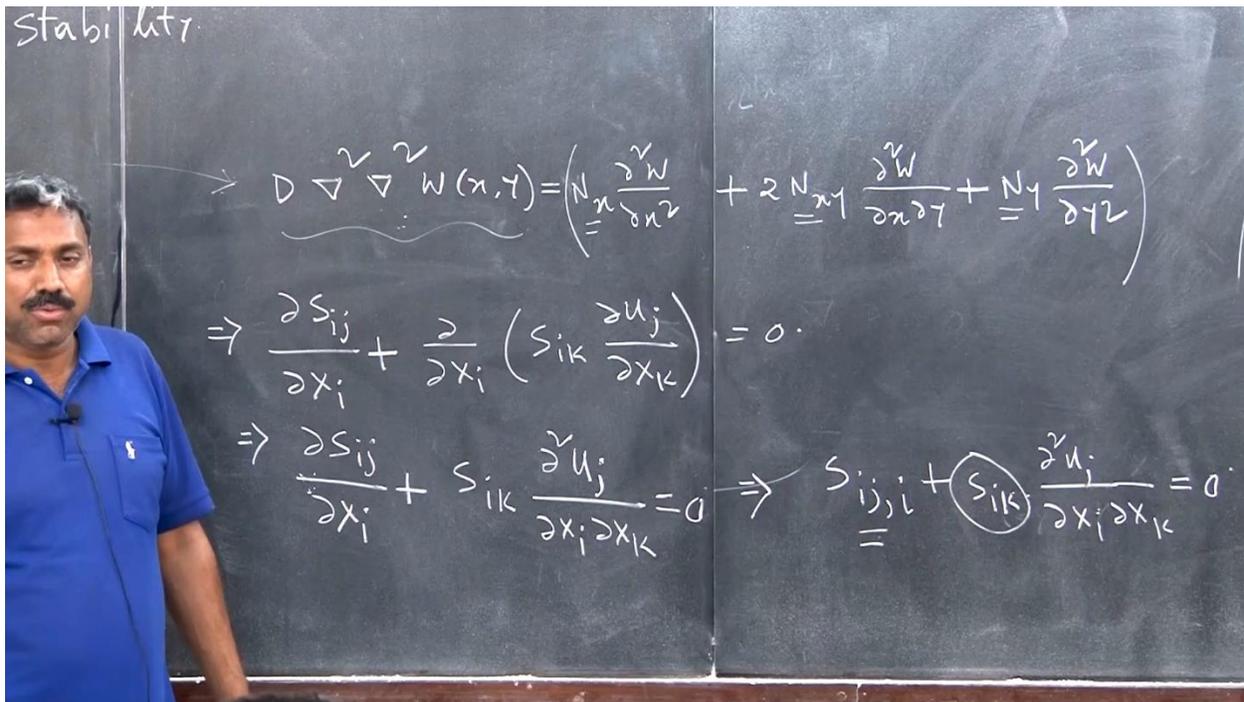
Now, I'll explain to you this second Piola-Kirchhoff stress tensor. This is nothing but what? This is nothing but $S_{ij,i} + S_{ik}$. So, this one, if you don't consider this, gives you this part, and this one

gives you this part. Do you see that S_{ij}, N_x, N_y, N_{xy}, and $\frac{\partial^2 U_j}{\partial x}$, $\frac{\partial^2 w}{\partial x^2}$, $\frac{\partial^2 w}{\partial x \partial y}$? Do you see how? So, if you integrate once again over the thickness direction, that will give you the vertical equilibrium equation, force equilibrium, and then you multiply that by Z. And then you integrate it over thickness; then you will get two moment equilibrium, and ultimately, you'll get this thing. So, this is a much smarter way to derive this directly from the continuum equation. Whatever I have done is a little ad hoc way to derive this term, but nevertheless, that is more physically relevant at least. Because we can physically understand what the hell is going on, right? So, I prefer to derive this using the previous approach. So, this equation will allow us to study the stability of a plate or the buckling of a plate. For the time being, we are assuming that N_x, N_y, and N_{xy} don't change along the length of the plate. They are constant, right? First thing, so, this is a boundary; this will lead to an eigenvalue problem.
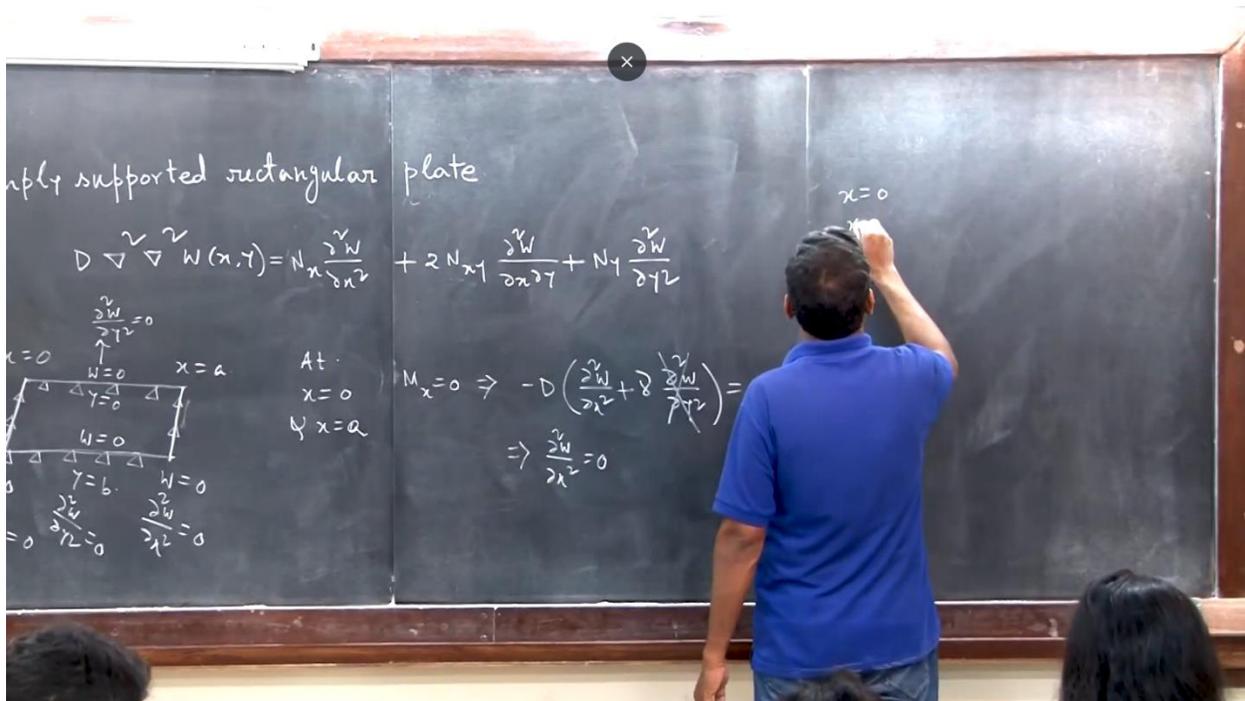

This is the governing equation; it must be accompanied by sets of boundary conditions. Right, that we will see. The first problem that we are going to consider is buckling, specifically uni-axial buckling of a rectangular plate. Along X a and along Y b length a and b, okay, and I'm assuming this is simply supported or these are simply supported; these are all simply supported. Simply supported means you can understand that now, these are all simply supported. Okay, simply

supported means you know something like this. But please note one thing. These are all simply supported. Huh? But please note one thing: in order to allow it, we must allow axial deformation, you know, buckling, right? So, they are simply supported, which means they are restraining out-of-plane deformation, but not in-plane deformation. Because you have to allow the thing to get compressed properly, it will not restrain the in-plane things, okay? That means it must be something like a roller here, okay? Roller understands that right. Okay, good. So, then what are the boundary conditions? If it is simply supported, what are the boundary conditions? So, if you require simply supported means two boundary conditions here and two boundary conditions there. All edges have two boundary conditions. So, at $x = 0$ and at $x = a$, at $x = 0$ of course $w = 0$, and right, what else? Of course, all the ages are basically at $y = 0$ and $y = b$. W must be zero. Right? What else? Another thing is that we are considering the bending moment along the x direction. That means M_x must also be zero, right? And m_x is what? $-D\left(\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right)$ must be zero. Now, at $x = 0$ and at $x = a$, M_x. Now here bending moment, here bending moment. Okay. This bending moment. Huh? Right. So, you see that this is anyway supported along Y. So, deflection is zero all along Y. So, there cannot be any variation along Y. So, $\frac{\partial^2 w}{\partial y^2}$ is anyway zero.

Right? So, what does it mean? That means this; this implies that $\frac{\partial^2 w}{\partial x^2}$ must be zero. So, along with $w = 0$, in both cases, $\frac{\partial^2 w}{\partial x^2}$ must be zero, and similarly, here M_y should be zero; for that, $\frac{\partial^2 w}{\partial y^2}$ must also be zero. Along y, is equal to $\frac{\partial^2 w}{\partial y^2}$ must be zero. Understand, right? So, the boundary conditions are that x here, once again, if you want to write at $x = 0$, $x = a$, $w = 0$, $\frac{\partial^2 w}{\partial x^2} = 0$, at $y = 0$, $y = b$, $w = 0$, $\frac{\partial^2 w}{\partial y^2} = 0$.

Fine. Okay. So, now and then, of course, because it is only axially loaded, all other things are not there. All right. D and N_x, $D\frac{\partial^2 w}{\partial x^2}$ right. So, what we will do is see if we can have some expression that will allow this; we have solved the Plate equation, right? Navier's approach, right? Can we use the same approach here, using double Fourier series? Yes, can we assume some function that will satisfy the boundary condition? So, how can we, let us assume $W(x, y)$ is the summation of $A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$. Where this is a double summation, right? $m = 1$ to ∞, $N = 1$ to ∞,

right? $m = 1,2,3$; $N = 1,2,3$; double Fourier series. If we see whether it really satisfies $x = 0$, yeah, it will be zero. If you take the double derivative, then it will be sign, will be cosine, will be sine once again, and then all the boundary conditions are satisfied, right? So, that basically satisfies the boundary condition now. So you substitute in the equation that, assuming you're substituting in this expansive expression here, so what you will get, See this: if you differentiate, the way you are going to get $\{ \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \} = \frac{N_x}{D} \frac{\partial^2 w}{\partial x^2}$, right? So, you'll see that here you know E. So, you differentiate it with respect to, of course, four times. So $\left(\frac{m\pi}{a} \right)^4$, right? Huh. And then $2 \left(\frac{m\pi}{a} \right)^2 \left(\frac{n\pi}{b} \right)^2$ and then $\left(\frac{n\pi}{b} \right)^4$, A_{mn} is equal to $\frac{N_x}{D}, \frac{\partial^2 w}{\partial x^2}$ will be, of course, $A_{mn} \left(\frac{m\pi}{a} \right)^2$. Okay, both sides $\sin \left(\frac{m\pi x}{a} \right)$, $\sin \left(\frac{n\pi y}{b} \right)$; you can cancel out both sides, right? And then, of course, A will also cancel out. A will also cancel out, right? So, then what we are going to get is: From here, you see here. I think there will be a negative sign here, right? Because, huh. Isn't it? And N_{xy} will be $-D \left(\frac{a}{m\pi} \right)^2 \{ \left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \}$. Right. this negative sign is coming, because we initially assume tension but it is compression right. that's why, then minus D, you know I will just see the simplification you know in what format they presented okay So this π^2, if you take one π^2 out, will remain.
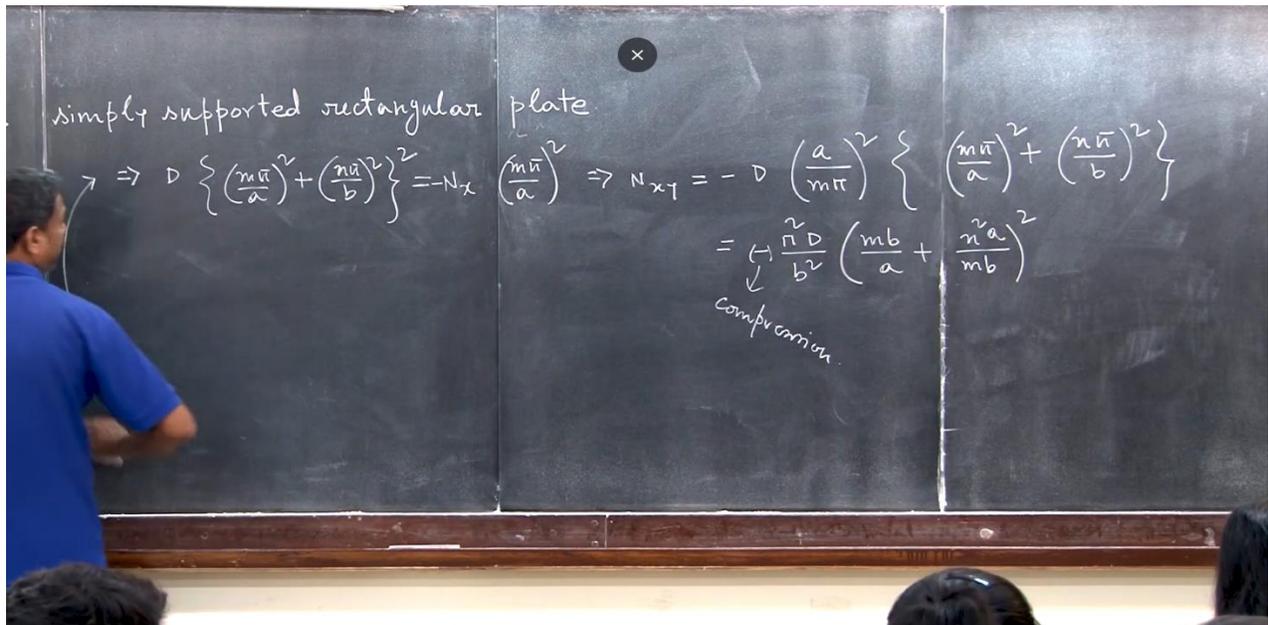
So, $-\frac{\pi^2 D}{b^2}\left[\left(\frac{mb}{a}\right) + \left(\frac{n^2 a}{mb}\right)\right]^2$, okay. That this kind of expression will come out okay. I don't know how; please check. So, of course, the negative is because it is compression, you know. Now we will do a little simplification. What kind of simplification are we going to do? So, we'll assume that we'll let us have the plate in which the plate is assuming. We'll assume that this plate along the X direction has whatever, but along the Y direction, it will buckle in a single buckle, right? So, that's what I am assuming; you know that's what I'm assuming, something like this.
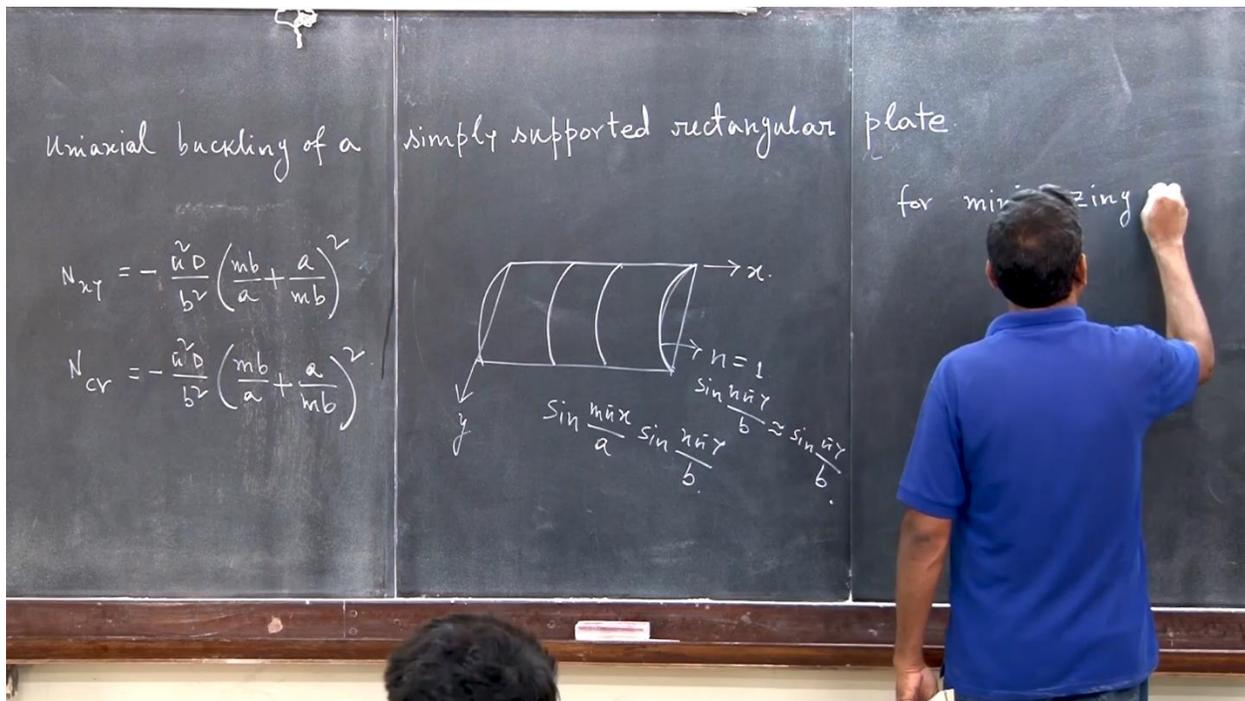

So, when I'm using this expression, $\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$, this $n\pi y/b$ is equal to 1 because when it is buckled in the y direction, if it is buckled and a single buckle, then it is one. So, here n is basically one. So, $\sin\left(\frac{n\pi y}{b}\right)$ is basically always taken as $\sin\left(\frac{\pi y}{b}\right)$. You see that? So, for this. So, the simplification is that along x direction, we are not constraining, but along y direction we are constraining that it is buckling in single buckle. So, we are basically converting the problem; instead of dealing with two dimensions, we're dealing with only one dimension. What is happening? Let us see, and that will reveal many important conclusions, right? So, N_{xy} will be, in that case, $-\frac{\pi^2 D}{b^2}\left(\frac{mb}{a} + \frac{a}{mb}\right)^2$, okay. So, this is the critical load considering this thing: $-\frac{\pi^2 D}{b^2}\left(\frac{mb}{a} + \frac{a}{mb}\right)^2$. Now we'll even see some other things over here. Okay. See how we'll try to

minimize $N_{critical}$ with respect to m. So, treating, of course, m as an integer, right? So, treating this, if you minimize $n_{critical}$ with respect to m, let us see what happens. Okay. So, for minimization, you know N_{cr} with respect to m, right? So, then $\frac{dN_{cr}}{dm}$ must be equal to zero. And when you differentiate, you see that what you are going to get is that m is b/a. Well, I mean, so $2(\frac{mb}{a} + \frac{a}{mb})$, and there is an m that is basically $(\frac{b}{a} - \frac{a}{bm^2})$ equal to zero, right? So, this cannot be. From here, what we get is that $\frac{b}{a} = \frac{a}{bm^2}$; that means $m^2 = \left(\frac{a}{b}\right)^2$.
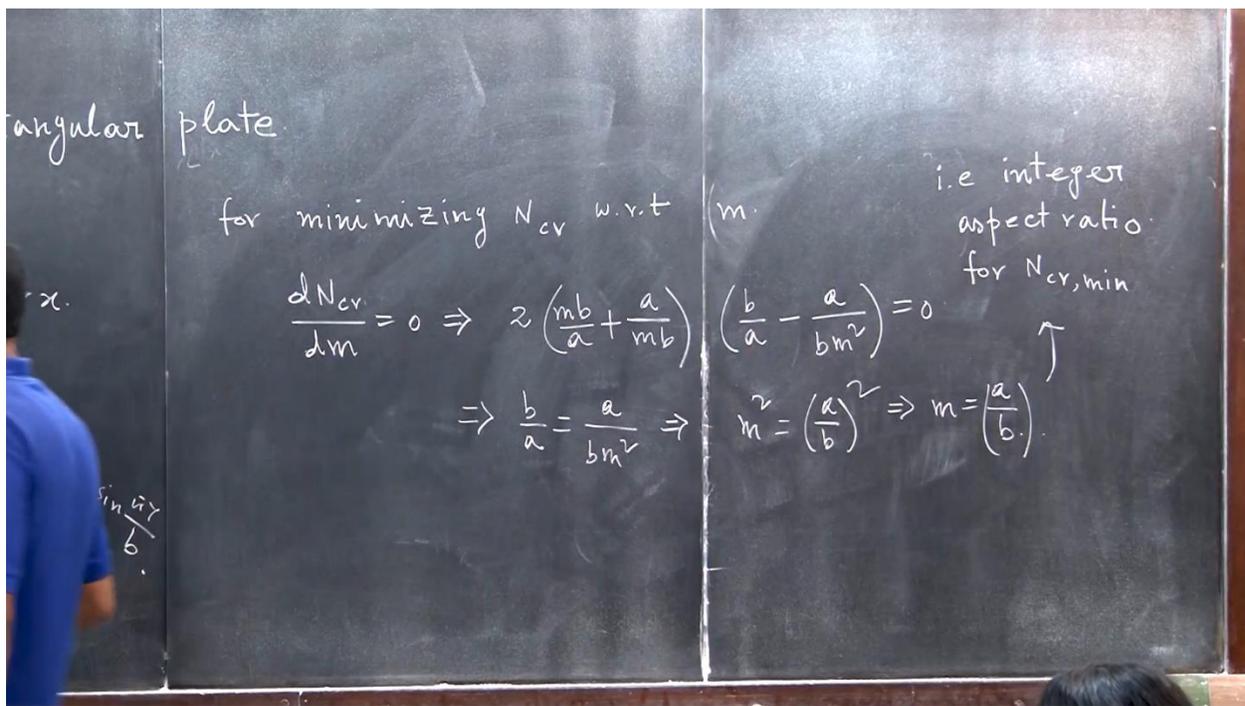
So, $m = a/b$. What does that mean? This mean that, the critical load will attain minimum, when this a and b, you will occur you know a/b is basically appears in a multiple okay, an integer multiple means the aspect ratio is an integer, you understand. So, for critical, that means that it is an integer aspect ratio. So, this is the condition of the minimum integer aspect ratio for the $N_{critical}$ load to be minimum. Okay, and what is the value? Minimum value. The minimum value, as you see, substitutes N_{cr}, which is equal to $\frac{\pi^2 D}{b^2}\left(\frac{mb}{a} + \frac{a}{mb}\right)^2$, and then m is equal to a/b, you know. So, m is equal to a/b, right? a/b times b/a divided by b, and $1/m$ means d/a. So, this is $(1+1)^2$. So that means you know $(\frac{4\pi^2 D}{b^2})$, you see that. So, the critical load depends on the aspect ratio B, and you know the dimension over the other direction, the smaller direction, right B? Because what

we can see is that M is equal to A/B, that condition we have obtained. So, when the aspect ratio of the plate is an integer, then the theoretical load is $\frac{4\pi^2 D}{b^2}$. It depends only on b, the smaller dimension. Okay, now what happens if it is not minimum? Then what happens? Then $N_{critical}$ is $\frac{\pi^2 D}{b^2}\left(\frac{mb}{a} + \frac{a}{mb}\right)^2$. Now, what we see is another thing I would like to put here. So, if you plot the $N_{critical}$ over a/b, you know here you have N_{cr} and here you can put a/b. So, here are two, here are three, four, five, six. Okay. So here it is, four. Okay. Four. Huh. So that's $\frac{4\pi^2 D}{b^2}$. $\frac{4\pi^2 D}{b^2}$. Okay. So, what will happen here? here It will be, see between one and two, and this will be $\sqrt{2}$; between two and three, this one corresponds to $\sqrt{6}$, and this one corresponds to $\sqrt{9}$, and then in here it will correspond to $\sqrt{12}$, something like that. So, what happened is that, you see, I'm plotting this here: $\frac{4\pi^2 D}{b^2}$.
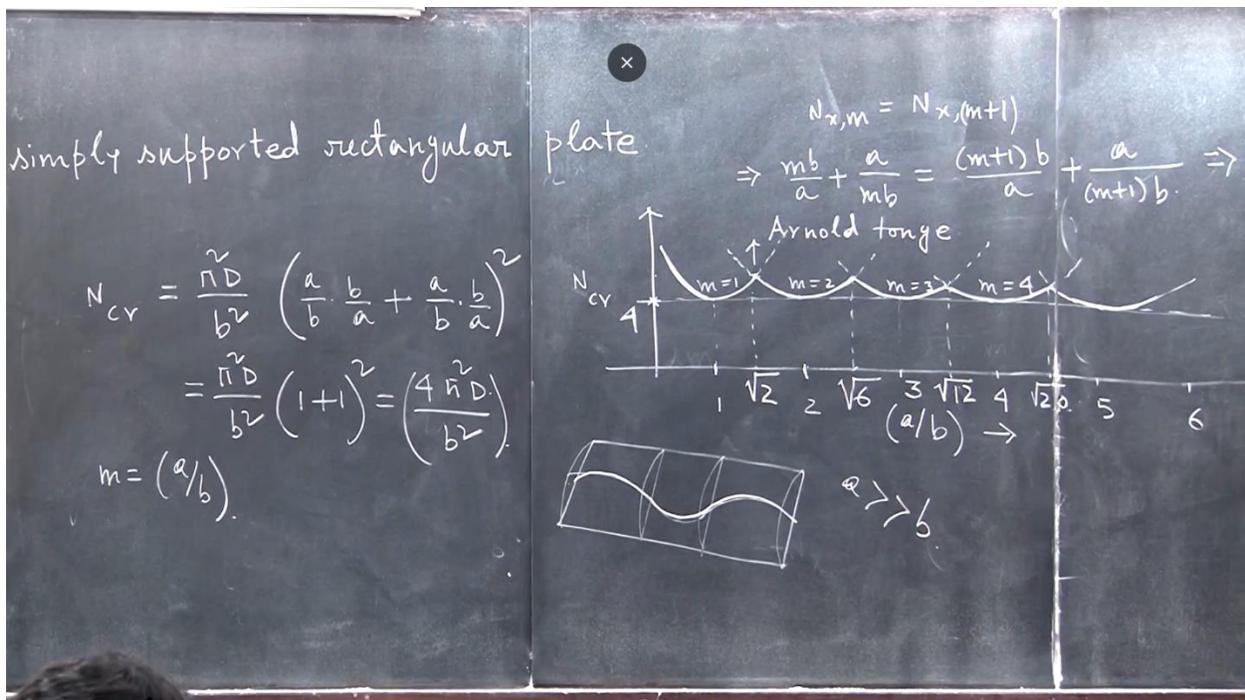
So, this is always four, right? So, what happened is that first this is buckling; since the aspect ratio is one, then $a = b$, and the minimum is attained, right? On either side, when the aspect ratio is not one, not an integer on both sides, whether it is decreasing or increasing, it will increase, and then when it is coming to some point, you know, if it is crossing $\sqrt{2}$. If it is slightly less than $\sqrt{2}$, then of course it will buckle only with the first mode $m = 1$. So, this is $m = 1$. This $m = 2$. This is

$m = 3$, $M = 4$. Then as soon as it crosses $\sqrt{2}$ aspect ratio then it will buckle. There will be two buckles because from here there is a sifting mode transition. Then it will remain; this, of course, critical load will reduce when it is in perfect integer, and then once again it will rise. Then once again, there is a mode shifting at $\sqrt{6}$, and this $\sqrt{2}$ $\sqrt{6}$, you know how you can obtain this thing; let me show you. So, $N_{xm} = N_{x(m+1)}$ means that the critical load at m mode and the critical load at $m + 1$ mode should be the same, right from this point. So, here you can clearly see that the $\frac{mb}{a} + \frac{a}{mb}$ is equal to $\frac{(m+1)b}{a} + \frac{a}{(m+1)b}$, right? And on simplifying, you know a/b will be $\sqrt{m(m + 1)}$. Huh? It will be $\sqrt{12}$. Sorry. $\sqrt{12}$. So, you can clearly see here. From 1 and 2, $\sqrt{1 \cdot 2}$ then $\sqrt{2}$. Okay. Here, $\sqrt{2 \cdot 3}$ means $\sqrt{6}$. Here, $\sqrt{3 \cdot 4}$ is $\sqrt{12}$. Then here, $4 \cdot 5$ is $\sqrt{20}$. Okay. Here it will be $\sqrt{20}$. Okay. Something like that. Okay. Why is this happening? Because you see, when the plate is able to buckle in a perfect square safe or in a safe in which you know that a and b occur as integers, right? That is being energetically favorable over others; you see that because basically, it is conducive to the perfect buckling mode—that's what we assumed, right? $\sin\left(\frac{m\pi x}{a}\right)$, $\sin\left(\frac{n\pi y}{b}\right)$ okay. So, your perfect sine curve is possible only when a and b appear as integers. That's what is essentially happening. So that's what aspect ratio one it attains, minimum aspect ratio two; three, of course, when the aspect ratio two, then there will be two buckles along. m is what? m is the number of buckles along the x direction. So, then the minimum will be attained when the number of buckles is two because it is buckling in the second mode. The aspect is your third; the buckling mode that is giving the minimum is, of course, the buckling mode, specifically the third buckling mode, which has three web numbers. I mean three, you know, sign, you know, half-period functions, right? And then, similarly, four; and from there is a point of transition whether mode transition is occurring from mode one to mode two. That basic or mode two to mode three happens when a/b is $\sqrt{m(m + 1)}$, okay? So, you see that this is nothing but m, and a/b is the same, okay? This is okay, fine. Now it's not that, so these places, you know, this looks like what? This looks like what? This looks like a tongue now, right? This is called Arnold's tongue; you can also call it Ma Kali's tongue. Okay, if you don't want to call it Arnold, why is it called Arnold tongue? Because Arnold is the scientist. He is basically studying some problems. This thing is not common here. I mean if you want to study the stability domain, there are many cases that appear not only in elastic stability but also in hydrodynamic stability and magnetoelastic stability. I have seen some

recent papers appearing in PRL, where this Arnold tongue is just demonstrating quantum optics. There are some phenomena involving quantum optics, so they conducted some experiments to see how Arnold's tongue appears in the parameter space.
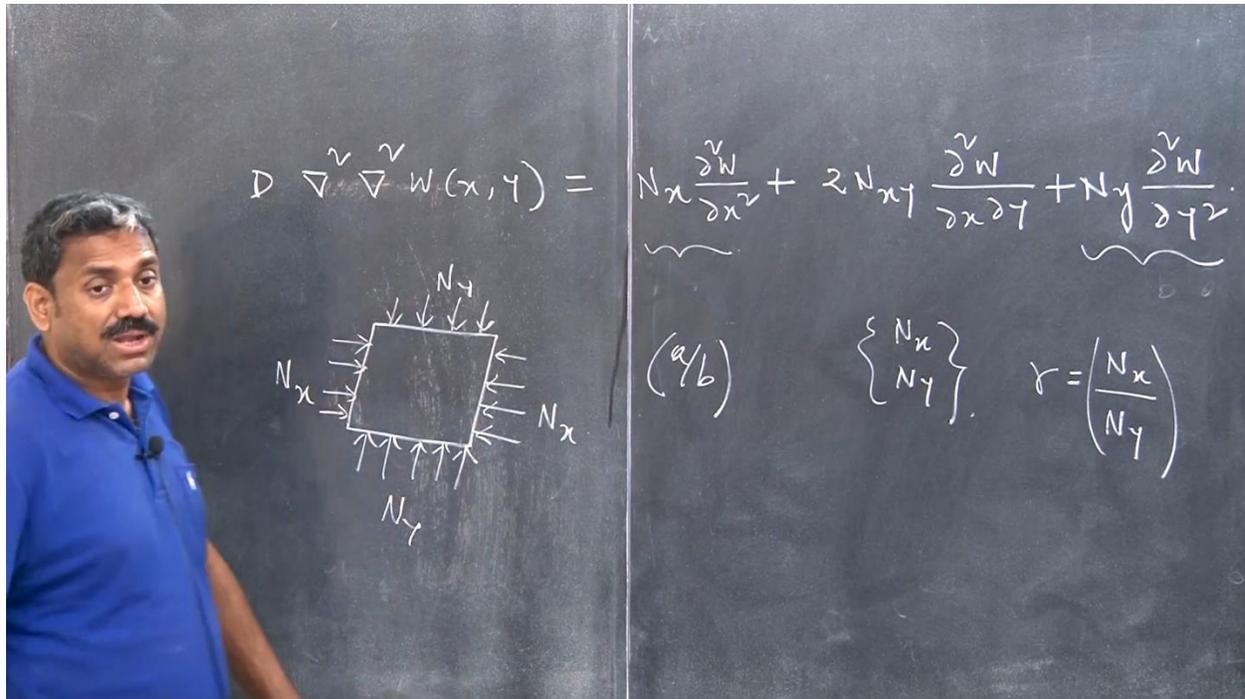
So, how it is experimentally, how they're showing the disparity between the experimental and the theoretical one. You see these look-like strung structures called Arnold's tongue. So, what we see is that, and how it will buckle you can clearly understand that, well I mean if you are having a you know like this kind of thing, you know and then it is having you know a perfect ratio of three, right. So, in this direction, it is always like that, always like that, always like that, right? In the other direction, you know it will be something like this; it will go, and then it will be something like this. So, you see. So, there are three complete waves, okay? Three numbers, okay? Here is a single wave number. Because a prime, we have assumed for simplification then. The plate will buckle only in single mode because the other direction is assumed to be much smaller; b is assumed to be much smaller than a, which is the higher dimension. So, along the shorter direction, we are assuming that the single mode prevails. Okay. And in the other direction, basically, the minimum critical load is attained when it is appearing as a perfect integer; that's what we can see. So, this is for the uniaxial buckling. You can also study the biaxial buckling. Of course, things will be a little more complicated, but you know that, and you have to plot the stability diagram here; it has only

one controlling parameter. Okay. So, m or a/b, but there it will have more, and then instead of n, if you just write the governing equation $D\nabla^2\nabla^2 W(x,y) = N_x\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x\,\partial y} + N_y\frac{\partial^2 w}{\partial y^2}$. So, if you want to study, now I'm considering all other things as zero; you can consider N_x and N_y. So, for a plate, you consider the bi-axially loaded plate. So, a plate is simply supported and it is bi-axially loaded. That's a regressive by b. So, with a bi-axially loaded plate, of course, you have to consider the combination of N_x and N_y. And then, if you want to study the stability, you have to plot everything to see when it is buckling or not.
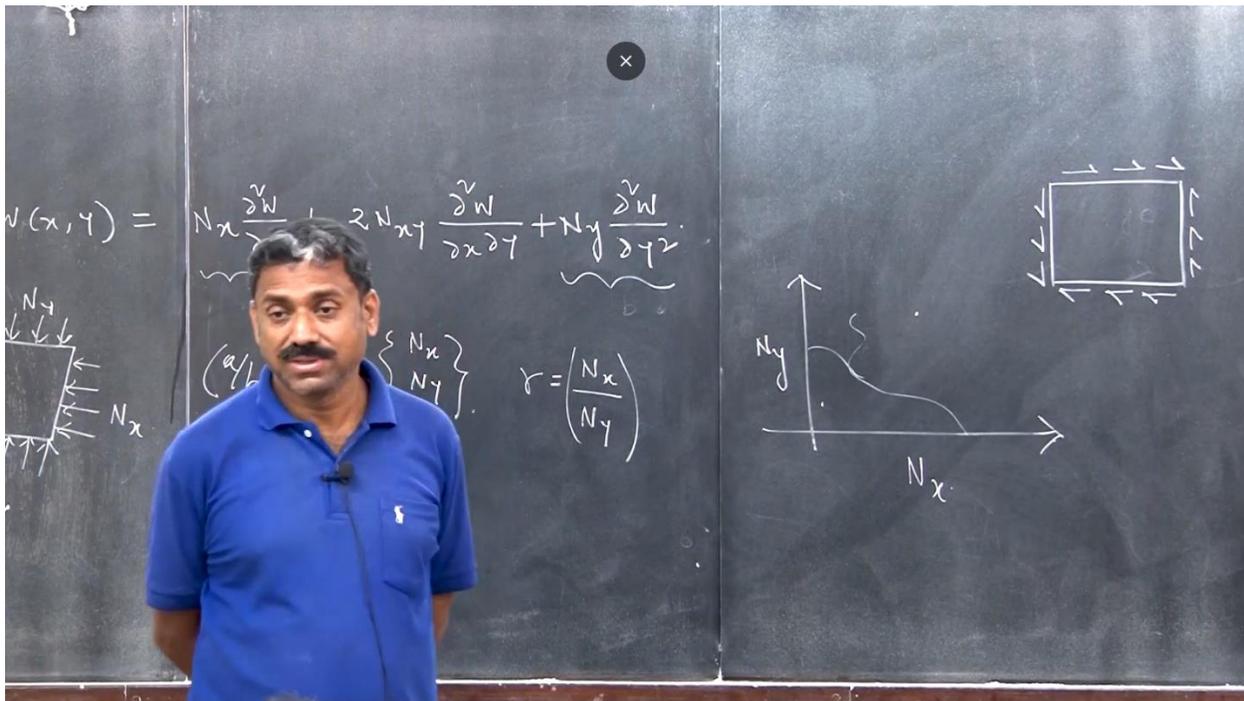
You know different combinations of N_x and N_y, and there you can distinguish what the stable and unstable domains are. But you know, you can always consider that there is some ratio maybe equal to N_x/N_y, and then you can solve the differential equation. You can do okay, but I'm not going into details, okay? But it's possible; you can also have shear buckling, which means you know N_x, N_y, no compressive forces, but you have shear forces, and then the plate is subjected to shear. So, the plate can also buckle under shear, and this is basically a precursor. So, you know this is subjected to some kind of shear, you know, something like this, you know. Why is shear buckling important? Because, you know, in reinforced concrete design or steel design, you have learned about tension field action, right? Tension field action involves designing the web against shear

buckling, right? So those theoretical concepts are what we can learn here. Okay. How web buckles under shear, shear buckling. Okay. So, then a web panel, that's what we provide: stiffener, right? No transverse stiffeners, longitudinal stiffener, all these things. So, shear buckling we will consider. So, this is basically the way, we consider the, you can see once again the analogy between the column and the plate as far as buckling is concerned not much difference except, there is another addition of a dimension, $\frac{\pi^2 EI}{l^2}$ right. here is $\frac{\pi^2 D}{b^2}$, and there is some four right, π^2 instead of π^2 the four is coming. π^2 will be around 9.80, whatever. However, there will be some important things if you go beyond buckling, okay? That's what the column plate also shows; similar behavior, I mean, the plate shows a little distinct behavior that we will see. Now, this approach, whatever we have done, is based on the equilibrium. We have written down the equilibrium equation in the deformed configuration, and that's what we have solved: an eigenvalue problem. This eigenvalue problem is a trigonometric eigenvalue problem; to obtain the eigenvalues, that will give you the buckling load vector that will give the buckling mode shape. But you can also solve the same thing using the energy approach, and you will see that for shear buckling, we will consider using the energy approach. So, let us pave the way to do the energy analysis for buckling. What's, you know, in all the previous examples, we have followed okay. So, in the next class, we are going to continue on that. Thank you very much for today's class.