Stability of Structures

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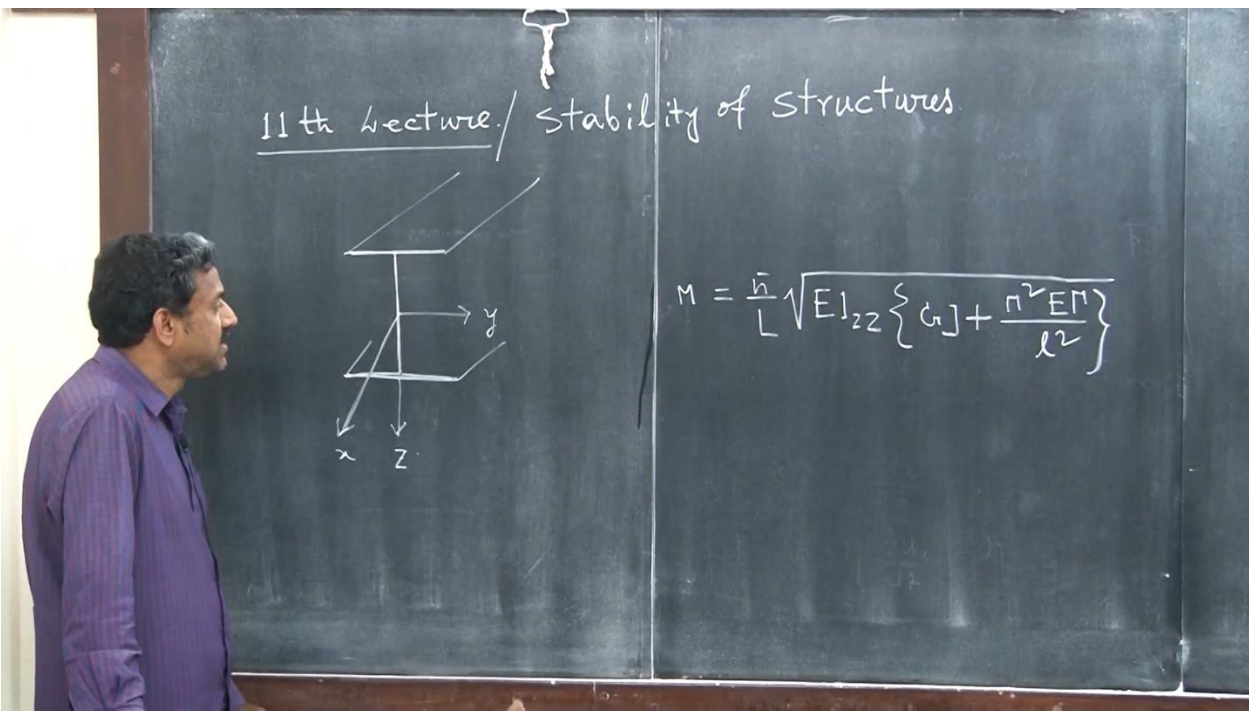
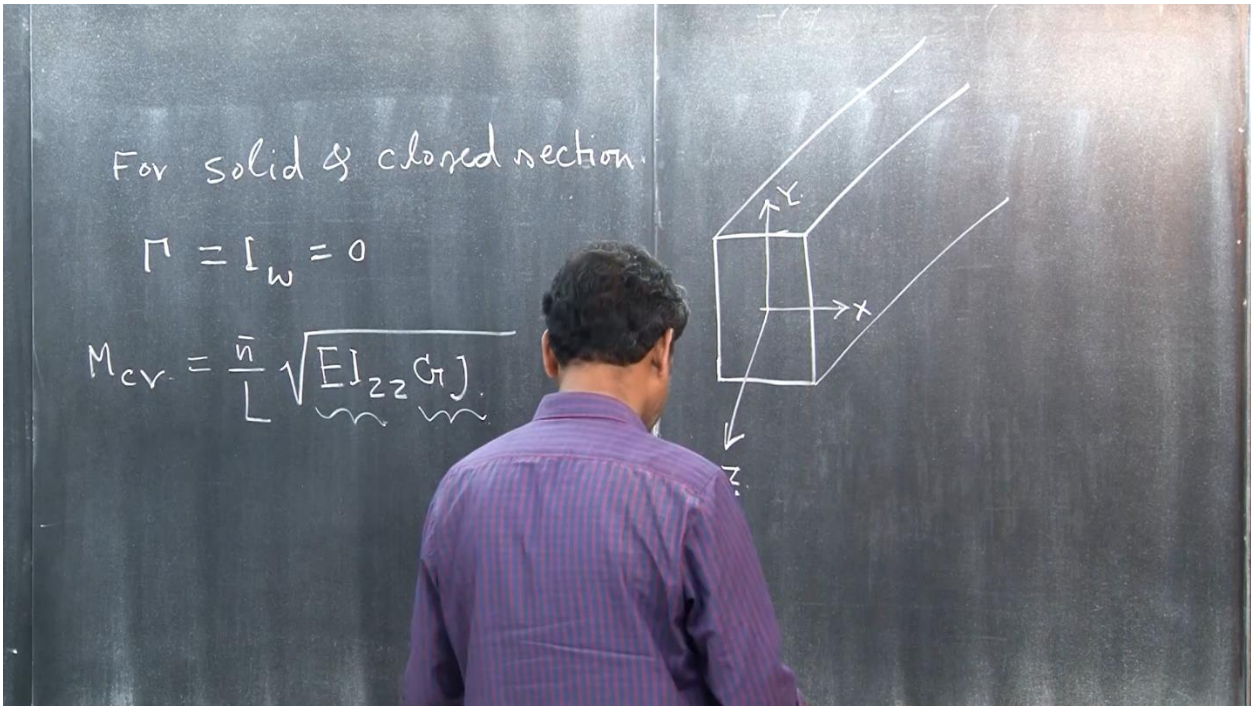
WEEK-06

Lecture 11: Matrix Method for Stability Analysis

Okay, welcome to the 11th lecture on the stability of structures. So, what we are discussing is lateral torsion and buckling of beams. So, that involves both the flexure as well as torsion. So, we have derived the formula for the, you know, critical moment. So, this is the critical moment. Okay. This is the critical moment, at which lateral buckling occurs for the beam. So, we have assumed the I-section beam to be subjected to pure moment and then M.



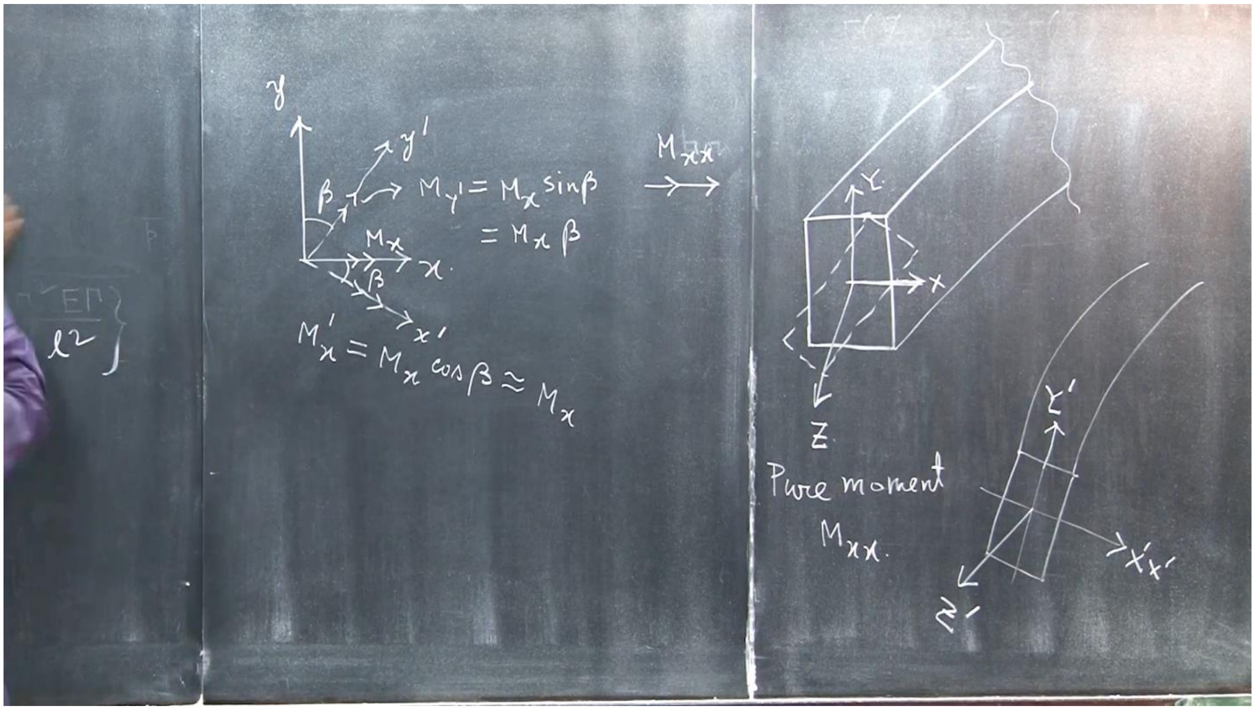
So, that expression is similar to the expression that you see in various standards. Okay, for design. When you design laterally unrestrained, you know I mean unrestrained flange, you know, top flange especially, which is under compression, right? So, of course, if there is a gradient in the bending moment diagram and the bending moment diagram is varying, then there will be some modifications to that, okay. So, we have derived that we have seen how we have derived the potential energy function. The potential energy function has, you know, strain energy and work done, you know. And there was little consideration in finding out the end rotation, that is happening at the end rotation of the two ends, right?



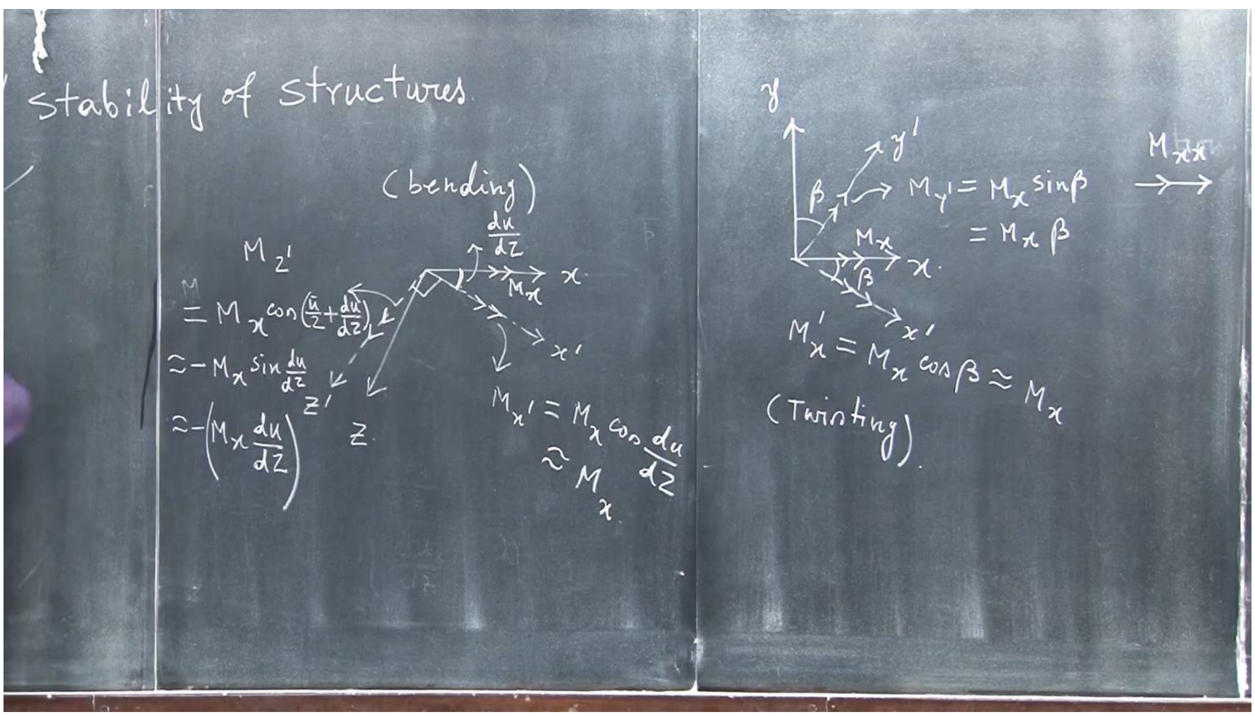
Subjected to how to plane, you know, bending, lateral bending, okay? So, bending was happening in the basically XY plane with respect to the Z axis, right? Here now, you see that for the rectangular section and others for the solid section or closed section. Okay. Closed section: what I mentioned is that the warping constant is very, very small; it's tending to nearly zero, okay? And then this M is a critical moment. M critical can be simplified as

$$M_{cr} = \frac{\pi}{l}\sqrt{EI_{zz}GJ}.$$

So, there is no warping. So E, I_{zz}, GJ. So okay. So, it is only flexural stiffness as well as Saint-Venant torsional constant. Okay. So, for a rectangular section, this can actually be derived in a much simpler way, which I can show you a little bit. Okay. For example, if you consider, you know, a So, you consider a slender, you know, kind of rectangular section, and then you define your coordinate system the way we have defined XYZ. Similarly, X, Y, and I think we are finding Z, right? Then you know how we will arrive at what I am showing. So, at the other end, maybe it is whatever fixed; whatever is happening. So here you see. This is going to, you know, I mean it will be twisted, right? So, you see that the way it will deform, this end will be twisted. So, maybe it is going to— I mean, say, you know the deformed configuration is like this. Okay, something like this. Huh? So, it is essentially happening like this, and then you know something like this, something like this, you see that, okay? Due to lateral B like then, what will happen? So, I'm considering that it is also subjected to a pure moment, okay?

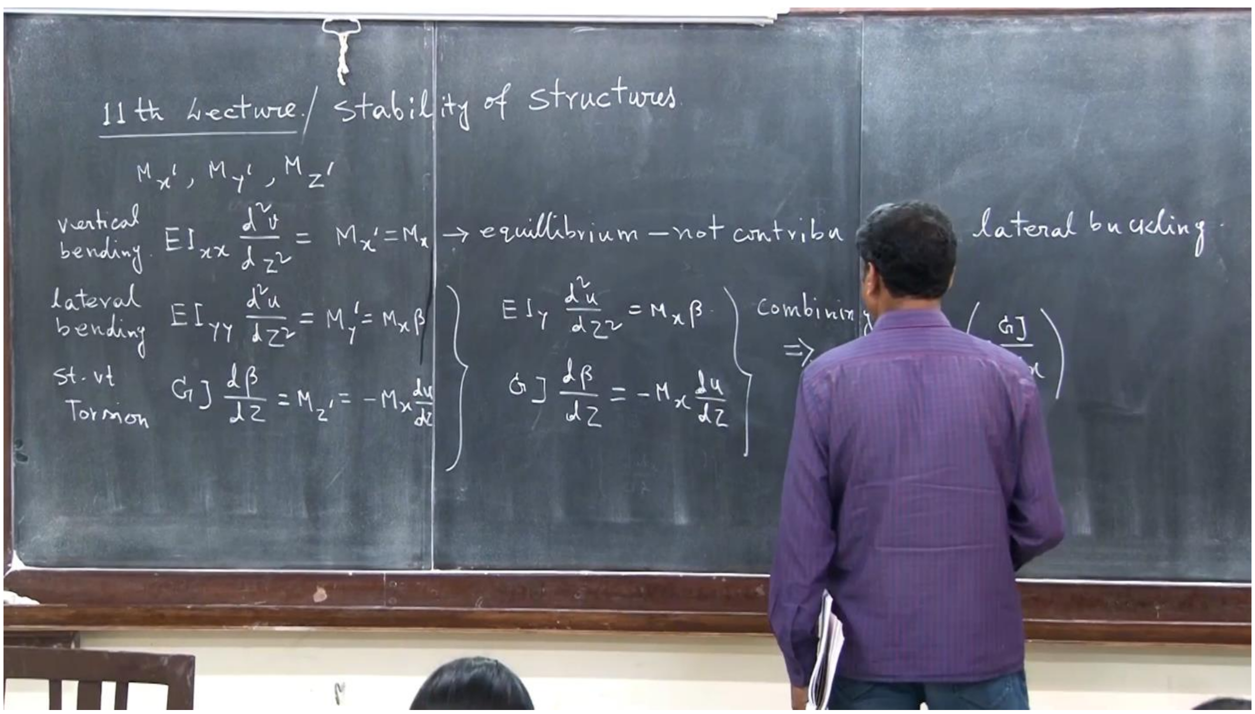


So, the pure moment with respect to the X-axis, okay? So, pure moment bending moment. So, M$_{xx}$ is okay with respect to the x-axis. So, now xyz is going to deform. And this is going to have, I'm considering that this is being x dot x dot, this is being y dot, and then this is being z dot. So, z dot and z are parallel; z and z dot will be parallel. So that's not a problem, but there will be some change, you know. So, from XY, it is happening that, you know, it is coming to be X dot, Y dot, and okay. So please note that this bending moment I am denoting. So, using the arrow diagram, M$_{xx}$ is okay. Because if it is, you know the screw is going to come like that, okay? So, I'm just denoting this as M$_{x}$. Okay. So, if they're getting twisted, that means this angle is β and this angle is also β, right? So, if this, so then it will have two component M$\dot{x}$ is going to be M$_{x}$ cosine of β and β we are considering in very small deformation. So, M$_{x}$ cosine β mean M$_{x}$ you know β tending to zero. So, essentially, M$_{x}$, right? And then what is happening here, in this moment, that moment is M$\dot{y}$. M$\dot{y}$ is going to happen. So, M$_{x}$ sin of β. So, M$_{x}$ sin β means M$_{x}$ multiplied by sin β is equal to β, okay, fine, right? Now, considering the component Hm, if I consider the component, you know, this is in the xy plane. The xy plane means this is the transverse cross-section, okay? If I consider in X-Z what is happening, X-Z after deformation xz is coming to x dot j dot, right? So, it is horizontal. Here it is x, and here it is Z; it was in the horizontal plane. It is coming in x dot. So, x dot and z dot, that is happening basically here, you know, I'm going to consider this one to be dot and here is going to come; it is z dot, okay. Now see, you know that on the vertical plane the section is undergoing a twist, right? But here in the xz plane, this is going to undergo lateral bending. This is β; this is because twisting is happening in this vertical plane, right? Twisting, right? But here, what is happening? What is happening is bending, right, in the horizontal plane, bending in the X-Z plane. So, bending with respect to the y-axis, right? Sorry, not with respect to the z-axis, with respect to the y-axis, right? So, if we consider the deflection along x to be U, and the deflection along X to be U, V, then what is this? This is nothing but $\frac{\partial u}{\partial z}$, right?

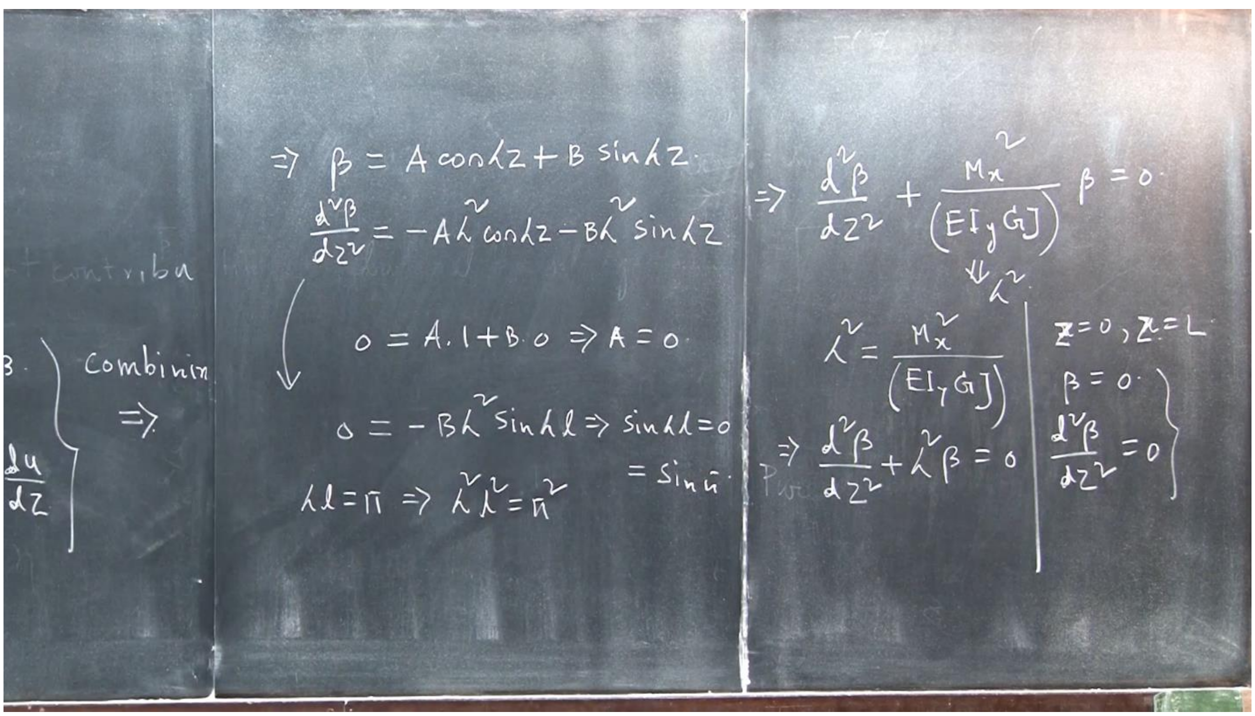


I'm considering that, because of flexure, there will be deflection along x, right? So, U and V are the components, right? Along x, this is u; along y, this is v, and w is fine, you know, u and v. So, when it is deflecting, the flexural deflection happening in the x-z plane bending with respect to the y-axis is the deflection along y. So, this angle is nothing but what? $\frac{du}{dz}$, right? So, now this bending moment, M_x, the component of this along xx', is what? This one is $M\dot{x}$, which is nothing but

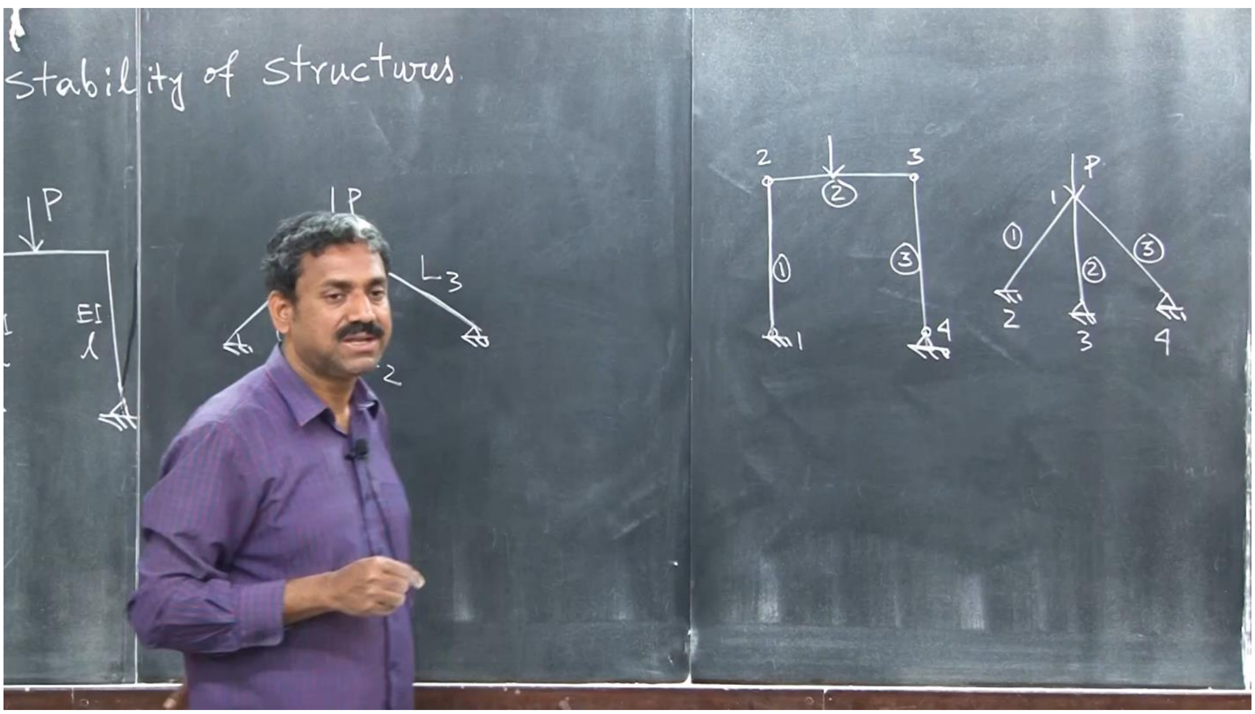
$M_x$cosine $\frac{du}{dz}$; you know it is very small. So, M_x is right, and what about this component? This one is $M\dot{z}$. $M\dot{z}$ is nothing but $M_x$cosine $\frac{du}{dz}$; this angle is $\frac{\pi}{2}+\frac{du}{dz}$. So, $(\frac{\pi}{2}+\frac{du}{dz})$ means - M_x sin $\frac{du}{dz}$, right? That means it is - M_x $\frac{du}{dz}$, right? Now, so you see, even in the deformed configuration, when X is coming to X dot, $M\dot{x}$ is not going to change; $M\dot{x}$ is M_x in both cases, right? Because there is a common X-axis, the X dot axis is correct. However, what is happening is that there is a twisting moment that is occurring. That there is, you know, this moment when $M\dot{y}$ is coming M_x into β, right? And then, we are having another twisting moment that is coming, which is M_x $\frac{du}{dz}$, right? Okay. So, now we can write it down. There are three moments, you know: $M\dot{x}$, $M\dot{y}$, and $M\dot{z}$. $M\dot{x}$ is nothing but the bending moment with respect to the X-axis, right?



So, EI with respect to the x-axis means it is xx, you know, $\frac{d^2v}{dz^2}$; it is nothing but M_x, that's what I'm going to write, right. And then EI_{yy}, that is $\frac{d^2u}{dz^2}$, is going to M. That is with respect to Y, right? $M\dot{y}$, right? And Z then, there is only Saint Venant torsion $\frac{\partial U}{\partial \beta}$, nothing but $M\dot{z}$ or M_z, okay? These are the three things happening: see $M\dot{x}$ is the bending moment with respect to the x-axis, right? So, it's causing bending with respect to the x-axis; the bending moment y dot is going to cause a bending moment with respect to the y-axis, which means it is going to cause lateral bending. So, this is vertical bending in the vertical plane. This is, you know, lateral bending. And this is, you know, that Saint Venant torsion, right? Now I'll just substitute M$\dot{x}$; this is always x, right? M$\dot{y}$ is what? M$\dot{y}$ is M$_x$ times β, right? And M$_z$ is nothing but - M$_x$ $\frac{du}{dz}$, right? So, now I will just remove this. I'm removing this one. Huh? Now you see, we don't care about bending in the vertical plane because this is already for equilibrium, right? Huh? And this is not contributing to lateral buckling, right, or instability. This is not contributing, right? So, you just have to consider these two; now you see, there is a coupling, okay. So, what is going to write you $\frac{d^2u}{dz^2}$ is M$_x$ β. And GJ, you know $\frac{d\beta}{dz}$ is equal to - M$_x$ $\frac{du}{dz}$. So, here is what I will do now. So, do dz if you differentiate these two eyes right. So, what is going to happen? So, I'll combine these two, okay? So, I'm going to combine what I can, right? EI$_y$ $\frac{d^2u}{dz^2}$, $\frac{d^2u}{dz^2}$ is nothing but - $\frac{GJ}{M_x}$. So, because this is β β, I can eliminate U from these two expressions. Look, this is in terms of β β, okay? We totally eliminate this. So, this is E I$_y$ and then - $\frac{GJ}{M_x}$ and $\frac{d^2\beta}{dz^2}$ β is equal to M$_x$ into β. Okay.

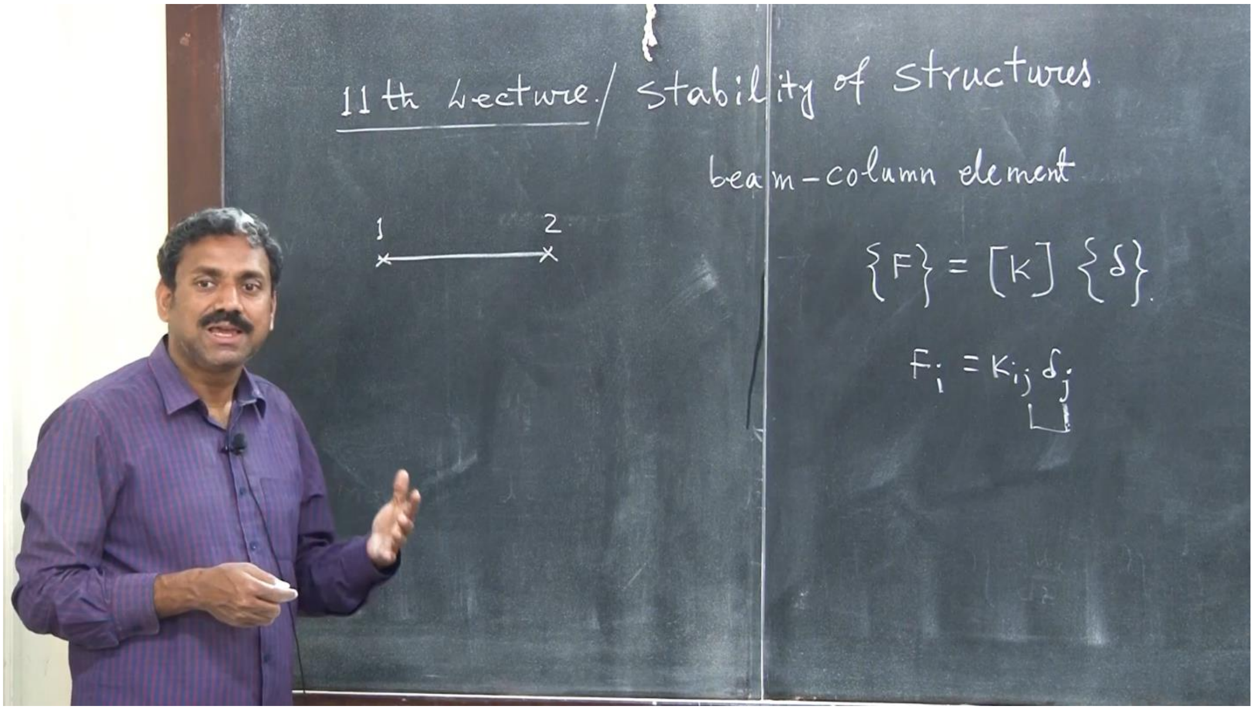


Right now, you may wonder why I'm not considering warping. Yeah, I have already mentioned that. Right? So, you see that this way, through the equilibrium approach, we can also find out that we have deformed it. We have considered the equilibrium approach. We have deformed it, and then we are writing the equilibrium equation. We are not using the energy approach. So, this is an alternate way to derive the lateral fraction buckling as well. Okay. Now this equation solves a very, $\frac{d^2\beta}{dz^2} + \frac{M_x^2}{EI_yGJ}\,\beta$ = 0. And this one we denote, as you know, λ^2. So, $\lambda^2 = \frac{M_x^2}{(EI_yGJ)}$. So, $\frac{d^2\beta}{dz^2} + \lambda^2\ \beta$ = 0. And now, we can put the boundary conditions that at x=0 and at x=l, both ends have the boundary condition in a way. So, β = 0 and $\frac{d^2\beta}{dz^2}$ is to be zero. So, β = 0, meaning it is restrained against twisting and unrestrained against warping. We are allowing it to work freely, which leads to this equation, right? This boundary condition, Now, I'm removing it. Okay. All of you have noted down this one. So, if you solve it, then if you solve this second-order equation, it will be nothing but $-\mathrm{A}\cos(\lambda \mathrm{Z}) + \mathrm{B}\sin(\lambda z)$, right? So, $\frac{d^2\beta}{dz^2}$ is nothing but -Aλ^2sin (λ z) +, this will be sin, will be cos, and then here it will be - Bλ^2sin (λ z), right? So, you see if you substitute β = 0, you know β = z = 0. Sorry, this is not x; it's z, right? So, Z = 0, A times 1 + B. So, A is basically zero. And then, if you substitute 0 = - p λ^2 sin (λ L), this one is λ L. So, that means sin (λ L) must be equal to zero and sin (π) is okay. So, λ L = π and then $\lambda^2 L^2 = \pi^2$. So, for λ, what is λ^2? $\frac{M_x^2}{EI_yGZ} = \frac{\pi^2}{L^2}$, or M$_x$ critical moment is nothing but $\frac{\pi}{L}\sqrt{EI_y}$. The same thing, you know, only the Saint Venant torsional constant is there, and lateral flexural buckling, right? M$_x$ critical, right? So, you see how we are figuring it out, right? So, that is the way you solve it, huh? So, this is an alternative approach to doing this. In summary, what we have learned is that we started this in the second chapter. So, in the first chapter, we have covered all the idealized systems and the two-system approach. We have also demonstrated all the basic behaviors, including stability behavior. They individually, you know, all possible modes or, in combination, those are ported by the elastic system. And in the second chapter, we have considered the beam, you know, the beam-column system. We have seen how it is leading to the amplification of deflection and moment, you know, and other quantities as well. And then we have considered that you know it's since we are all aware of how to find out the buckling load for the column. That we have done in our graduate work, by solving the differential equation of equilibrium for the deformed configuration and then solving it. We have obtained the eigenvalue problem, but is the trigonometric eigenvalue problem correct? Trigonometric, why? Because it leads to a transcendent equation, right? Okay. And then we considered the elastic finite deformation of the column to demonstrate that the column hardly has any post-buckling, post-critical strength, and then we demonstrated the behavior by considering that flexure and torsion are coupled. So, we have considered the torsional flexural buckling of the column and the lateral buckling of the beam. You know those are very close to our hearts as civil engineers, at least because we have, you know, seen this formula and have used it while designing steel structures, right? And I think it is also very useful for aerospace structure design, okay? We have seen that for solid sections, because we have very, I mean, very low warping rigidity, warping is very negligible there, and we do not need to consider it. For solid sections and closed sections, in closed sections, J is what? Okay, 4^2 integration $\frac{ds}{dt}$, okay. Those are Saint-Venant constants, but the warping constant is different, okay. So, for the I section, the warping constant is found out as $\frac{I_F h^2}{2}$, right? I_F h is what you have seen, right? So, it basically depends on the sectional area of the flange as well as the distance between the two, okay? Anyway, we covered that. Okay, so now we are going to start a third chapter, and it is a very simple chapter where we are going to demonstrate how we can use finite element analysis to find the critical load.

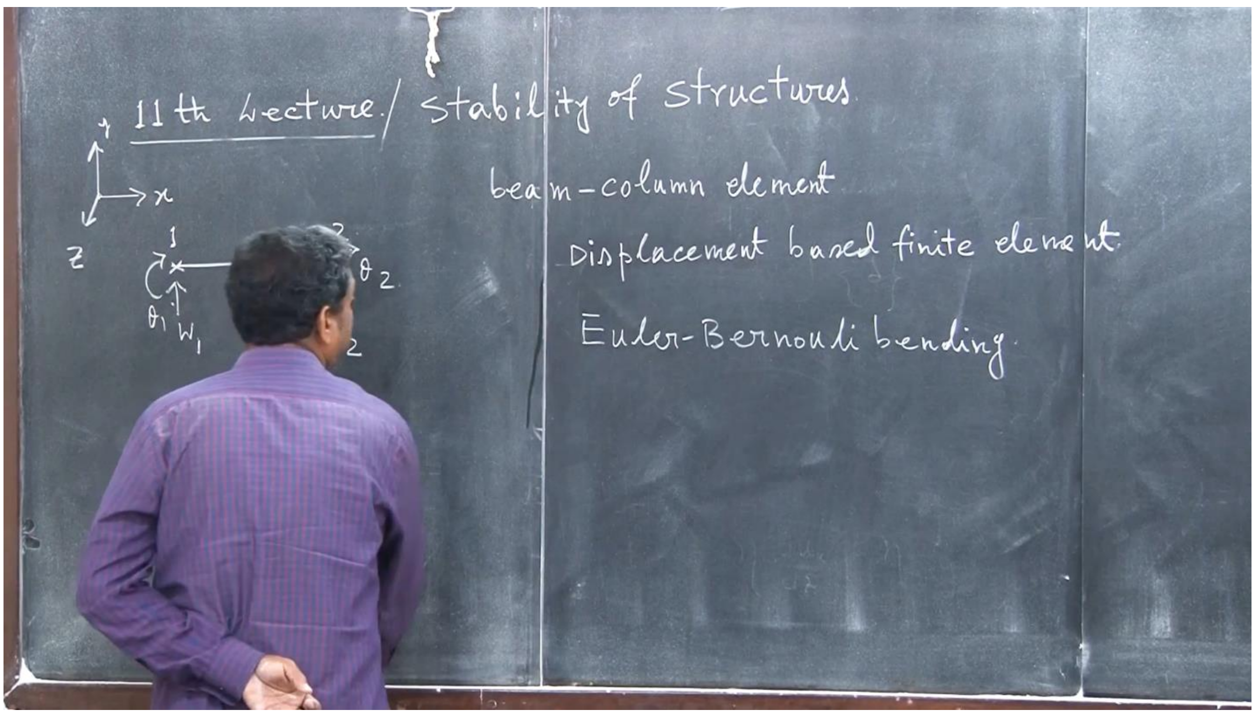


So, we have all learned about numerical methods, specifically the matrix method or the finite element method, which is a more advanced method for structural analysis. So, we will show a simple example for the beam-column problem, demonstrating how to derive and convert the eigenvalue problems in terms of matrix formulation and finite element formulation. So, essentially, whatever the eigenvalue problem you're solving can be, you know, either for the closed-form solution, which is leading to a trigonometric eigenvalue problem that Can be converted into an algebraic eigenvalue problem through discretization because the differential equation is essentially being discretized when we are doing infinite element right. Or matrix method, right? That's what I'm going to demonstrate. Now, although I'm demonstrating only for the column. But please note that the approach is general, and since all of you have kind of been exposed to a finite element course. So this approach can, well, be extended to all kinds of buckling analysis, including the buckling of plates, buckling of shells, buckling of any other structures, and frames, you know, things, right? Okay. So, that's what I'm going to demonstrate to you, huh? So, many of you know about finite element analysis. I'm going to follow the same approach. Okay. So, there are two distinct approaches. One is, of course, based on energy formulation, right? We can directly perform the minimization of potential energy, or we can follow the Galerkin approach. Galerkin means Bubnov Galerkin, okay? So, I will try both, actually. Okay? So, just to demonstrate that, this chapter is very simple, okay? So, you see how we are going to do this. You have learned that in the matrix method or finite, for example, you know there is this column, okay? Right, okay. So, we want to find out the buckling load P for this. We know the close contribution for that; either we can discretize it into finite elements or in matrices. So, we know how to discretize using beam column elements; these are defined as node 1, 2, and 3, and these are defined as elements, right? We have this nodal connectivity, right? Now, the same thing is more even; this is more important for frame structure because sometimes you will see, although for beam-column it is easy, for plate and shell we also derive the governing equation. For the frame, it is difficult to have a single differential equation correct. So, it is most handy for the frame. For example, if it is subjected to some critical load P, you want to find what the value of P is. So, these two are column buckles. Okay, this is length L. So, for the analysis of the frame right, you will sometimes see, say, that you are having this kind of, you know. Space trusses are right. So, there is some load P being applied. We can find out what the critical value of P is so that, you know, for the frame and this problem, it is a space frame and a two-dimensional frame. The best method to analyze the critical load is, of course, that there is a possibility that some members will be under compression, right? Because we are considering buckling as the only form of instability as of now, right? So, for this problem, it is easy to use the matrix method or finite element method for the analysis of stability, which will eventually result in the eigenvalue problem being solved as an algebraic eigen value problem.

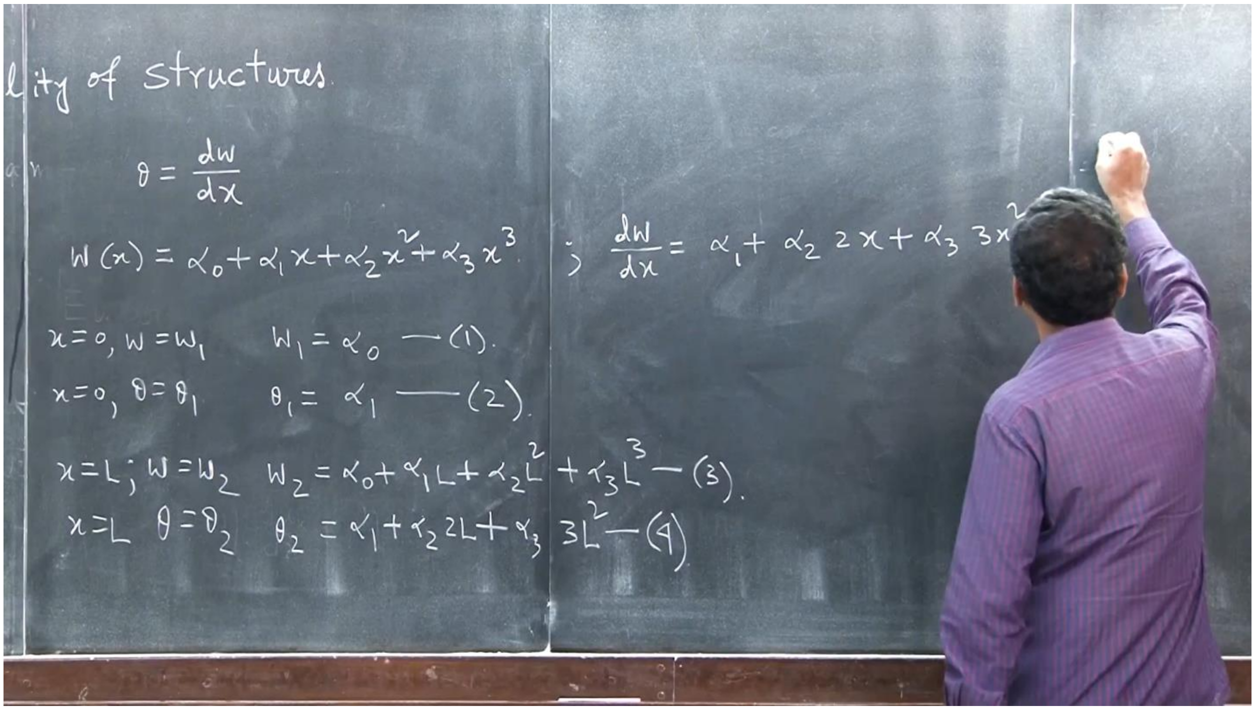
Because the differential equation will be converted into a set of algebraic equations, okay, linear simultaneous equations, right?



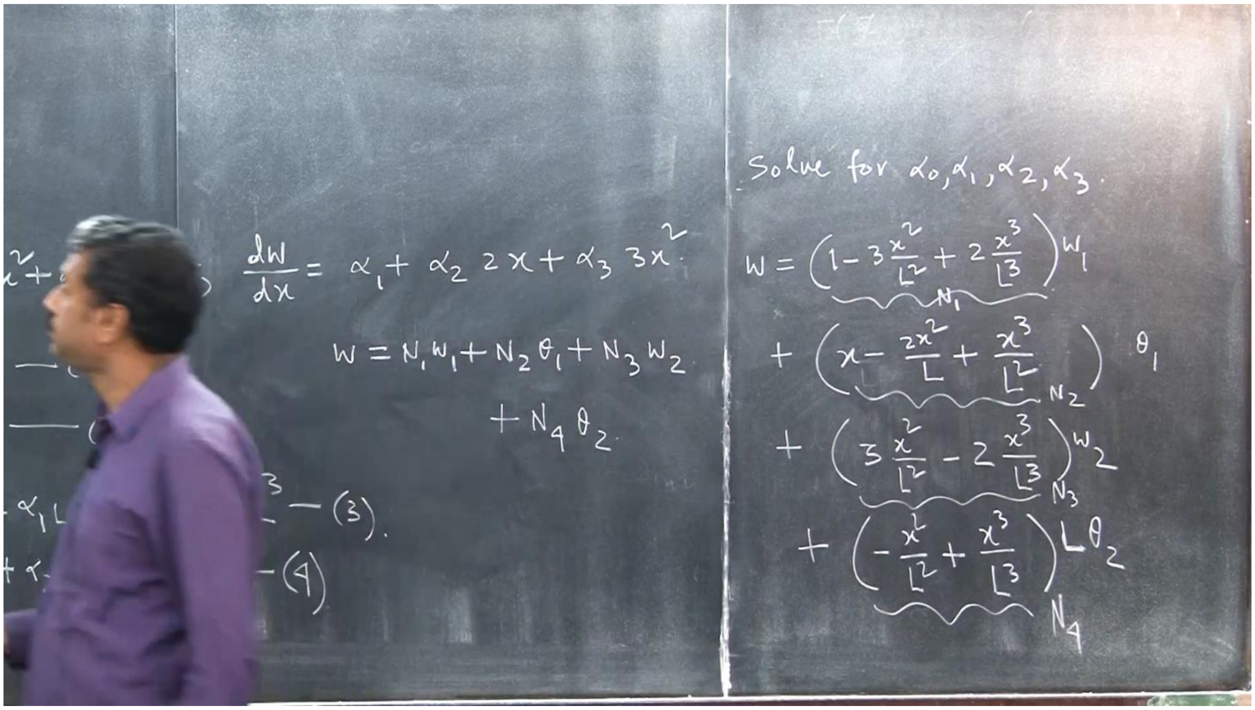
Of course, if you want to analyze post-buckling strength and post-buckling reserve, then you have to do a full nonlinear analysis, including geometric nonlinear analysis. That is discussed; that is the subject matter of a non-linear finite element course that I'm not going to discuss here. Okay. Fine. Okay. So, as we see in any matrix method or any finite element method, for example, you have a frame structure or some kind of structures. These are all things you know, and it is subjected to some force P. So, the first thing to do is to discretize the structure. We define the nodes here; we are discretizing. These are node one, node two, node three, node four, and then there is global nodal connectivity. So, element connectivity. Member one is connected between one and two. Member two is connected between two and three. Member three is connected between three and four. In finite literature, we call them finite elements. Right? So, it's a finite element. This element is a beam-column element, right? Here, you can also discretize it similarly: node one, node two, node three, node four. And then these are element one, connected between node one and node two. Element two is connected between node one and node three. Element three is connected between node one and node four. Similarly, when you consider a plate element, you will see that we can discretize it using finite element methods for buckling analysis. So, here you can use a four-noded element, or there can be higher-order elements, eight-noded elements, nine-noded elements, things like that. Similarly for Shell, as you know, you can also have shell elements; of course, this will have curved linear coordinates, and then you will see that you can have this node. So, before doing that, since all of you are already exposed, we have to find out the element-level matrix, the stiffness matrix. For each element, we need to assemble them to form the global stiffness. All of you are aware of the global element-level assembly to form the structural stiffness matrix, right? So, first we will see that we will try to derive the matrices at the element level that are required to solve for finding the critical load. We are not considering the post-buckling strength. We are considering how to find the critical load for buckling. So, for that, we are going to consider. Here, let us consider a beam-column element. I'm considering a beam column element because unless you have out-of-plane deflection, you cannot analyze for buckling. So, only a pure truss element, if you have a truss element with only axial degrees of freedom, will not allow you to analyze for its buckling; it does not allow for out-of-plane deflection. So, you have to essentially formulate it in terms of a beam element. You understand? So, I am terming it a beam column element. It has two nodes and this is a beam column element. Okay. So, when we consider the beam-column element, what we are trying to find is the stiffness matrix. Right. So, we are going to follow the procedure. There is a simple procedure to find out the stiffness matrix following the, you know, when we write down the stiffness matrix, you know it is basically f is equal to k times δ, right? This kind of matrix form will come out right. So, f_i essentially is K_{ij} times δ_j, right? So, K_{ij} is nothing but the force developed at node i due to unit deformation at j, okay? Right. So, for a simple beam we can restrain all other degree of freedom. We can give unit deformation here and you can find out from what is the force or moment developed. Then we can find out the stiffness matrix, right? The term for the stiffness matrix, right? That is very simple. That's what we did in our undergraduate studies. But here I will not do that. Why? Because I want to emphasize the formulation of the finite element once more. I want to make it general so that it can be followed for the plate, as well as for Shell and all other things. Fine. So, I will do finite element not like this adopted way to find and determine the stimulus. Okay. So, in finite element, what is the way we are going to follow the displacement finite element, right? Almost displacement-based finite element, huh? Now, are you aware that there are alternate formulations of finite element? All of you are aware of why we are considering only displacement, but there is another stress-based finite element. There are mixed formulations. Are you aware? You did finite element analysis, right? You know about stress in finite element analysis. No, you're not even aware. Why do we use displacement versus finite element over stress space? Huh, no. Because the displacement finite element formulation is very general, it can, if you learn one, be easily extended to the others. Okay. But based on stress, you require some special input from its behavior and others. Okay. But stress-based finite element was pioneered by THHP and MIT. Okay. There is also a vast literature, and there are advantages to stress space. Sometimes in nonlinear analysis, the tangent stiffness matrix can be obtained in closed form in stress space, but we are not talking about those steps. Okay. So, what if displacement, then what are the degrees of freedom? They are not Y and Z. The degrees of freedom, of course, here are vertical deflection and then rotation; you know, deflection and then rotation, right? Okay. So, W_1 θ_1 W_2 θ_2, right? Two deflections and two rotations, right?



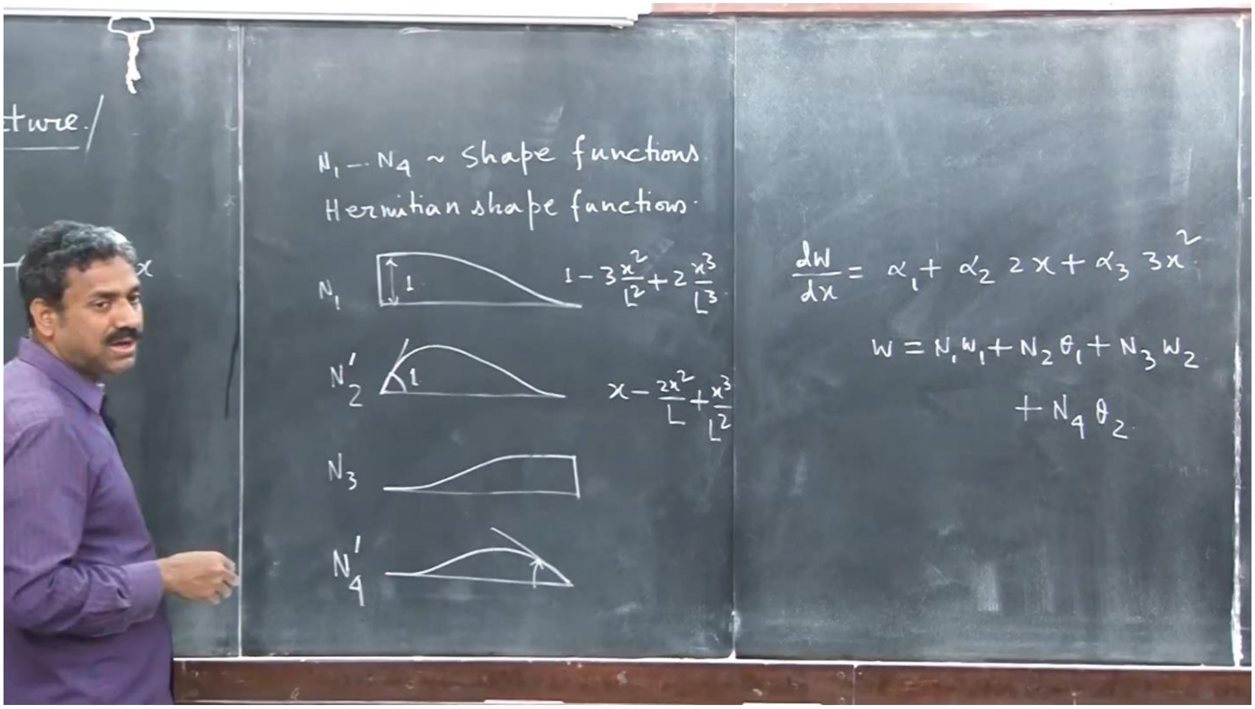
Fine. Now we are going to consider simple; we are not going to consider shear deformation. We are going to consider Euler-Bernoulli bending. But all of you are aware of the importance of shear deformation, right? We are not considering the importance of shear deformation, okay? We are going to consider only Euler-Bernoulli beam bending, okay? So, then what is going to happen? So, we can essentially write that θ can be expressed as θ is nothing but $\frac{dw}{dx}$. That is possible, right? Because plain sections remain plain before and after deformation, right? $\theta = \frac{dw}{dx}$; we don't have additional shear deformation, right? So, the first step is the interpolation of the displacement field, right? So, what is the displacement field w? I can express that in terms of the constants I know: w_1, θ_1, w_2, θ_2, and four. So, we evaluate at the maximum of four constants. Right? So, may I write α_0 + α_1 x + α_2 x^2 + α_3 x^3 right. four then, I am going to put at x = 0, w = 0 at x = 0 sorry w = w_1, θ = θ_1 and x = l, w = w_2, x = L, θ = θ_2. This we are going to do. So here, it is $\alpha_1 + \alpha_2 2x + \alpha_3$ $3x^2$ times.



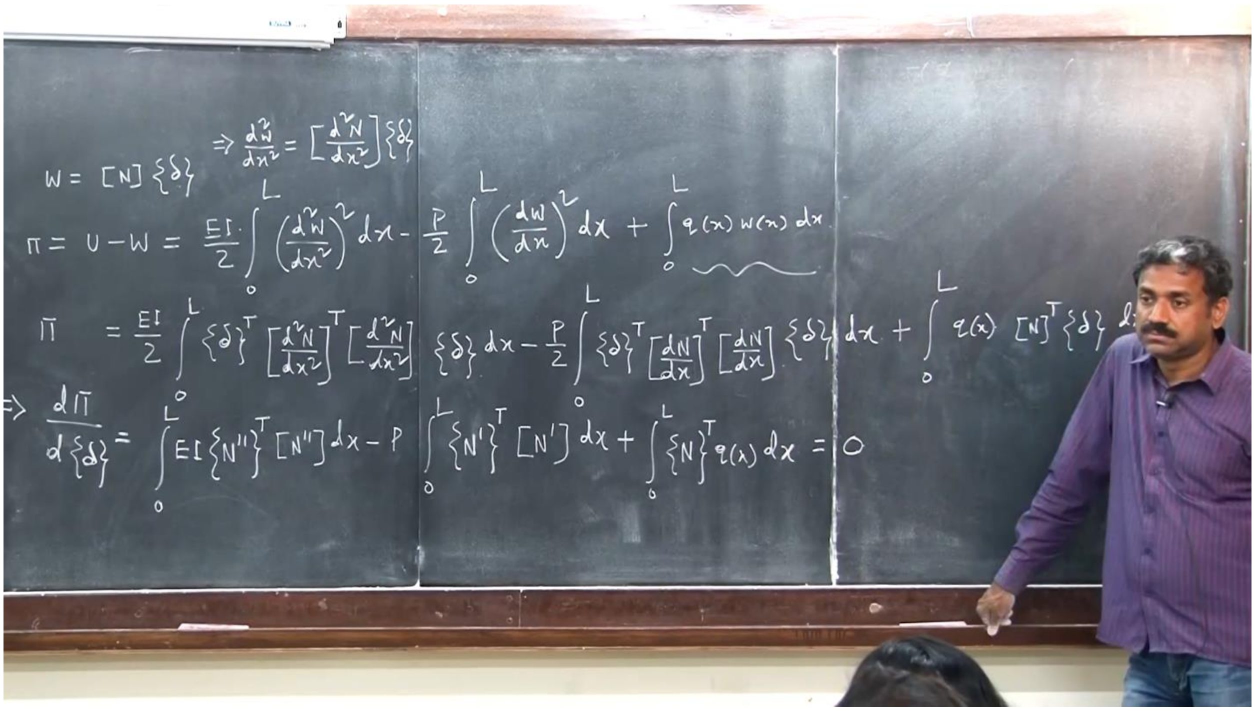
So, x_0 w is equal to w_1 when x = 0. So, α_0 is the first equation. And θ = θ_1; θ_1 at x=0, α_1, then w_2, θ_2. So, now we can solve for four equations and four unknowns. We can solve for W_1, θ, α_0, or we can solve for α_0, α_1, α_2, and α_3. These are constants, right? Okay. So, this can be. So, we can write w is equal to, thus we can write w = α_1 is nothing but 1 - $\frac{3x^2}{l^2}$. Let me write it. Right. So, you see that we can write N_1 W_1 + N_2 θ_1 + N_3 W_2 + N_4 θ_2.Right? N_1, N_2, N_3, N_4. This is N_1. Right? This is N_1. This is N_2. This is N_3. And this is N_4. Right. Right. And what are these called? N_1, N_2, N_3, N_4. These are called safe functions. Right. Why are they called safe functions? Because they are basically safe functions, because they Help in interpolating the, you know, from discrete, you know, they help in interpolating this nodal displacement to the displacement field inside the element. Right? So that's. Why are they a safe function, and are there some properties? Right? These step functions are called Hermitian shape functions because they are Hermite polynomials. There are some characteristics, okay?



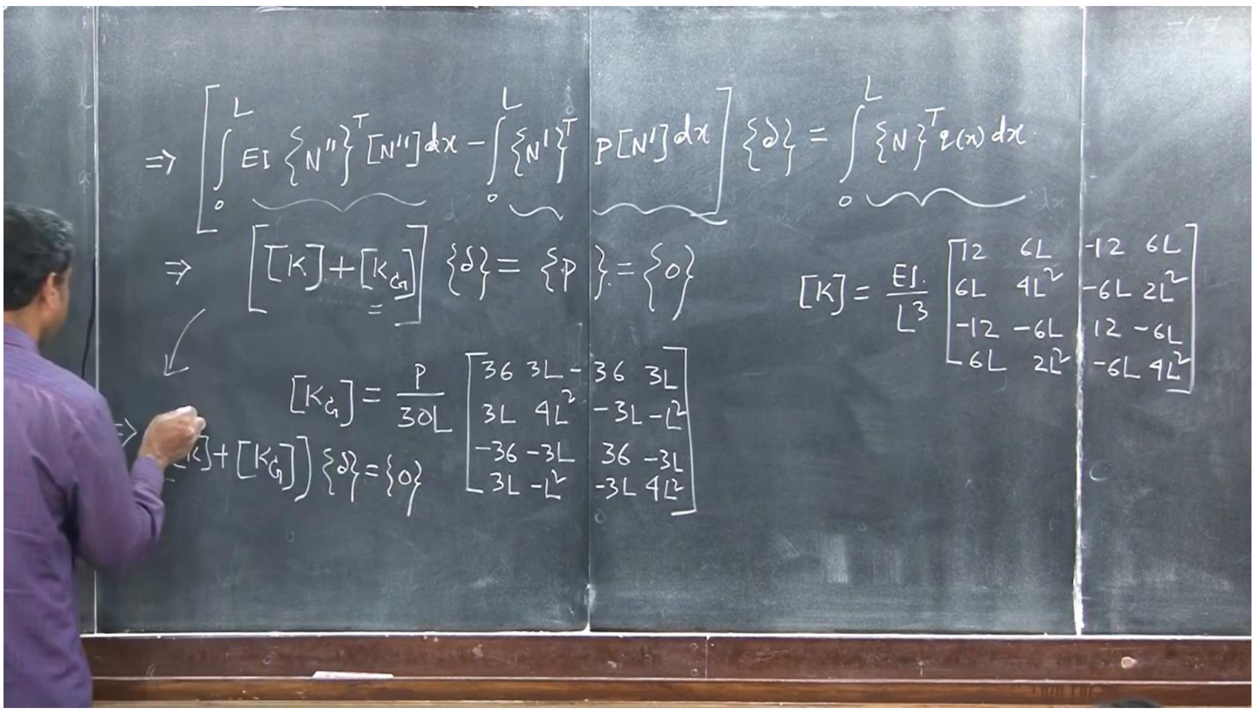
They also form a legitimate basis for a certain different class of differential equations, okay. So, anyway, we can draw it in one; N_1 will be something like this. So, it is not a slope but its value is 1; you know, 1 - this one, 1 - $\frac{3x^2}{l^2} + \frac{2x^3}{l^3}$ and N_2. For N_2, it is different; you cannot plot N_2 directly, but you have to plot N_2 dot derivative of N_2, okay? So, then you know if you plot the derivative of N_2, x - $\frac{2x^2}{l} + \frac{x^3}{L}$, you know the derivative of that. Why derivative? Because W is related to θ_1. So, don't worry. Okay. So, then if you take the derivative, you will see that it has. So, here it will have one value, one slope is one, and then similarly, the other two safe functions you can plot will also look like this, you know. So, at node one, you know node one; they are one, and elsewhere these are all zero. Okay, here it is. You see that? N_1, N_2, N_3, and then if you plot N_4 as well. So, this property is called safe functions, has some property you know it is, its value for the displacement, its value takes one at that node; elsewhere it is zero. So, this is called the Kronecker delta property. Okay. So, N_i N_j is δ_{ij}; δ_{ij} is the Kronecker delta. Okay, this is called the Kronecker delta property. Okay. And then, if you sum up the shape function at any point, it will sum up to a value of one because it is an interpolation function. That is the reason. Okay. And another thing is that, Hermitian shape functions, any shape functions, all of you know that, perhaps, they must be complete polynomials.



That means you first have to, you know, have this constant term. Okay. And then you will have this first term. Okay. And then the second order term and things like that; you cannot do arbitrarily. So, you know, so in one dimension it should be 1, x, x^2 this all these terms should be there in the okay. and in two dimension it must have 1, x, y then x^2, xy, y square some. So, if you want to have three terms, it is like that. If you want to have four terms, then it should be like that. So, this is called Pascal's triangle. Okay. So, we must, so it must be complete polomial. must okay, have some constant to allow for the rigid body mode and thing I will discuss more, rigid body mode is very important in terms of the geometric stiffness matrix you will see okay. Now, we have interpolated the displacement field using shape functions, right? Now everything is very simple. So, may I remove these things? All of you know these things; I'm just refreshing and brushing up your little knowledge on finite element analysis, okay? So, let us remove it, okay? So then, what we are writing is that W is nothing but, you know, we can write N into δ; N is N_1, N_2, N_3, and δ is W_1 θ_1, W_2 θ_2, right? Fine. Now, once you have it, you see that in nodal, discrete values are converted into a continuous displacement field. So, that's basically the role of the safe function. So, for any element, that's what it does. So now I will see that I go to the strain energy function. The strain energy function is what? You know potential minus work done, right? Sorry. So, potential energy is strain energy minus work done. So, for this one, it is $(\frac{EI}{2} \int_0^L (\frac{d^2w}{dx^2})^2$dx, and what is worked on $-\frac{P}{2}\int_0^L (\frac{dw}{dx})^2$ dx, right? I can have other load vectors, etc., that I am not considering at the moment.



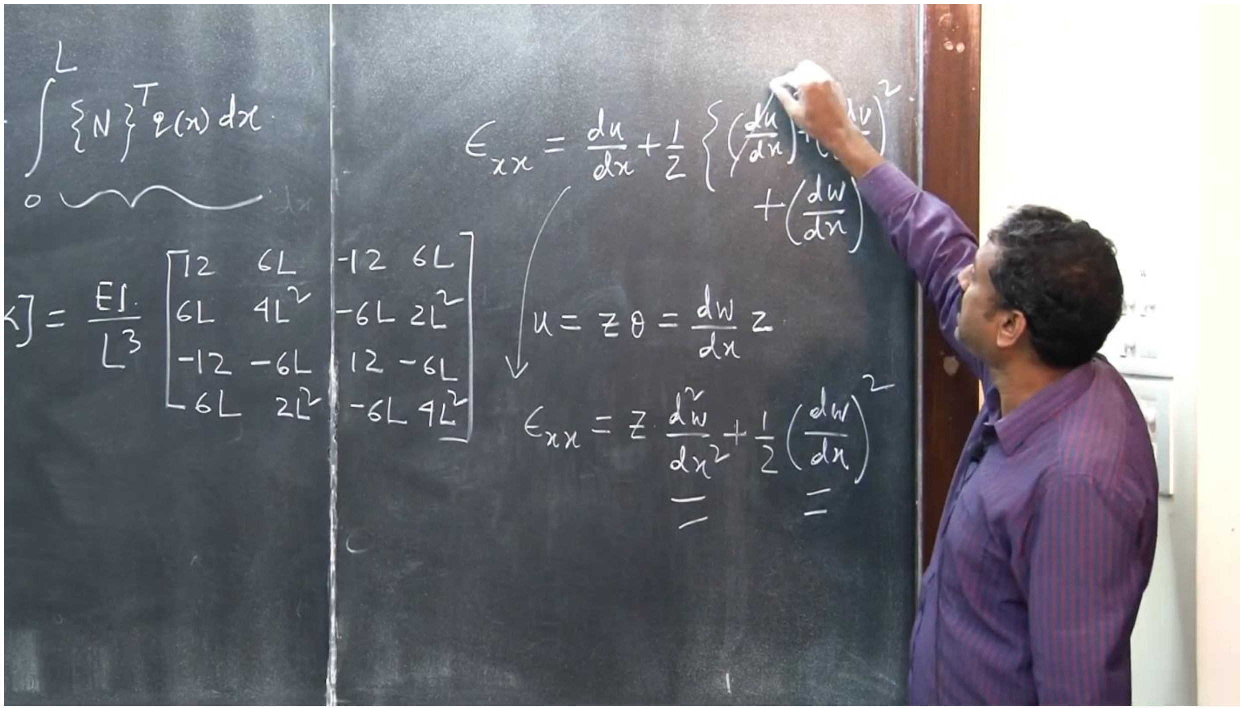
If you want to consider the load, you can definitely consider, you know, integration from 0 to l of q(x) wx dx, no problem for the load vector. Not considering that you know how that results in a consistent load vector and things like that, now I'm going to consider only two. Okay. So now, so then I'm going to write it, $\frac{EI}{2}$ intrigation from 0 to L. What I will do? If W is like this, then from here you see that $\frac{D^2W}{Dx^2}$. So, $\frac{D^2W}{Dx^2}$ is $\frac{d^2N}{dx^2}$ like this, right? So, may I write it this way? Oh, right. Yeah. Is that fine? Because what I have to explain is why I'm taking $\frac{d^2w}{dx^2}$. I'm basically saying that you have to ultimately result in a square. So, you have to have the confirmed meeting in the multiplication term right. So, that's what $\{\delta\}^T \frac{d^2w}{x}$ into $\frac{dw}{x}$; this one I'm just taking the transpose so that it becomes a row vector and this is a column vector, right? So, ultimately these two will be a column vector and these two will be row vectors, right? So, that's what I am doing, okay? Fine. Okay, now integration dx. Now what? This is what we are getting, so this one you know, similarly, we are going to write. Load vector you can put it, but you know it's not required for now. I mean, you know that. How do you know if you integrate from 0 to L? If you want to put this qx, you know. I'm writing q(x), and then this one is, you know, in N^T δ, this is okay. now we always try to minimize the potential energy right. So, what I am going to do is minimize $\delta\pi$. What is my nodal displacement δ right δ model? So, you know I'm differentiating; there are two, so 2, 2 will cancel out: δ^2, δ^T, δ. So, it will be 2 δ; two twos will be cancelled out in the integration from 0 to l of EI, and here I'm going to write this one because this is a row vector. So, N double dot and special derivative, okay. and this I'm going to write N double dot dx minuZ, here also 2 will cancel out, P integration 0 to L See this equal to zero, energy minimization? Okay. You have done finite element analysis. Huh, all of you noted it, huh? Okay. Now Which term? Yeah, the δ term will be there. All δ terms, one δ term will be there. The δ term will be there. Here there will be no δ.



Yes, ultimately. So, what we obtain Now what is this term? All of you know, what is this? This is the stiffness matrix for the conventional stiffness matrix for the beam, right? I can write this as K; what is this term? Load vector, right? I can write it to be, you know, Q or whatever load vector P, maybe you, and then δ is there. Okay. What about this new term, which is coming due to work done by the von Karman nonlinear and axial force, the function of the axial force? And then, this matrix is called the geometric stiffness matrix. Why is it called geometric? Because it depends only on L and P. The length of the member, as well as the axial force P, is a function of P; you see that. It is also called the stress stiffness matrix. Initial stress stiffness depends on the value of the axial force, whatever is there. So, all of you know what the term k is, I'm writing, so k is basically, you know, the double derivative and then do the integration. So, you know you'll get the term. I am writing to you about these terms. So, $\frac{EI}{L^3}$, and you know here, it is $\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$

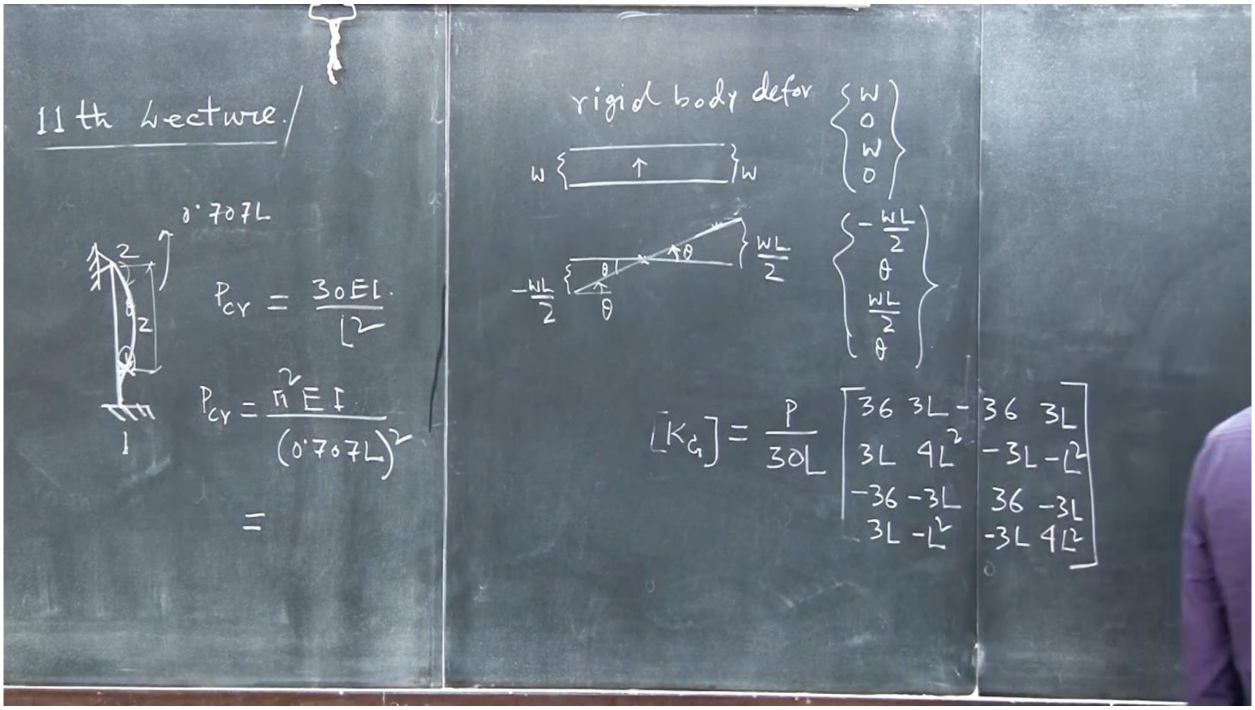
So, it's a function of flexural rigidity, and this is the stiffness matrix, which depends on the flexural stiffness length, all this. and this new A k_g this geometric stiffness, I'm just writing geometric stiffness, this will be $\frac{P}{30L}\begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix}$

And P, you know P sees, either you can have a consistent load vector or you can have a lambda load vector, okay. It's not for static analysis, you know that, but for when there is no external load, you know this will be zero, and that basically results in an eigen value problem, right? So, the eigenvalue problem. You see that this is basically the eigenvalue problem given by A + K_g H δ = 0. This is the eigenvalue problem: the algebraic value problem. And if you find out the eigenvalues, those eigenvalues will give you the value of P and the P critical, right? The eigenvalue. So there will be a number of eigenvalues, and the lowest eigenvalue is basically the lowest critical load since there is a number of eigenvalues. Higher eigenvalues will give you what? Higher modes. Okay, the way you have solved the eigenvalue problem for the pre-vibration problem in structural dynamics, right? That involves the stiffness matrix and the mass matrix, but that was a generalized eigenvalue problem. So, K ϕ is equal to, you know, λ ω^2 a m, but here it is not generalized; it's an eigenvalue problem, right? Instead of the mass matrix, the role of the mass matrix is basically taken by the geometric stiffness matrix. And this geometric stiffness matrix is a function of p, and from where you know, so that's what it is related to the eigenvalue; you know eigenvalues. So, λ will give you the critical load; by finding out the eigenvalue problem, you know. Finding out the critical load for a beam-column frame, whether it's a plate, a shell, or anything else, is basically leading to an eigenvalue problem. When it is leading to an eigenvalue problem. Please note that this is a linearized analysis, which means we are neglecting the higher-order terms and nonlinear terms. When it is an eigenvalue problem, it is still the linearized system, as you can see. Nevertheless, please note that in von Karman's equations, nonlinear terms are present when we are finding the geometric stiffness matrix. Its origin is heading into nonlinearity because we are considering a part of the geometric nonlinear term. So, if you see ϵ_{xx}, ϵ_{xx} is basically $\frac{du}{dx} + \frac{1}{2}(\frac{du}{dx})^2 + (\frac{dv}{dx})^2 + (\frac{dw}{dx})^2$, right?

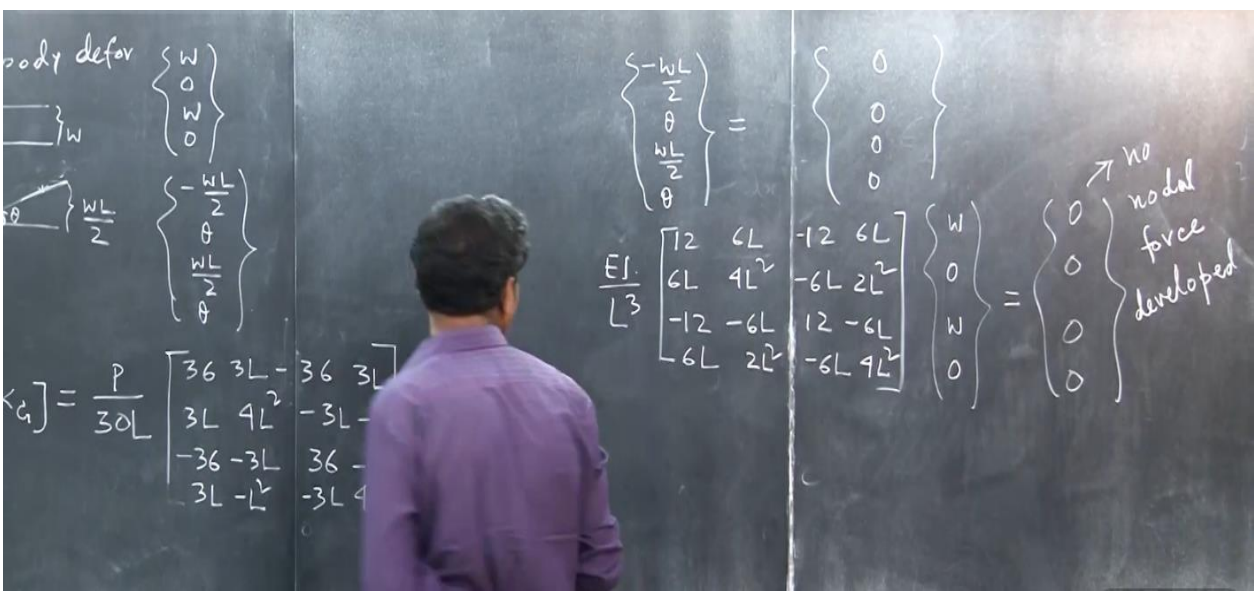


Now for the, you know this, what is happening you know $\frac{d}{dx}$ of, U is nothing but, u is what? $\frac{dw}{dx}$ into, I mean θ into z, right? θ into z and θ is nothing but, is not the u, for the bending u is what? Z into θ, and Z is nothing but you know, θ is $\frac{dw}{dx}$ into Z, right? You see another von Karman. So, here it is coming, ϵ_{xx} is coming as you know; if you just take it, you know Z $\frac{d^2w}{dx^2}$, and from here, if you take half, the out-of-plane deflection here is $(\frac{dw}{dx})^2$. So, this term is contributing to bending; whether this term is linear or nonlinear, it is happening $(\frac{dW}{dx})^2$. Other terms are dropped because this will be negligible, right? This and this. So, the origin of the geometric stiffness matrix lies in nonlinearity. But it is not capturing the fully geometric nonlinear; it's a part of geometric nonlinear that we are capturing, at least as far as this beam formulation is concerned. Of course, we can have a generalized formulation for the geometric stiffness. We consider all the terms; if you consider 3D, you know, maybe you consider a 3D body. And maybe it is subjective from both ends; it is subjected to some compression. It is coming like the way folded mountains are formed. Okay.

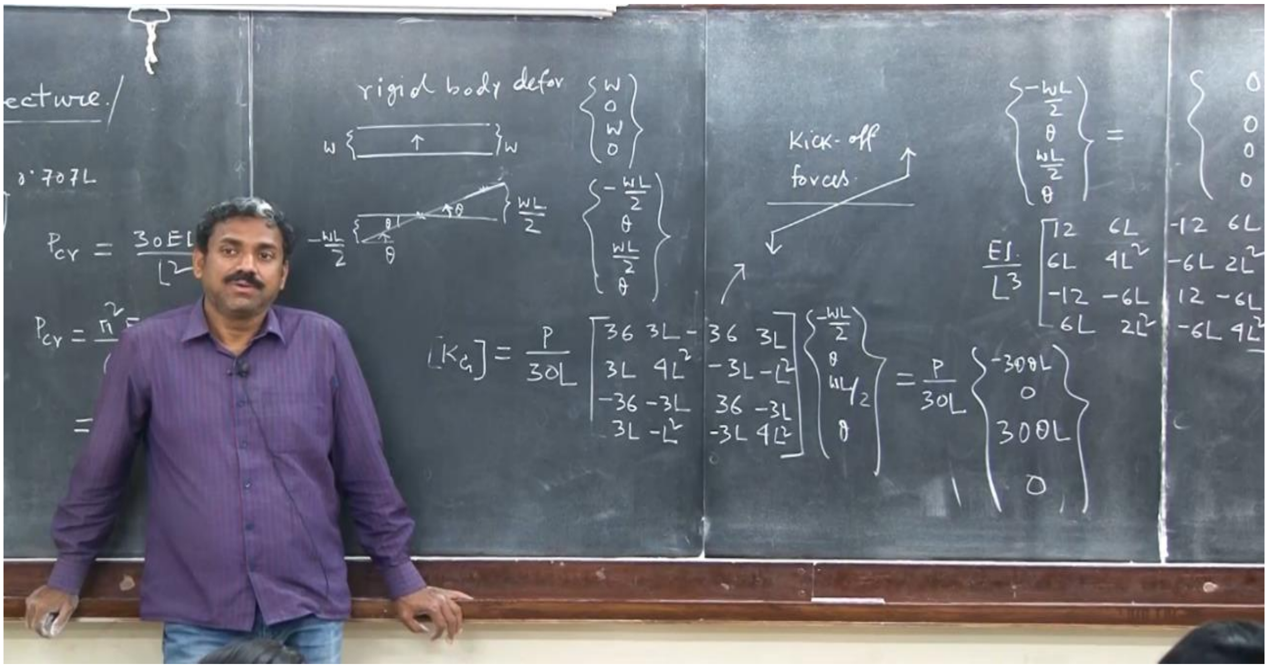
Similarly, maybe you're considering a case where you have a 3D dam, and at both ends, there may be movement of the mountain and things, and it is being compressed. So, you see the folded mountain will form, right? I mean geologic material is folded, but here there will be, maybe, and then you may consider that, while you know, 3D this will buckle. Okay. So, all these components will come in 3D formulation. Okay. So, we must now, you will see that the geometric stiffness matrix, since its origin is hidden in geometric non-linearity. But please note that when you do a full geometric nonlinear analysis, if you take a nonlinear fundamental course, the geometric stiffness matrix will also be a part of the tangent stiffness matrix. But not all. I mean, please note that the tangent stiffness matrix, one part of the tangent stiffness matrix, will be contributed by the geometric stiffness matrix, which will contribute to the nonlinear aspect of the tangent stiffness matrix. But that is known; there will be additional contributions from the nonlinear strength and the Green Lagrange strength term. All of you did green L when I'm writing this; it is Green Lagrange strength, right? Now, if I want to solve a simple problem, then I will go into some stuff. Let us see whether we can. We discretize it; this is one and two, okay? This is only a single element discrete; we can discretize it into multiple, right? By inserting another node, but we are not doing it. So, what is this? This is fixed, and this is a hinge, right? So, what are the, I mean, what are the degrees of freedom? Here, there is only one degree of freedom, which is only θ, right? So, if I consider this as node one, node two, element one, w_1 θ_1 will not be there, only w_2 θ_2; only the last term will be there, right? Last term. So, let us do that. So, whereas $\frac{EI}{L^3}$ and then $4L^2$, right? $4L^2$ and then, here - $\frac{P}{30L}$ $4L^2$, you know this into θ = 0, θ_2 = 0, right? So, this must be zero, you see. So, essentially only a single term, you know, assembly, we have done right. Single term assembly right, because there is only one θ. So, θ_2, I am just node 2, that's why θ_2 θ. So, this, when it is zero, then it is $\frac{4EI}{l}$ - what is this? $\frac{2PL}{15}$ = 0, right? Huh? So then, is it fine? So, what are you getting here? You do not need to solve for the eigenvalues because it's only a single term. Right? Only single term approximation. So, here you will see that P will be, what? $\frac{4EI}{L}$ multiplied by $\frac{15}{12}$ now and then it is, so this is I know $\frac{15}{12}$ and then it will be, no problem $\frac{30EI}{L^2}$. Now you tell me, if it is $\frac{30EI}{L^2}$, what is this? P critical for this. So, what are we using this matrix method or finite element method for? I'm getting $\frac{30I}{L^2}$, right? Now, what do you get for whatever you have learned from your undergraduate course? What you have learned is that P$_{cr}$ was $\frac{\pi^2 EI}{L^2}$. How much is L for this one? 0.707, right?



Because you know it is something like this, and then it will come like this. So, there is a point of counter flexure. I think this distance was around 0.707. I think something like this is right because here it will be hugging curvature, and then there's a change in curvature. So, this can be an ideal hinge over there. So, 0.707 L^2, do you see how much it is? Could you please tell me how much it is? I think it will come in 28 components or something. Okay. So, there is a little difference. Okay. And the difference is that, you know, the finite element is an approximate technique, right? The second thing is that the finite element makes the structure stiffer, right? Because the displacement finite element is a formulation, you are assuming the displacement field correctly. So, that makes you constricting your structure to deform in some way, right? So, that makes the structure stiffer; displacement in finite element analysis always makes the structure stiffer. So, it is making the structure stiffer; that's what you know. You will see maybe a little less, but the error can be reduced by having higher and higher distinctions. You can have two elements, three elements. and that thereby you can capture higher and higher mode. Right? So, you can do more problems. Okay. Now, before completing one more important thing, I'm going to show it to you. Okay. See, do all of you understand what rigid body mode is, right? In finite element analysis, did you check for the rigid body mode, even in the element stiffness terms? So, an element stiffness matrix, to check whether it is legitimate or not, you must check for the rigid body mode. So, that means this is a beam element. What is it? What are the two possible modes? Rigid body mode is when it is deforming by some amount here. This is a rigid body. So, maybe this is W, this is W. Another rigid body mode is if it rotates; you are rotating by some amount θ. Rotating by some amount θ. That means these two fellows you know will be $\frac{wL}{2}$. Here it is - $\frac{wL}{2}$, and here you see that the positive θ, and here it is negative θ, - θ, right? Okay. No, not like that, like that. If this, you know that will be the same θ, same θ, okay? Because you have to consider this. Sorry, sorry for that, okay? How can it be θ θ right? So, here you please see what the displacement is? W_0, W_0 right? And what are the displacement components here? Uh, $\begin{Bmatrix} -\frac{wl}{2} \\ \theta \\ \frac{wl}{2} \\ \theta \end{Bmatrix}$ Right. So, both of these are rigid bodies, right? Rigid body deformation, right? Rigid body deformation pattern. So now you tell me, you have done this thing already in your in your finite element. I'm going to put this; same if you put, you know, $\begin{Bmatrix} w \\ 0 \\ w \\ 0 \end{Bmatrix}$, then what is happening? You see this into w, this into zero, this into w, this into zero, this will be zero, right?



And then, if you are putting this into w at 0, 0, 0,0, there is no resistance force now. No nodal force is developed. Right? No nodal force developed because, of course, it should be that in rigid body deformation, there shouldn't be any force developed. Right? The force must be developed due to elastic deformation. Right? Now you come here. Okay. This is for this one. Okay. What about this one? If we put this one $\begin{Bmatrix} -\frac{wl}{2} \\ \theta \\ \frac{wl}{2} \\ \theta \end{Bmatrix}$ what is happening? 12 - WL, you know, please see that you know this, and 6L θ - 6 θ; this will all be zero. Multiply this by this; it will all be zero. So, that means both this rigid body moves; it is not developing any nodal resistance for resisting force. But now you have come here. So, when you are coming here, when you come here, then what you see that, $\begin{Bmatrix} w \\ 0 \\ w \\ 0 \end{Bmatrix}$. So, then what is happening? This is into w; this is into zero. Yeah. Well, I mean the first one is being zero. What about this one? T into w, this into zero minus 3. Well, 0 is coming. Not a problem. Right? Now, if this geometric stiffness matrix acts upon this one, then what is happening? $\begin{Bmatrix} -\frac{wl}{2} \\ \theta \\ \frac{wl}{2} \\ \theta \end{Bmatrix}$. What about this - $\frac{3WL^2}{2}$, this 3L, and this, and then 4L^2? So, you will see this; I will just directly write, okay? 30 and 30. For the rotation one, you will see $\frac{P}{30L}$, and it will be $\begin{Bmatrix} -30\,\theta\,\mathrm{L} \\ 0 \\ 30\,\theta\,\mathrm{L} \\ 0 \end{Bmatrix}$. What does it mean? It is not; it is not resulting in zero forces. There are additional forces that are being developed. So, what does it mean? What are the forces developed?



This one is minus thirty; you know whatever forces you developed, this one, and then here also these forces are developed, right, these forces. So, that means rigid body modes are not shown by the geometric stiffness matrix. These forces are, you know, kick-off forces. That explains why it is happening, why the geometric stiffness matrix is not allowing it, and why the geometric stiffness is not allowing the rigid body mode. The reason is that, you know, we are only considering the $(\frac{\partial w}{\partial x})^2$; we have not considered the whole geometric nonlinear term. So, there is an inconsistency in geometric compatibility. See, what is happening is that this fellow is not able to determine that when we are considering $(\frac{dw}{dx})^2$, it means this is because of the von Karman nonlinear term. Whenever that is $\frac{\partial w}{\partial x}$, it is basically incorporating; it is taking. It is unable to distinguish whether this is due to the rigid body mode or due to the elastic mode, you understand? Okay. It is unable to distinguish. Huh? However, had the complete geometric stiffness term been included, it would not have shown this kick-off force. that means the rigid body modes it would have demonstrated you understand. It is the truncation of the one term, the Von Karman nonlinear term, out of the whole geometric nonlinear term that is responsible for that. So, the takeaway is that unlike the stiffness matrix you know, the geometric stiffness matrix prevents the rigid body modes. It doesn't show the rigid body; at least one rigid body mode should have two rigid body modes: one is shown, the pure translation part, but it is not showing the pure rotation. Why? Because it is unable to distinguish, it cannot differentiate between the von Karman nonlinear strain and the strain due to rigid rotation. Okay. Thank you very much.