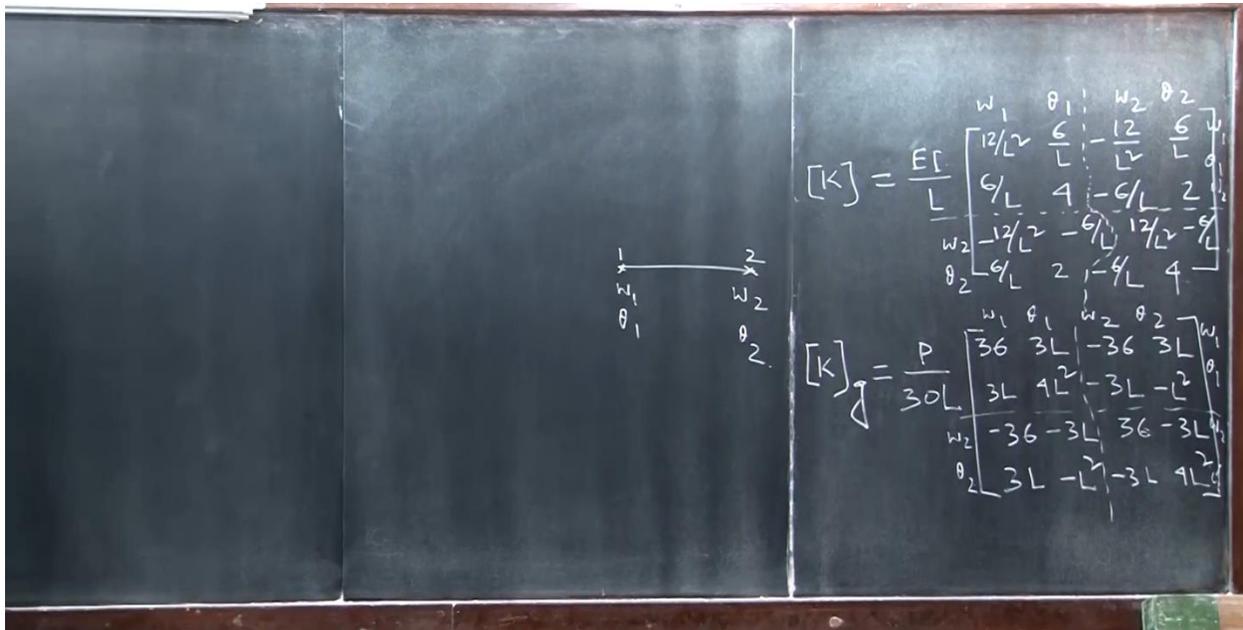


**Stability of structure**  
**Prof: Sudib Kumar Mishra**  
**Department of Civil Engineering**  
**IIT KANPUR**  
**WEEK-06**

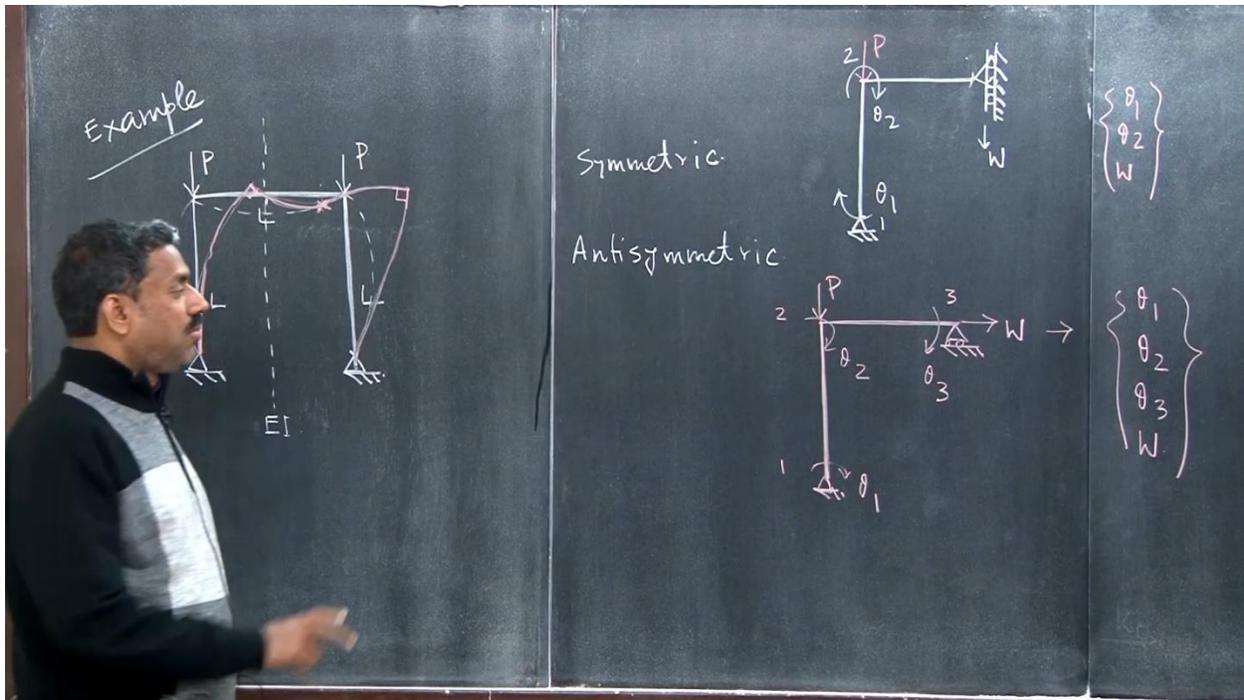
**Lecture 12: FEM solved example and Review of plate bending**

Okay. So, what we are discussing, let us briefly recapitulate. We are discussing the matrix method or finite element method of frame analysis for buckling. If we want to study the global buckling of the frame and we have derived the stiffness matrix for the beam-column element, right? So, a two-noded beam-column element; this is node one, this is node two, and then they have  $w_1, \theta_1$ , and  $w_2, \theta_2$ .



Deflection, vertical deflection, and rotation are right, and then, you know, four degrees of freedom; this was the stiffness matrix contributed from the elasticity of the, you know, column element. And then an additional thing for buckling that is required is the geometric difference matrix. So, we have written these terms here once again. Okay. Then we will, as we have also demonstrated, explain how we have taken a simple example of the cantilever beam or any boundary condition you can consider, such as fixed or hinge. With a single element discretization, we can solve for its

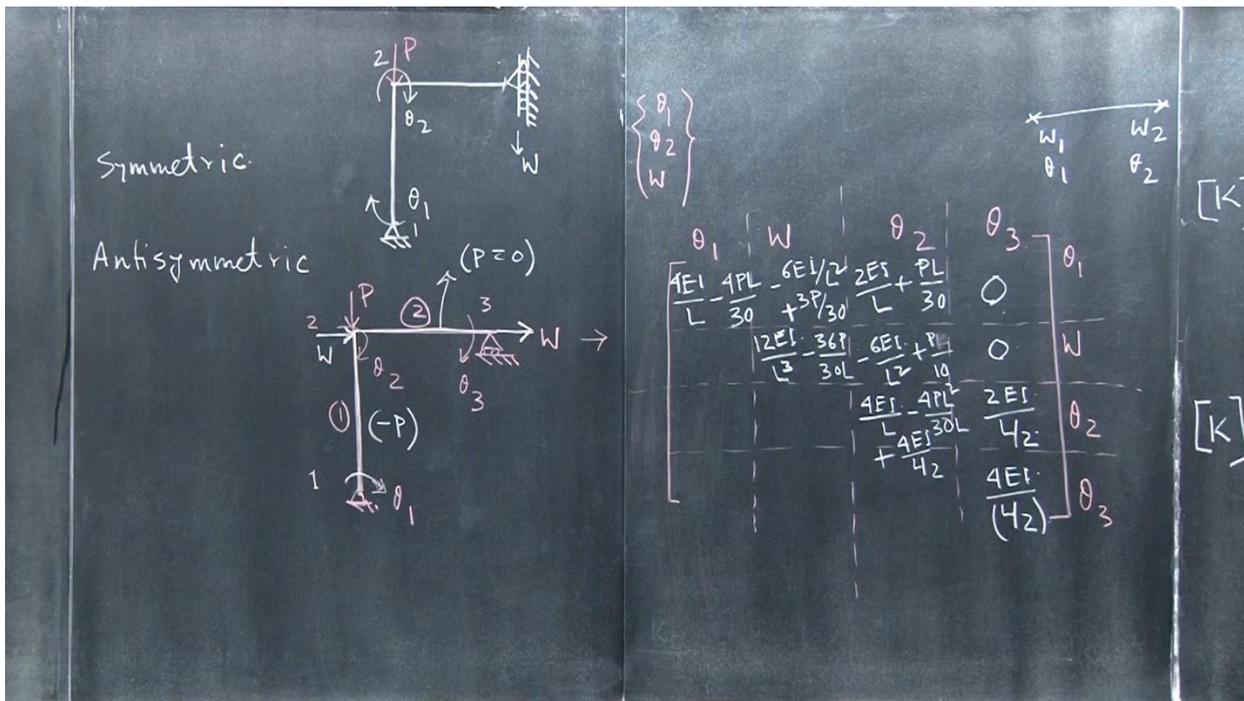
critical root that we have found, and we have seen that whatever we obtained from solving the algebraic eigenvalue problem is valid. Matches well with the one we derived analytically in our second-year strength of materials class. Now we are going to solve a problem in which the buckling of a frame, maybe some of you will definitely remember from the experimental methods class, you will verify this, okay? But I've seen that the buckling of a frame; there is one experiment designed for that. How will we find out the buckling load for the buckling of a frame?



Okay, that's what I'm going to demonstrate. This is a frame; I'm assuming that this is hinged, and then all elements are length  $L$  for simplicity and flexural rigidity, subjected to this. Two, of course, there needs to be compression, right? So, subjected to this compression. now we can consider you know, because there is symmetry right, symmetry. So, either we can consider, or we can exploit the symmetry or anti-symmetry in that matter to, you know, analyze it, okay? So, it can buckle either in symmetric mode or anti-symmetric mode. If we don't consider, symmetric mode means there will be one in which you know what symmetric will be. When it is buckling in symmetric mode, what is going to happen? You know this will deform like this, right? This is the way it will deform; this is the symmetric mode, right? So, then what can we see? When it is deforming in symmetric mode, the vertical point you know is the axis of symmetry; this is deflecting, so here the slope will be zero, right? And there is no lateral translation. Right? So, there is no sway of the

frame. So, for the symmetric one, we can approximate it like this. We can approximate this frame something like this. This one. And here we can allow vertical deflection, you know. But we can rest horizontally. So, we can provide it with a shear hinge over there. This is a shear hinge. The vertical deflection is allowed, but there is no rotation. You see, this is the idealization half of this; we are exploiting symmetry, you know. Okay, so the degrees of freedom here, you can see for this one will be here; there will be  $w$ , you know. There will be rotation; you know there will be rotation, there will be rotation and deflection. So, maybe here it is 1, 2, 3 this way; you are going to put it. So,  $\theta_1$ , this will be  $\theta_2$ , this will be  $\theta_3$ , and this is maybe right. this also only four degrees of freedom right. Now, if it is deforming under an anti-symmetric mode, then what is going to happen? You see that this will go there, you know, and then this will come here, and then it will be something like this, okay? This will be something like this. Okay, so this is the anti-symmetric mode. Maybe something like this, you know, and this will go like this, and this will go like this, and then this will be something. So, in anti-symmetric mode, what we can see is that there will be a point of counter flexure here, because from the positive curvature it will go to negative curvature, right? And here you know a  $90^\circ$  angle should be maintained; also, a  $90^\circ$  angle needs to be maintained, right? So, what we see here is that this point will not deflect, okay? This point will not deflect, right? So, but lateral translation will be allowed right. So, for the antiymmetric, what kind of idealization, we can consider exploit anti-symmetric mode is we have to, we cannot allow you know well I mean there will be, so we will allow rotation, but something like this is okay. Something like this, isn't it? So, you see that Sway is allowed. Sway is allowed, and then we are considering only half of it. So, rotation is allowed. So, okay, you understand that. And so, this will be subjected to force  $P$ . You know this is also subjected to force  $P$ , and then here you see that there will be one rotation: rotation  $\theta_1$ , rotation  $\theta_2$ , rotation  $\theta_3$ ; you know  $\theta_3$ . Please note that there will be no rotation; sorry, there cannot be any rotation, only vertical deflection. Okay, then there will be  $W$  deflection. Okay. So, there are only three degrees of freedom here. Whether in the antisymmetric mode there will be four degrees of freedom. Right. Okay. Here the degrees of freedom. So, symmetric mode will be  $\theta_1$ ,  $\theta_2$  and  $w$ . Whether in anti symmetric mode you know will be  $\{ \theta_1 \theta_2 \theta_3 w \}$  symmetric mode and anti-symmetric mode. Okay, do you see that? So, you see how we can reduce the, you know, complexity of the problem. So why are we doing this? So, because we want to reduce the number of degrees of freedom, we have to solve by hand. We can write a computer program, and we can solve any problem. But this is what I want to show you:

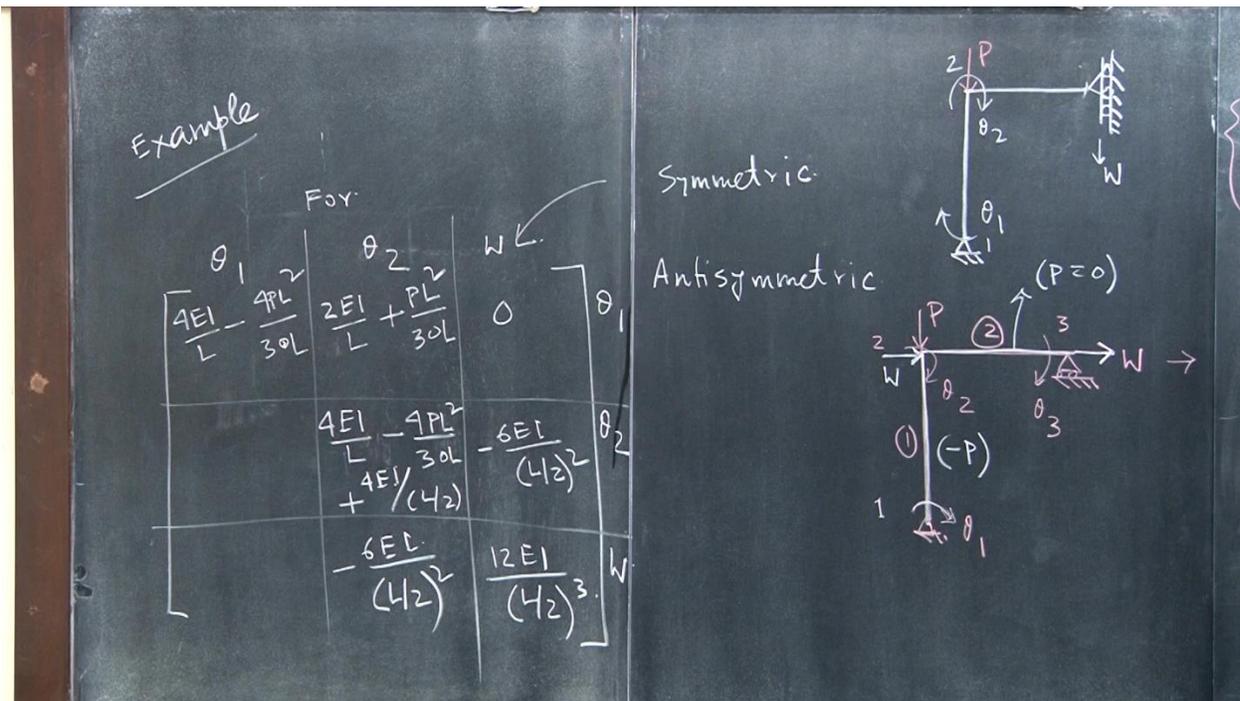
how we will assemble the element-level stiffness matrices to form the structural stiffness matrix. So now we will first consider the antisymmetric one. Okay. So, for the antisymmetric one, how will you do that? You see the, let us  $\theta_1$  and  $\theta_2$ , I'm going to write. So,  $\theta_1, W, \theta_2, \theta_3, \theta_1, W, \theta_2, \theta_3$ . You can see that. Now let us do the assembly. Okay. So, you see that this is element one. This is element two; that is what we are putting here. So, you see that  $\theta_1, \theta_1$ . This member, member one, is subjected to a compression load of  $P$ . Okay. Whereas there,  $P$  is zero. There is no compression load, so  $P$  is equal to zero. So, only this member is subject to compression. Right? Now how will you do that? So,  $\theta_1$ , this degree of freedom, node one. So, which member is going to constitute, in this stiffness, only member one, right? So, member one, what is the material stiffness matrix,  $\theta_1, \theta_1$ ? So  $\theta_1$  will be nothing but  $\frac{4EI}{L}$ , right? So, this length  $L$  is okay.



So,  $\frac{4EI}{L}$  is that fine, and so that means this term, right? And the respective term here is what? Four  $\frac{4L^2P}{30L}$ . So, here it will be minus  $\frac{4PL}{30}$ , right? This is okay. Fine, there is no contribution from the two, right? Because this note has nothing to do with this one. So that is the  $\frac{4PL}{30}$ . Similarly, now  $W$  and  $\theta_1, W$  and  $\theta_1$  will look the same, although we are saying  $W$  here; this  $W$  is the same as whatever  $W$  is here, right? You see, because this member we are considering is axially rigid, right? So, whatever  $W$  here, this  $W$  and this  $W$  are the same. So, when we consider  $W$  and  $\theta_1, W$  and  $\theta_1$  are

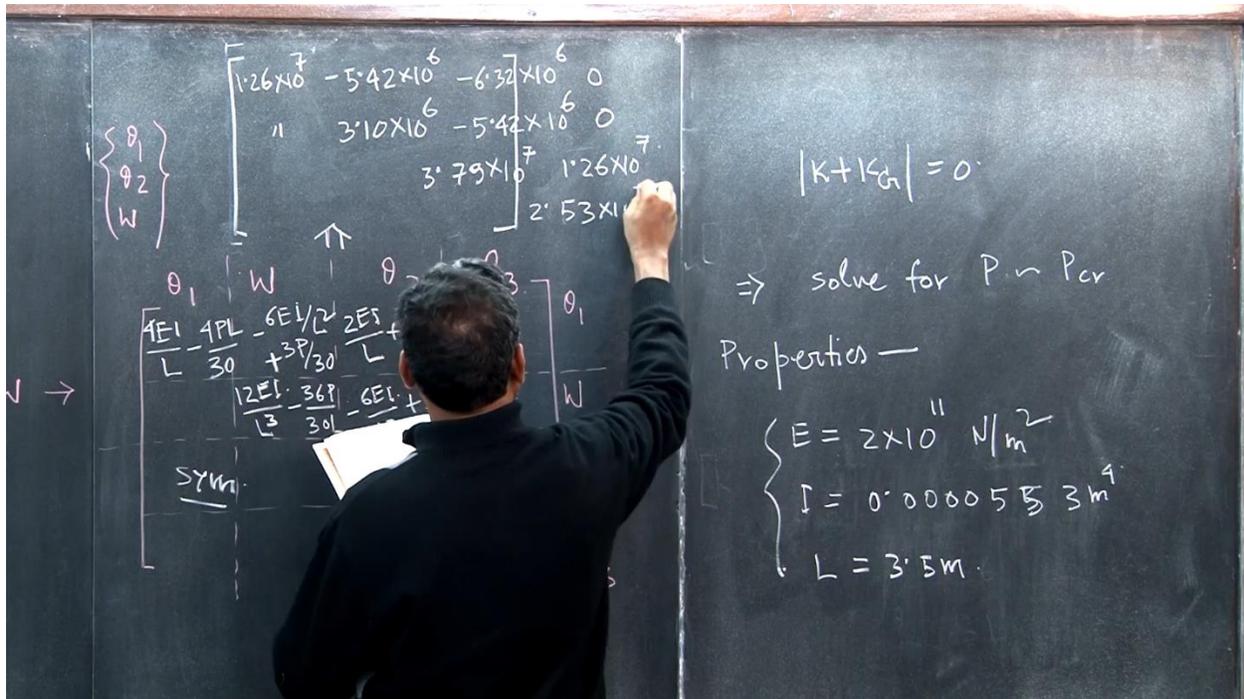
what we are going to consider; you know it is cross difference, right?  $W$  and  $\theta_1$  mean, "This is the two degrees of freedom that belong to two different nodes," right? So, when we are considering cross stiffness  $w$  and  $\theta$ , only the member that is connecting the two loads will contribute, understand? Isn't it? Look, this " $w$ " is the same as this " $w$ ," right? Because this is axially rigid. Now that we are considering the stiffness, you know that this block is basically between  $W$  and  $\theta_1$ , right? So  $W$  and  $\theta_1$  mean this is  $W$  and this is  $\theta_1$ . So, it belongs to two different nodes, right? Now all of you have done matrix method structural analysis, right? All of you are okay with matrix method structure. Then, if you did that, only then will you understand. If you didn't do a matrix structure analysis, you won't be able to understand. So,  $W$  and  $\theta_1$ . So, the cross stiffness, right? So, cross stiffness is because these two degrees of freedom belong to two different nodes. Only this member connects the two nodes. So only this stiffness will be contributed by member one. Right? Because this is connected between these two loads. Right? So, first you must understand that this  $DW$  is basically coming because of the vertical and lateral deflection of this member, not because of axial deformation in this member. Because when you consider this, you know this node, this  $W_1 \theta_1, W_2 \theta_2$ . We do not consider any axial degrees of freedom; must understand, right? So, for this,  $W$  and  $\theta_1$  will be what? Well,  $\theta_1$  is this and  $W_2$  is this;  $\theta_1, \theta_1, W_2$  is  $-\frac{6EI}{L^2}$ , right?  $-\frac{6EI}{L^2}$ , isn't it? The respective meaning of this one is right; this one isn't it; this is the respective member here. What is that? This one is so  $-\frac{3PL}{30}$ , right? Please know that. M Huh? It will be  $+\frac{3P}{30}$ . Yes, you are right. Because here you know this one. Yes. This, it will plus, right? This one it will be plus, that's true. So, what about  $\theta_1$ ? Do you understand why this is right? Okay. Then  $\theta_1, \theta_2$ . So,  $\theta_1, \theta_2$  will be what?  $\theta_1$ , you know, when we are writing  $\theta_1, \frac{4EI}{L}$  right by  $L$ . Okay, this one is fine. Okay, so now what are we going to do? Let us do  $\theta_2$  and  $\theta_1$ . So, what are  $\theta_2$  and  $\theta_1$ ?  $\theta_2 \theta_1$  looks like this: this is  $\theta_2$ . This is  $\theta_2$  and this is  $\theta_1$ .  $\theta_2 \theta_1$  is also a cross stiffness, which will only be by one, right?  $\theta_2 \theta_1$ . So  $\theta_2, \theta_1$  is  $\theta_1$ ; this is  $\theta_1$ , and  $\theta_2$  is this one, right? So,  $\theta_1$  and  $\theta_2$  will be nothing but  $\frac{2EI}{l}$ , right? This  $\frac{2EI}{l}$ , and then the respective member here is nothing but  $-\frac{EI}{30}$ , right?  $+\frac{3EI}{30}$ , right? Fine, now  $\theta_3, \theta_1$ .  $\theta$  means this one, and  $\theta_1$ , there is no positive, right? So here should be zero. Now, we will be what?  $W$  is here. Who will contribute  $WW$ ? Only this one, right? Because one  $W$  and  $W$ . So, of course, for member one, the second node  $W_2$  is this one,  $\frac{12EI}{L^3}$ , right? And for the respective member here, yes, nothing but.  $-\frac{36P}{30L}$ , right? Okay. So, you can also find out other

things; I am writing them, the final things. Okay. So, when I'm writing down here, please check yourself that.  $-\frac{6EI}{L^2}$ , you know,  $+\frac{PL}{10}$ , you know,  $\frac{P}{10}$ . Okay, I will cancel out. So, this is zero. And then it is  $\frac{4EI}{L} - \frac{4PL^2}{30L} + \frac{4EI}{\frac{L}{2}}$ . Let me explain to you why it is. So,  $\frac{4EI}{\frac{L}{2}}$ , so here what happens is that, look, when I'm considering this member,  $\theta_2$ ,  $\theta_2$ .  $\theta_2$  will be this one;  $\theta_2$  will be contributed by both member one as well as member two. So, for member one, what will be  $\theta_2$   $\theta_2$ ? For member one, it will be this one:  $\frac{4EI}{l}$ . So, that's what  $\frac{4EI}{l}$  is, and then minus this one,  $\frac{4PL^2}{30}$ , right? But then, for that, I also contributed 2; there is no axial force. So, there is no contribution from the geometric stiffness matrix, but the contribution of this one material will be there. So,  $\theta_1$ ,  $\theta_1$  is what? This is  $\theta_1$ . So,  $\frac{4EI}{2}$  is still there;  $\frac{4EI}{2}$ , instead of  $l$ , it is  $\frac{l}{2}$ . Because this is  $\frac{l}{2}$ , right? But something like that. So, this is the way we did it, okay? Now this has to be symmetric. So, the other term you can also write here is symmetric. Right? Similarly, I can also write down the stiffness matrix you know for the symmetric mode. In a symmetric mode,



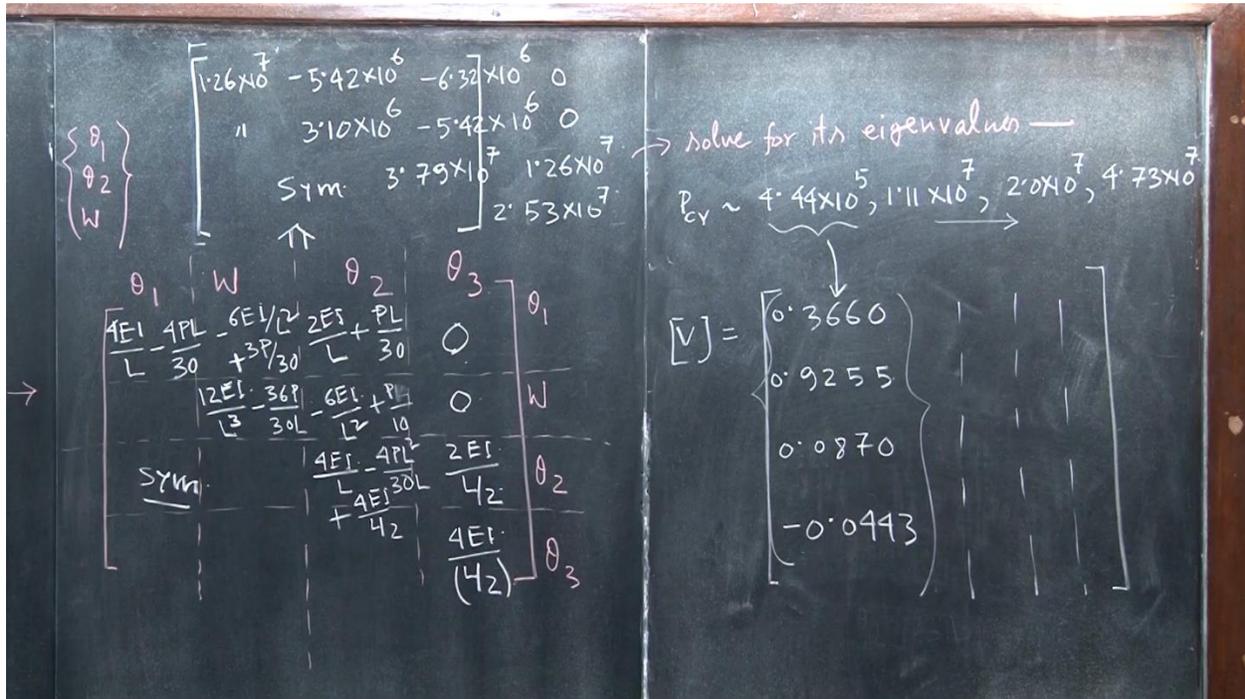
if I write it, then it will look like this. Huh?  $\frac{4EI}{L} - \frac{4PL^2}{30L}$ , this is  $\frac{2EI}{L} + \frac{PL^2}{30L}$ , zero whatever symmetric. So, this is  $\frac{4EI}{L} - \frac{4PL^2}{30L}$ . Well, I have to write it nicely; it is 2 okay. Anyway, I will write it here for the symmetric mode. This one, for the symmetric one, the stiffness matrix will be  $\theta_1$ ,  $\theta_2$ , and  $W_3$ ,

$\theta_1, \theta_2, W, \frac{4EI}{L} - \frac{4PL^2}{30L} + \frac{PL^2}{30L}$ , zero. Wow. Okay. You please check, for this one also. Similarly, now we can adopt, I'm removing. We can adapt typical value of this parameter. Now let us find out. So, you can see that, for this case, see what happened is that, I think whether you have taken it positive or negative  $P$ , it will not matter, because even if you take positive, there in the result you will get it negative.



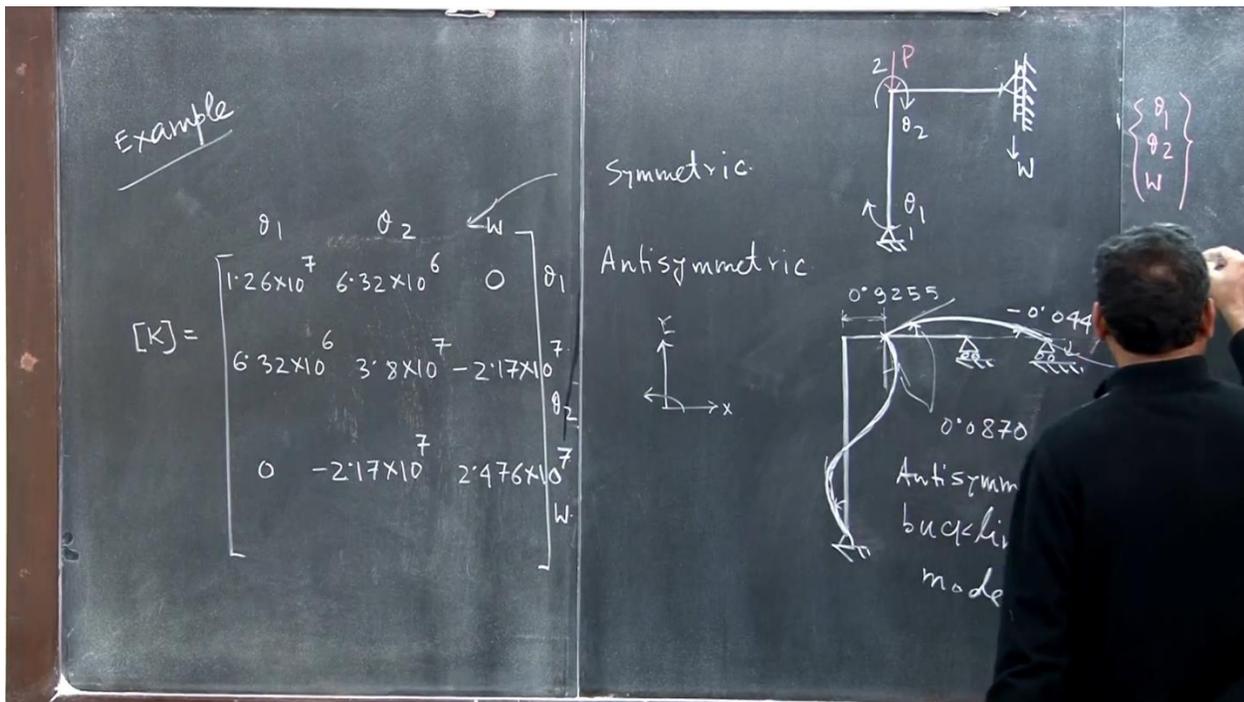
So, if you assume it is tensile, you will get a negative value. If you assume that it is compression, it is positive. Okay. So, you solve the eigenvalue problem. So  $k + k_g$  will be zero. And then to solve for what is unknown? The unknown is nothing but  $p$ . All other things are solved for  $p$ , then basically solve for  $p$ ; that is, that will give you the critical load  $P$ . So that is  $P_{critical}$ ; it will give. So, the input properties that we are going to give are You take  $E = 2 \times 10^{11}$  Newton per meter squared,  $I$  to be the second moment of area, 0.0000533 meter to the power of 4, and then  $L$  is 3.5 m, okay? If you take and then substitute, okay, then I will just remove it, but then this matrix is being, then this matrix, the term, I'm just writing with this; if you substitute, then what you are going to get. It is basically let me write it down. So,  $1.26 \times 10^7$ , then  $-5.42 \times 10^6$ , then  $-6.32 \times 10^6$ , and then zero. And then those things you can write down:  $3.10 \times 10^6$ ,  $-5.42 \times 10^6$ , 0,  $3.79 \times 10^7$ ,  $1.26 \times 10^7$ , and then lastly  $2.53 \times 10^7$ . And all these things are symmetric. Okay. Now, if you write it and find it for the eigenvalues, solve for its eigenvalues.

For its eigenvalues will give you, the eigenvalue you will get is  $P_{critical}$ . Let me write down  $4.44 \times 10^5$ ,  $1.11 \times 10^7$ ,  $2.0 \times 10^7$ . You can use a MATLAB command, okay? It will directly give  $4.73 \times 10^7$ . So, the fundamental critical load is important. So, this is you see that, all others are increasing you see that okay, and Then vector, if you write 0.3660, 0.9255, 0.0870,  $-0.0443$ . this is the other mode.



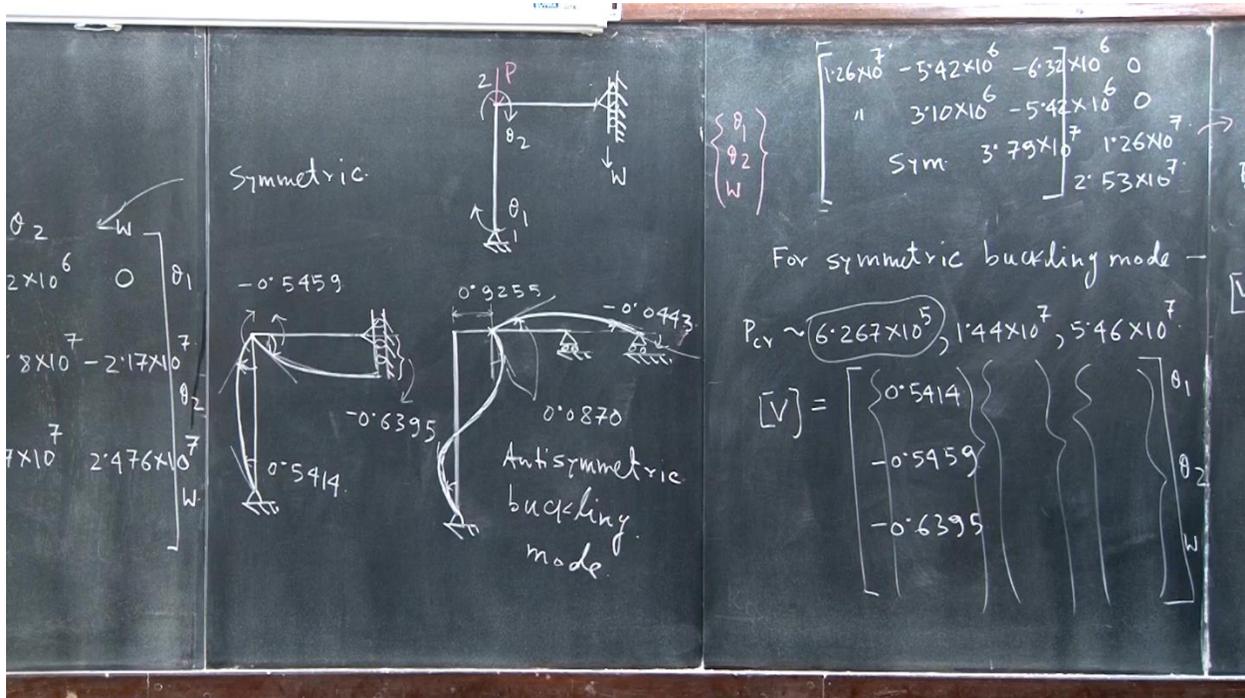
So, this is the first mode corresponding to the fundamental critical mode right. so, this mode we going to plot okay. So, when you are plotting it, you'll see that you know, let us plot it, how it looks like, okay. So, the first one is what were the degrees of freedom? This is  $\theta_1$ ,  $\theta_2$ . This is  $\theta_1$ . This is  $\theta_2$ . This is  $W$ , and then this is  $\theta_3$ . So, the first one is 0.366. Okay. So, if you recall, that was the choice of coordinate system. So, anticlockwise was positive. So here you know what anticlockwise positive means; here it should have been something like this. Okay, because this rotation is positive. Okay, 0.366. So here it will have something like this. Okay. Now what about the next one?  $\theta_2$ .  $\theta_2$  is basically, sorry,  $w$  is 0.0870, right? No, I think  $W_2$ . I think we have put  $W_2$  first here, right? Please see that  $\theta_1$   $W$   $\theta_2$ . Right? Something like this:  $\theta_1$ ,  $W$ ,  $\theta_2$ , and  $\theta_3$ , right? So, here  $W$  is positive. So, it will come in this direction. All right? So, it will come in this direction. So, this is 0.9255, okay? So, it is coming here, huh? Okay. And then the rotation here,  $\theta_2$ , that is what? Positive. It is also anticlockwise. Right. So, anticlockwise means here, if we consider this

one anticlockwise. From here, you know it is coming anticlockwise, and here it is anticlockwise. So here it will be something like this, you know, maybe. Here, your things are going there. From here, it is going anticlockwise. It will go like this, and similarly, here it is anticlockwise. So it will go something like this: this  $90^\circ$  angle needs to be maintained, right? So here is what we can see; it will be something like this. Okay. And then there is something like this; now here it will come there, and there  $\theta_3$  is negative. Negative means what? It is clockwise. What does clockwise mean? Clockwise means you know it will come something like this. Clockwise, right? From here, it is clockwise. Right? So here it will go and this will come there. This one is going to be  $-0.0443$ , and here it is. This one is, you know, this rotation; this one and this rotation, these two rotations are the same, and this is  $0.0870 \theta$ . Okay. So, this is mostly, you know, this is the way it will look for the buckling mode. Okay.



So, from here it will go something like this: take this one. This is the first buckling mode, you know, the anti-symmetric buckling mode. Huh. So now what will happen, you know, if it is symmetric, okay, then I will just write down the symmetric one. What will happen for the symmetric one? Symmetric one, the stiffness matrix—let me write down  $K + K_g$ , okay? Here, if you write, I'm writing the values, okay? So, the values will be  $1.26 \times 10^7$ , then  $6.32 \times 10^6$ , 0 okay and  $\theta_1$ ,  $\theta_2$ ,  $\theta W$ , we have  $6.32 \dots$  Now here, if you solve for the eigenvalue, the critical load

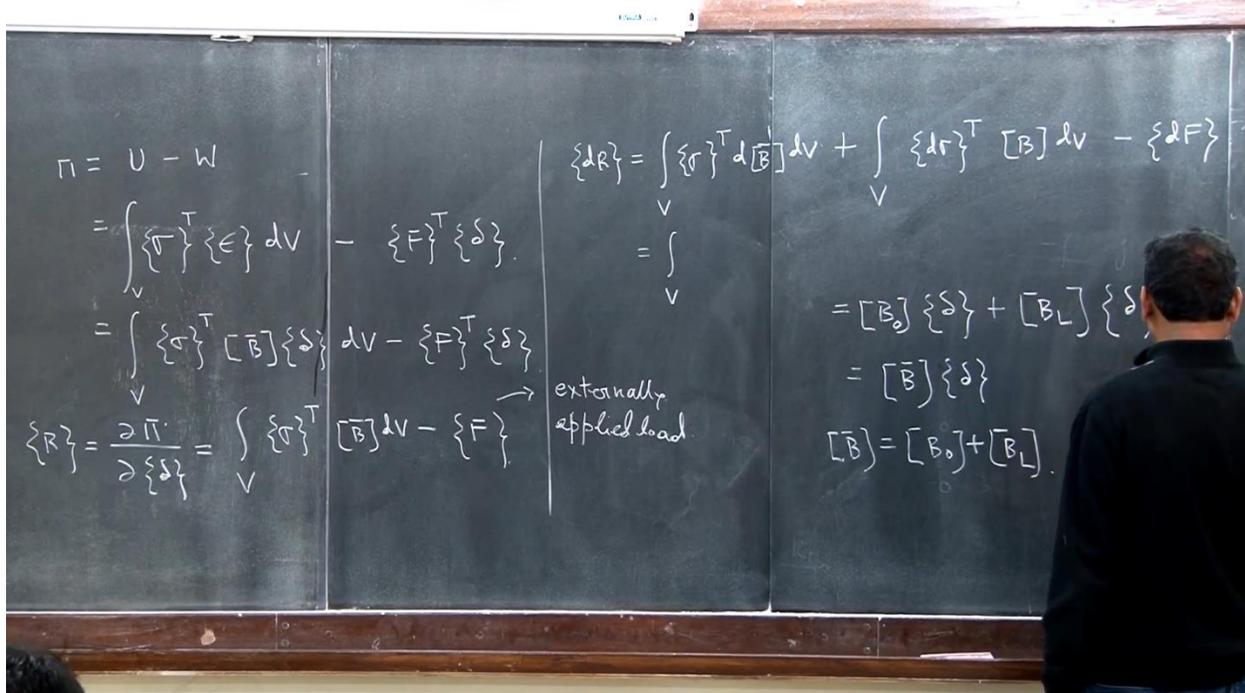
here, for symmetric mode buckling, once a critical load, I am just going to write it down. So, it is  $6.267 \times 10^5$ ,  $1.44 \times 10^7$ , and  $5.46 \times 10^7$ , okay. The eigenvectors are not all eigenvectors; there will be three eigenvectors, but I'm going to write only two, okay. Only the first one, okay? This is 0.5414 -0.55459 and -0.6395. So now once again, we can draw the buckling board for the symmetric one. The buckling board for the symmetric one, like the asymmetric, can be drawn. So here you see that  $\theta_1$  is anticlockwise, and you know this one; the way we have written it was, I think,  $\theta_1$ ,  $\theta_2$ , and  $w$ , right?  $\theta_1$ ,  $\theta_2$ ,  $w$ . So,  $w$  is downward; you know this is coming down, and  $\theta_1$  is positive while  $\theta_2$  is negative. So, it will be something like this. Now this is anti-clockwise, and here it is; and then here it will come down, like this. Okay.



Here, of course, it will be rotation zero, and this deflection is nothing but, you know,  $-0.0443$ . This is clockwise rotation. This rotation is, this is basically  $0.3660$ . Oh no, sorry. This is  $-0.6395$ . This one is  $0.5414$ , and this one is negative. This one is clockwise, and both of these values are  $-0.5459$ . Okay, something like this. Okay. So, do you see, now the important part here is basically, please note that the fundamental, you know, the lowest critical load for the symmetric mode is this, and for the antisymmetric, it is this. So, which one is more critical? The antisymmetric one is more critical because this is lower than this. Do you see that? So, at least for the antisymmetric mode, as far as the given frame is concerned, the lowest critical load is given by the

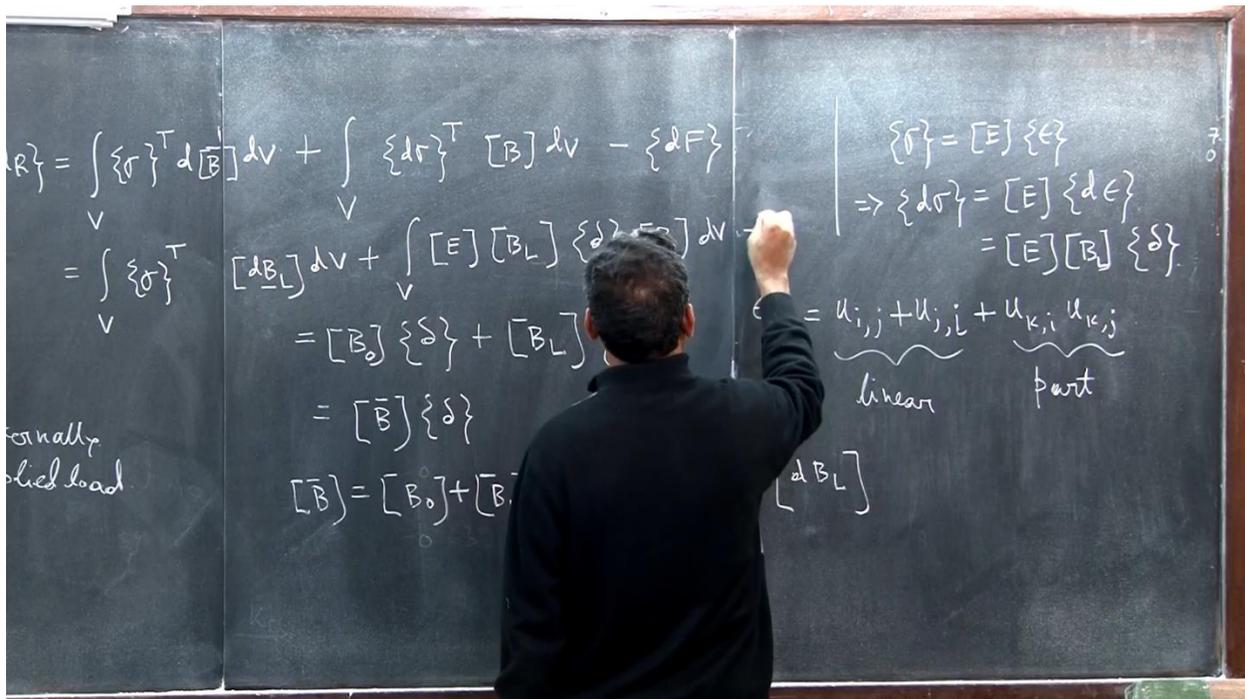
anti-symmetric mode, right? So, please note down the critical load as well as the respective buckling mode shape. Okay? Now, a higher buckling mode could have also been, I didn't do it. Okay? So, this way you can solve the problem. Now you can write code here; you know, the code can perform. I mean, of course, we have to have the approach similar; the only thing is that. The way you have learned in finite element analysis to find the stiffness matrix, similarly, you have to find the geometric stiffness matrix. And then you can solve the eigenvalue problem, okay? And you have a finite element code written. You should know, have you coded anything, any finite element thing? Have you coded? If you code it, you know. Then the only challenging part in coding that you know is doing the struct assembly of the stiffness matrix. So, from the element-level stiffness matrix, identifying them and putting them in the right place in the structural stiffness matrix is why it is a little challenging; otherwise, it is very simple coding. So I hope I am not explaining that one, but there needs to be one loop over elements, one loop over degrees of freedom, one loop over the element nodes, and another loop over the degrees of freedom. Then, because you have I and J, two indices for the stiffness matrix, you will have another one: a nested loop over nodes and degrees of freedom. So, there will be a total of 1, 2, 3, 4, five loops, okay, for performing the assembly of stiffness matrices. Only a single loop over the element, but there will be two loops over the node and two loops over degrees of freedom. Okay, and you need to give the input as the nodal connectivity. First, you have to give the element connectivity. So, each element is connected between the two nodes for which you have to first do the discretization of the structure. So, each element is connected to a particular node, right? So, when you are picking up any element, you have to identify which element is connected to which node. So that means you need to have a connectivity array defined while doing the computer programming. Okay. And then, you also need to give the coordinates of the individual node, because depending on that, you will find out a length. Here we are not discussing further. But you have to do the transformation from the local coordinate system to the global coordinate system. You know those things; we learned them in your finite element course. So, those things I'm not doing, but in the same way, you know the geometric stimulus matrix is derived; you have to have this geometric stimulus matrix, okay? Now I will just show one more thing. From the previous class, you can recall that the geometric stimulus matrix doesn't show all the way; we were deriving for the beam mode, not showing the rigid body modes. It was showing one rigid body mode, but another one was not, due to some reason that I have explained in the previous class. Now I will show you the role of this

geometric stiffness matrix in nonlinear analysis. Okay. So, what we'll see is that the geometric stiffness matrix becomes a part of the tangent stiffness matrix, as you require nonlinear finite element analysis. But please note that in nonlinear analysis, there are also additional components. That's what I mean: what is the role of the geometric stiffness matrix in the tangent stiffness value? So, what is strain energy? You know potential energy is this strain energy minus the work done, right? So, strain energy potential energy, we can write, this is  $\sigma$  right and  $\epsilon$  right, you know  $dv$  right minus your external load into  $\delta$  right. That is how you write the potential energy, right? So, when you do this minimization of the potential energy. So, what now? We will write that  $\epsilon$  will be one linear stiffness, another will be nonlinear stiffness, and if you do nonlinear analysis, this is linear stiffness and nonlinear stiffness. Okay. So linear stiffness and this one is nonlinear stiffness. So, you can recall what I wrote,  $\epsilon_{ij} = U_{i,j} + U_{j,i} + U_{i,k}U_{j,k}$ , something like that, right? Huh? Oh, sorry, this is  $K, IK, J$ . So, this is the linear part and this is the nonlinear part, right? The linear part was this, and the nonlinear part was because it's a product of this displacement gradient term. So, now when you are doing the finite element analysis, we have learned that linear strain can be related to nonlinear strain by using this  $B$  matrix, right?



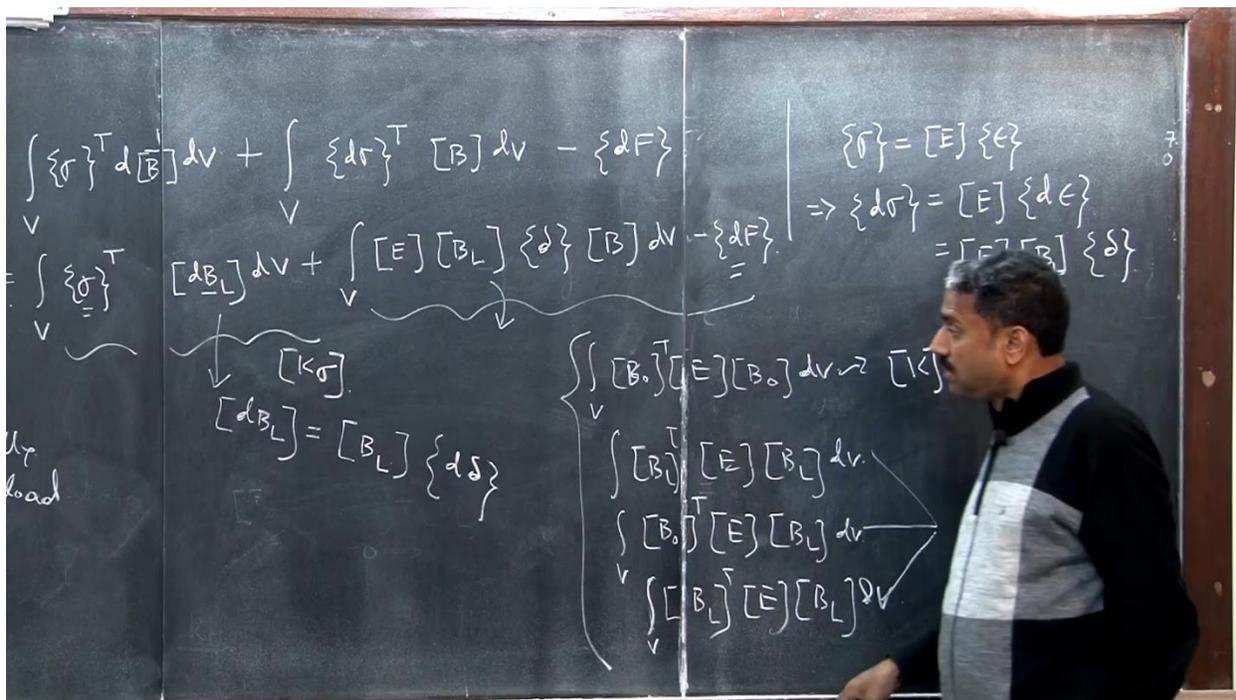
Strain displacement matrix, whether the nonlinear term, if you see that, you can also point out another nonlinear, you know,  $[B_L]$  and nonlinear one, okay,  $[B_0]$  and  $[B_L]$ , okay. So now, when

you are coming back here, you see you want to find out the, you know, when you  $\frac{\partial \pi}{\partial \delta}$ , okay. So, I will come later; let us write here integration  $V$ . You know  $\epsilon$  is nothing but you can combine the two. So,  $\bar{v}$  into  $\delta$  and  $\bar{v}$  is nothing but  $V_0 + B_L$ , both linear and nonlinear. Okay. So,  $\sigma^T$  and  $\epsilon$  is  $\bar{B}$  and  $\delta$   $dV$  minus. So, find out residual. Okay. So,  $\frac{\partial \pi}{\partial \delta}$ , when you  $\frac{\partial}{\partial \delta}$ , when you do here it will be  $\sigma^T$  will be  $[B]dV - \{F\}$ . Okay. Now you see, now what will happen if you, so this is the residual force; we know from the casting theorem that we are getting it. Okay. So, we're going to take an increment of this residual. Okay. Now we write this down in nonlinear analysis. So,  $DR$ , if we take the differential of this, then what will happen here? You know you will see that this one will be  $\{\sigma\}^T d[B] dv$ , and you have written it right, this one right. So, I can remove it. So then plus integration  $\int_V \{d\sigma\}^T [B] dv - \{df\}$ .  $F$  means the externally applied load. Please note that  $a$  is the nodal load.  $F$  is the externally applied load. Right? Applied load. Huh? And then, here is what we can write. Please note that. So here the strain displacement matrix consists of a linear part and a nonlinear part.



So, if you take the differential of the total sequence matrix,  $\bar{B}$ , okay, you know, this is  $\bar{B}$ , okay. What is going to happen is that  $D\bar{B}$ . Matrix reference will be, since it is linear, this will go; the only thing that will stay here will be  $DB_L$ , right? Isn't it  $DB_L$ ? So, for the nonlinear part one,

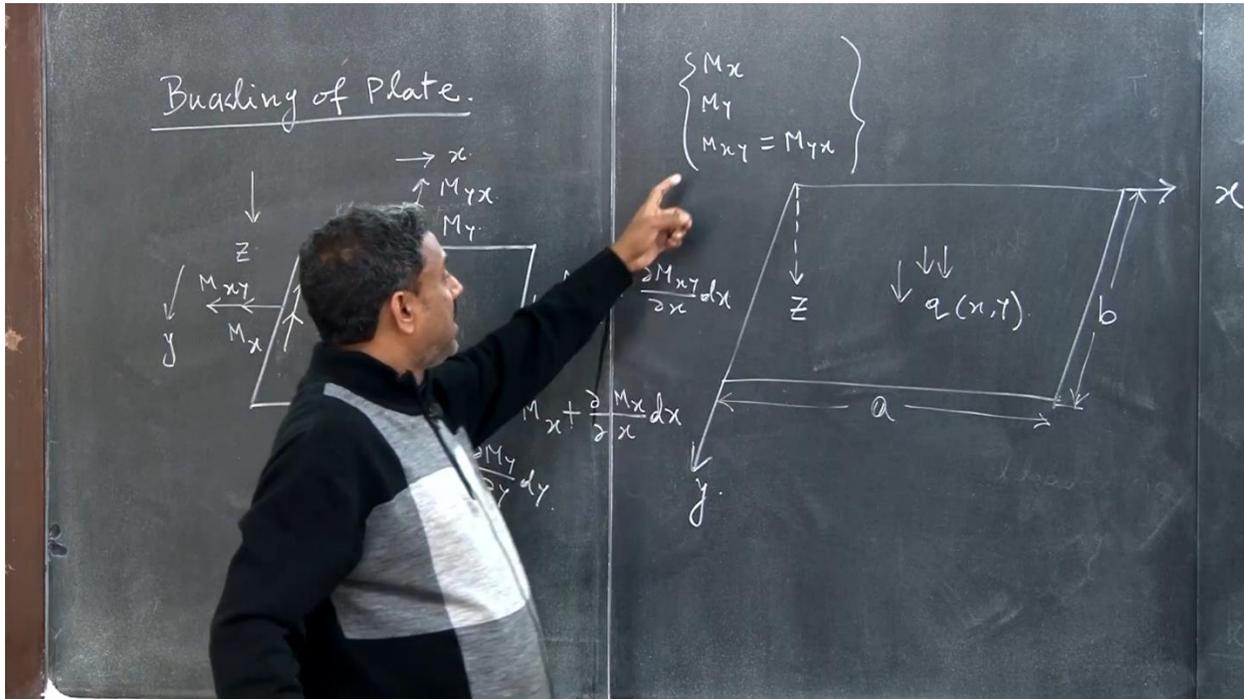
whatever, I'm just writing it as  $\sigma^T$ , and here I'm just writing  $\frac{dB_L}{dv}$  right, plus integration  $v$ . When I'm writing  $d\sigma$  you know bar,  $d\sigma$  is nothing but what, you can, right? You know  $\sigma$  is  $E$  times  $\epsilon$ , right? So then  $d\sigma$  is  $E$  into  $d\epsilon$ , right? Where you can write here, you know this is, you know, this  $\epsilon$  is nothing but, you know,  $EB\delta$ , and please note that this is nonlinear, right? Because if you take the differential, you know, the differential of the linear part of strength, then it will be zero. So, the nonlinear part will be there. So here, I can write  $E$ ; this is  $B_L$ , this is  $\delta$ , and then  $B dv - df$ , right? What do I see here? You say that this part you know; this part is the one you see that contains the stiffness and stress. Then, you know  $[dB_L]$ ; you can also simplify it later,  $[dB_L]$ , okay?  $[dB_L]$  can definitely be expressed as  $[dB_L]$  is equal to, you know,  $[dB_L]$  can also be expressed in terms of, you know, something like this. Okay, this one is nothing but the tangent stiffness matrix; you know, the geometric stiffness matrix. It's a function of the stress, and here you can see the rest of this part will consist of four elements. one will be of course, what will be  $k_0$  you know  $[B_0]^T [E] [B_0] DV$ . Another one will be  $[B_L]^T [E] [B_L]$ . Another one will be  $[B_0]^T [E] [B_L] Dv$ , And another one will be  $[B_0]^T [E] [B_L] Dv$ ,



So, this additional thing will come from this. Okay. And then, these are the incrementally externally applied loads  $DF$ . Okay. So, you see that this is the total; this  $DR$  is nothing but  $K_T$  into  $\{d\delta\}$ , and  $E_2$  into  $\{d\delta\}$  is nothing but  $DF$ . So, externally applied, okay. So here, the tangent

stiffness matrix is a part of the geometric stiffness matrix, and then there will be some additional terms. Some additional terms will be here. Okay. So, what is important to see here is that, when we do the non-linear analysis, the tangent stiffness, you know, the geometric stiffness matrix plays an important role there. And that's basically what we are getting here. Okay. So that's the point I wanted to make here: that the geometric stiffness matrix becomes a part of the tangent stiffness matrix for nonlinear analysis. But it is not everything; there will be additional stuff, other than the geometric mathematics. Okay. And when are we doing this, because when we do a fully nonlinear stability analysis. We have to solve for the whole system through a nonlinear analysis. In geometric nonlinear analysis, you have to do a full analysis. But when we are doing linearized stability analysis, that means we are linearizing the system of equations, only in a system of equations. To include only the linear term, that's what is leading to an eigenvalue problem, and that's why we are truncating this part. We're truncating this part. And we are just making the, you know, material stiffness matrix, linear stiffness matrix, and geometric stiffness matrix, and then solving the respective eigenvalue problem. Okay. So, do you see how the linearization is done? When we are doing the linearization, only this part will be there. See, this part is the one: the linear stiffness matrix. Okay, so  $K + K_\sigma$  will be equal to zero. Okay. If there is no externally applied load,  $DF$  is zero. So that becomes the linearized stability analysis, you know. And we neglect this part. These are nothing but nonlinear terms, which otherwise need to be included for a nonlinear stability analysis. So, we must understand and appreciate the regime of analysis. How are we doing the analysis? We are basically doing a linearized, you know, analysis. When we solve the eigenvalue buckling problem, an eigenvalue buckling problem is basically a linearized stability analysis. But if you include the nonlinear term, then you'll have additional stuff. Okay. Now, of course, when we do nonlinear analysis, we have to use some special methods; you know, we have to evaluate this  $[K_L]$  nonlinear. This is not difficult in nonlinear analysis. You know, people will explain to you how to find it out, but it's not very difficult. It's kind of okay. So, with this, I will conclude here the portion for the matrix method or finite element analysis for stability analysis. Okay. Okay. So now what we are going to do is study the buckling of a plate. Okay. So, all of you have already studied the bending of plates. All of you have studied correctly. What is a plate? A plate is a structural element that primarily carries transverse loads, and one of its dimensions is much smaller compared to the other two dimensions. Right? So, essentially its mechanics can be condensed into

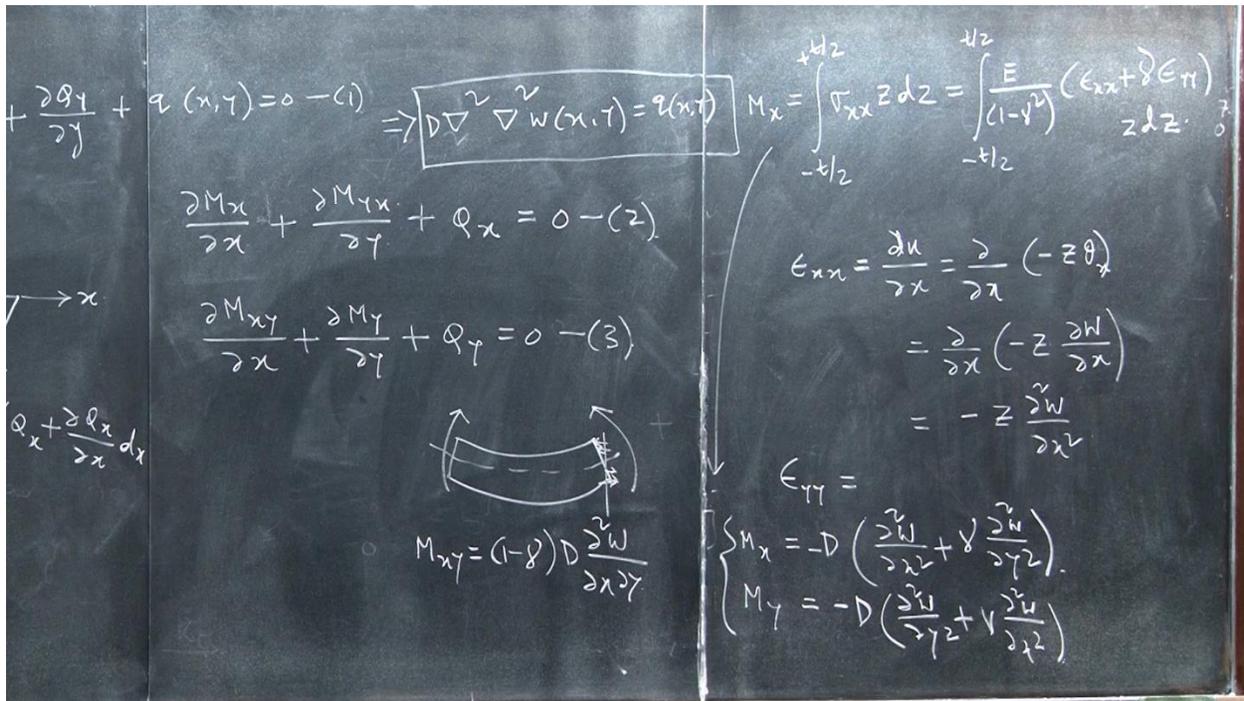
terms of displacement, which is concentrated at its middle surface. So, all the kinematics of deformation can essentially be expressed in terms of the deformation of the middle surface. Right?



So that's why we define a plate, right? We can take this as  $x$ , this one as  $y$ , and then downward you can take as, it's the right coordinate system, right? This one, huh? The dimension, we can take  $A$  for the  $x$  dimension and then for  $B$  the dimension  $y$ , right? Things like that, right? Now all of you are aware of the bending of the plate, right? And then you have derived the governing equation, right? What was the governing equation? The governing equation was  $\nabla^2 \nabla^2 w(x, y) = \frac{q(x, y)}{D}$ .  $w$  is nothing but the vertical deflection, of course, downward deflection, and  $Q$  is the load being applied, that is,  $Q(x, y)$ .  $D$  is the flexural rigidity, and this is equal to  $\frac{Et^3}{12(1-\nu^2)}$ . The flexural rigidity of the plate is okay. And nabla this operator,  $\nabla^2$ ,  $\nabla^2$  It is nothing but a you know  $\frac{\partial^2}{\partial x^2}$  laplacian operator plus  $\frac{\partial^2}{\partial y^2}$  and then if you just write it It is  $\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}$  that was the governing equation for plate bending, right? How did you derive it? that you all of you, just did it, I can briefly recap it, I mean two way you can derive, it one is directly integrating the starting from the equilibrium equation, as you have learned in continuum right and integrating the equations over thickness and another Okay. Is to show the you know stress resulted over here and you can go their right. Which

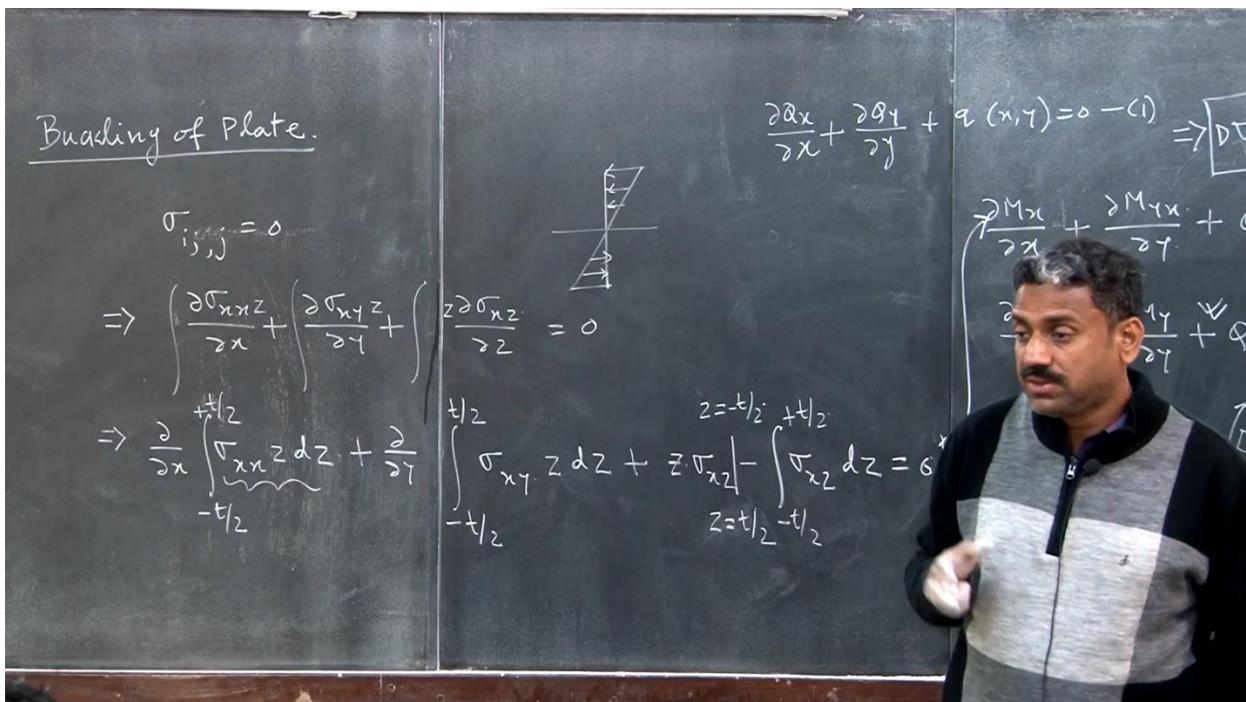
approach did you follow in your theory of plates and shells, huh? Did you directly start from the equilibrium equation as you learned in mechanics, or do you have the equilibrium equation? Yeah. So, the way I have derived, you know, the bending of the plate, I will not briefly recapitulate. So, if you can recall, assuming this is the plate. So you see, this is positive  $x$ , this is positive  $y$ , and this is positive  $z$ . So, I recall that you know. That was the double arrow; we have used the right one. So, we're using a double arrow for the bending moment  $M_x$ . So, here it is, this is  $M_x$ ,  $M_x + \frac{\partial M_x}{\partial x}$  to the  $dx$ , right? And then. here it will be in opposite direction. because this is positive base this is negative base. You understand now positive base in positive direction of  $x$  you know, this is positive okay that's what and then this is a negative base because it is normal to say this is negative, right? Similarly, here it is  $M_y + \frac{\partial M_y}{\partial y} dy$ , and here it is  $M_y$ ; then also I give that torsion, right? This is  $M_{xy} + \frac{\partial M_{xy}}{\partial x}$ , right? And this one is  $M_{xy}$  and then  $M_{yx}$ ; please note that  $M_{yx} + \frac{\partial M_{yx}}{\partial y} dy$ . Okay. And here it is  $M_{yx}$ . So, these are the bending moments. And then please note that there are too many bending moments per unit width:  $M_x$ ,  $M_y$ , and  $M_{xy}$ .  $M_{xy}$  and  $M_y$  are the same along the  $X$  axis. Bending moments per unit width.  $M$  is the bending moment per unit width with respect to the  $X$  and  $Y$  axes, and  $M_x$  is the torsional moment by  $M_{xy}$ . And you can also derive that this is the bending moment, and we are showing the moment, and then you can also show the forces. The forces, how we can show. The  $x$ , this is positive  $x$  right and downward  $X$  positive. So, this is  $q_x$ , then  $q_x$  by  $\frac{\partial q_x}{\partial x}$  into  $\partial x$  right, and here it was  $q_x$ , then here it was  $Q_y + \frac{\partial Q_y}{\partial y} dy$ , there it is  $Q_y$ . So, from there you can do the force equilibrium, and then you can do the moment equilibrium. You can recall the force equilibrium will give you the equation  $\frac{\partial q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = q + xy = 0$ . This is the moment equilibrium, right? Similarly, you can get that this is force equilibrium from the vertical  $Z$  direction, and then  $q_x$  was  $\frac{\partial M_x}{\partial x}$ . If you write here, you know from here; you can see you know.  $q_x$ , this one you can understand; this is the force equilibrium, right? Now, if you set the moment equal with respect to the  $x$ -axis and the  $y$ -axis, then here you see that  $\frac{\partial M_x}{\partial x}$ , right? Here, write this one:  $+\frac{\partial M_{yx}}{\partial y}$ , right? Hmm, plus then this one, plus  $q_x$  is equal to zero, right? And then similarly, you will also get  $\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yx}}{\partial y} + Q_x = 0$ . So, you are just combining all three equations; if you

combine, you substitute  $q_x$ , this expression here, this expression here, okay, and then you'll get what you write. and if you can recall by you know, we can derive, what is the  $M_x$ ?



$M_x$  will be, this is nothing but integration  $\sigma_{xx} Z dz$  and this is  $-\frac{t}{2}, 2 + \frac{t}{2}$  And when it is plate bending, it is basically the generalized plane stress, right? Generalized plane stress, especially when we are discussing this plate theory, is thin plate theory. Which is Kirchhoff's plate theory, right? There is no shear deformation, so plane sections remain plane, huh? after bending, so  $\sigma_{xx}$ , if you can recall can be expressed as  $-\frac{t}{2}, 2 + \frac{t}{2}$  and this is  $\frac{E}{1-\nu^2} (\epsilon_{xx} + \nu\epsilon_{yy})z dz$  Huh. And here,  $\epsilon_{xx}$ ,  $\epsilon_{yy}$ , and  $\epsilon_{xy}$ ,  $\epsilon_{xx}$  you can recall,  $\epsilon_{xx}$  was what? You know it is nothing but  $\frac{\partial u}{\partial x}$ , and  $u$  is nothing but what  $-Z$  into  $\theta$ . And  $\theta$  is nothing but  $-Z \frac{\partial w}{\partial x}$ . So,  $\theta_x$  is basically okay. So,  $-Z \frac{\partial^2 w}{\partial x^2}$ . Similarly,  $\epsilon_{yy}$  can also be written as, you know why this is happening, right? Because if you may recall, it's like this, huh? Isn't it? So, plate you know, so plate bending moment like there. Okay. So, you can see this normal section. So here it is coming there and this is there, so that's why this is minus sign right. Hm, similarly, from there  $M_x$ , if you combine these two, you will get  $M_x$ , which is nothing but  $-D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right)$ . This is the bending moment curvature relationship. Then similarly, you

will get  $M_y$  is you know,  $-D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right)$ . Please note that I'm there might be some inconsistency in terms of sign but that you know sign will then  $M_{xy}$  will be  $-D(1 - \nu) \frac{\partial^2 w}{\partial x \partial y}$ . So, you see that, so for the moment it is direct curvature  $\frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial y^2}$  and because there is two ways bending That's why this new term will be used for the Poisson ratio and for the cross curvature related to the torsion in unit width. Okay. So, if you take this and then substitute everything, then only are you getting the equations you know, which are  $\nabla^2 \nabla^2 W(x, y)$ , nothing but  $\frac{Q(x, y)}{D}$ , huh?  $Q(x, y)$  and here is  $d$ ; this is the governing equation. When you get it, when you substitute everything, combine.



So, this is, I know, this is the way to derive. But another way to derive the plate bending equation is, basically, we do not need to discuss all these things; you just start with the equilibrium equation. So, the way you can directly derive it, you know, is you see that  $\sigma_{i,j,j} = 0$ . So, you can see  $\int \frac{\partial \sigma_{xxz}}{\partial x} + \int \frac{\partial \sigma_{xyz}}{\partial y} + \int \frac{z \partial \sigma_{xz}}{\partial z} = 0$ , okay. Now, if you multiply by  $Z$ , okay, and integrate it over  $Z$ , when you are integrating it, you will see that here you take  $\frac{\partial}{\partial x}$  of the integration of  $\sigma_{xx} z dz$ , and then you multiply  $-\frac{t}{2}$  to  $+\frac{t}{2}$ . Here is  $C \frac{\partial}{\partial y}$  integration  $-\frac{T}{2}$  to  $+\frac{T}{2} \sigma_{xy}, z Dz$ , and then last, here it is basically when you are integrating it; here you have to be a little careful, you know. So, first, the integration

of the second minus the whole integration of the derivative of the first plus the integration of the second. Okay. So, here you see this is nothing but  $\sigma_{xx}$  multiplied by  $z dy$ ; is nothing but what?  $M_x$ . So,  $\frac{\partial M_x}{\partial x}$ , this is what this is nothing but  $M_{xy}$ . Because of what is happening, you see that there are stresses. You are taking the moment, okay?  $M_{xy}$ , and then this top and bottom surface, if there is no attraction, this will vanish. And  $\sigma_z dz$  is nothing but  $Q_x$ . So, do you see that this equation is nothing? This equation is nothing but this equation is leading to this equation. You see that, right? These equations are coming directly; we can obtain them without going into anything and can derive that just from the equilibrium equation that we have learned in mechanics, right? Directly multiply the equilibrium equation by  $Z$  and integrate over the thickness, and we get this equation. Similarly, if we write to integrate the same equation, which indicates the equilibrium along the  $y$ -axis, Similarly, you will get the second equation. This one equation you will get, and then if you integrate the equilibrium equation over  $z$ , you won't equal to multiply with  $z$ . Then you will get this equation. You see the three equilibrium equations and now all these relationships. So, then you substitute from this into there and assume the plate kinematics; you know all these assumptions, and you can directly go to the lowering. So, you see that there are two different ways to derive it. Okay. Now that the plate bending theory is known, we have to take one step ahead, because we now want to start with the stability of the plate. Right? So, for the stability of the plate, we have to consider all these things when we are considering equilibrium in the undeformed configuration, the unperturbed configuration. But when we want to study stability, we have to consider equilibrium in the deformed configuration; we have to consider a priority deformed configuration of the plate, and then you. Once again, when we write down the equations in the deformed configuration, some additional components of stresses will come that we will see, once again, can also be done directly from considering the You know the resultant force in unit width, or we have to consider the stresses; this is the equilibrium equation, but not this; now this is the Cauchy stress equilibrium equation. Which basically, in the deformed configuration, now we have to distinguish between deformed and undeformed configurations. So, we have to introduce an additional equilibrium equation, where this distinction is made; that's what we have to consider in finite deformation theory. And in finite deformation theory, the equilibrium equation is written in terms of an alternate stress measure. First, Piola-Kirchhoff's stress measure, second Piola-Kirchhoff's measure; that's what we are going to discuss in the next class. For now, let us stop here. Thank you.