Optimization Methods for Civil Engineering
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Lecture - 14
Constrained Optimization

Hello student. In the last class, we have discussed constrained Optimization problem with equality constraint, we have derived the necessary and sufficient condition for optimality for constraint optimization problem with equality constraint. So, today, we will discuss the problem with inequality constraint.

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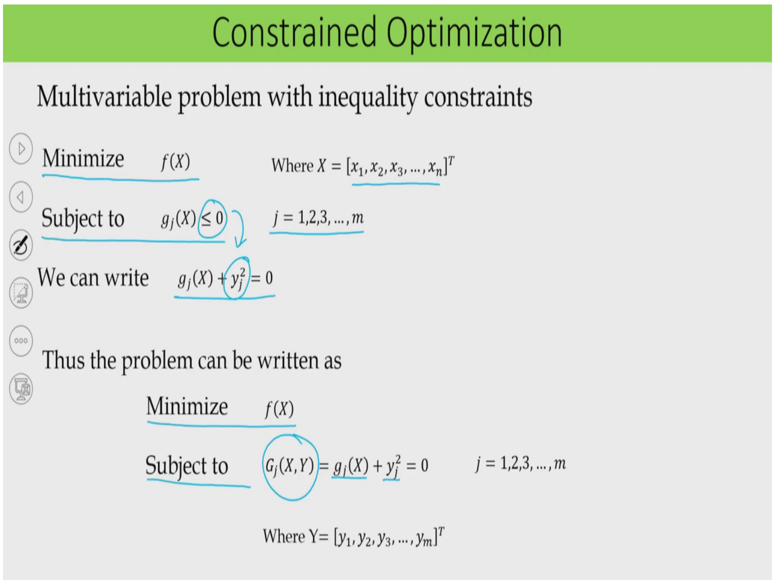

The problem with an inequality constraint can be defined as; so, we are considering a minimization problem here. So, this is a minimization problem. So, minimize f X subject to g j X less than 0 ok. So, we are not considering the equality constraint here. So, we are only considering the inequality constraint. So, this is a less than equality type constraint. So, we have number of variable here is n and number of constraints are m ok.

Now, we can write this constraint that g j X less than 0, we can write as g j X plus y j square equal to 0. So, what we are doing here? So, we are putting this y square term. So, this is another variable y just to make the constraint a equality constraint.

So, what we are doing here? We are converting the inequality constraint to a equality constraint so that I can apply the sufficient condition of optimality that is that we have discussed in the last class, so in this particular problem. So, the problem concept is I think the is very simple.

So, what we are doing here? We are just converting, we are just converting the inequality constraint. So, we are just converting the inequality constraint to a equality constraint by putting the y ok. So, we are getting that one and now, I can write the problem like this.

So, the problem if we use this y j variable here. So, in that case, this is a minimization problem. Minimization of f X subject to now I am putting this is as capital G j X, Y; capital G is a function of X and Y and which is equal to small g X plus y j square equal to 0. So, we are converting this problem to a equality constraint type problem. So, now, I can apply the necessary condition optimality in this particular problem. So, whatever we have discussed in the last class, so I can put that necessary condition on this particular problem.

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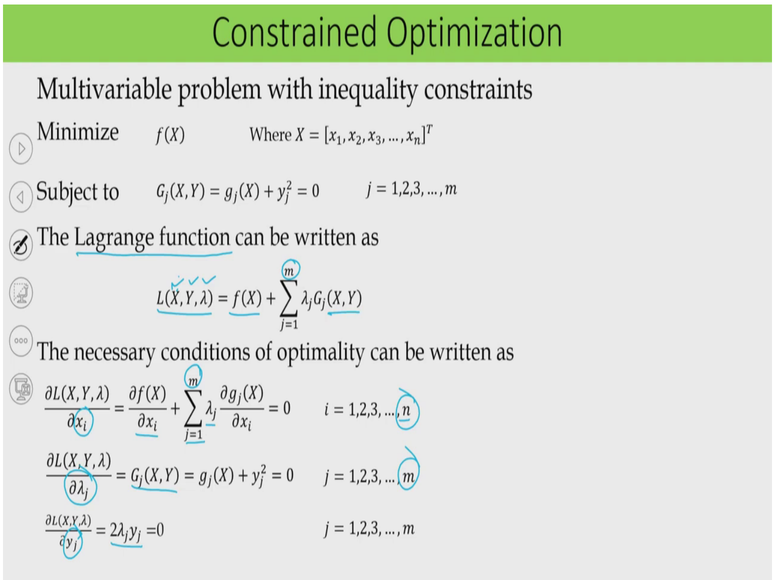

So, let us see now the problem is or problem is this that minimize f X subject to G j X, Y. So, this is a function of X Y as I said and this is small g X plus y j squared and basically, so j is 1 to m and we have n variable here. Now, if I write the Lagrange function here. So, Lagrange function, we have discussed in the last class. So, can be written something like that.

So, this is L, the Lagrange function. Now, this is a function of X; this is a function of X and Y and lambda ok. So, which is equal to small f X ok plus that we have actually total m constraint. So, summation of j equal to 1 to m and lambda j capital G j and this is a function of X and Y ok.

So, I am writing the Lagrange function and now, I can apply the necessary condition for optimality in this Lagrange function. So, what is the necessary condition? The necessary condition is the partial derivative with respect to the variable should be equal to 0 ok. So, let us do that.

So, if I apply necessary condition of optimality, so in that case, what I will get? So, I will get this equation. So, first equation is the derivative with respect to x i ok. So, we have total n variable. So, therefore, you are getting total n equation here. So, number of equation you will be getting total n equation ok ah. So, what is this?.

So, del L by del x i which is equal to del f by del x i and summation of i equal to 1 to j equal to 1 to m lambda j del g by del x i which is equal to 0. So, we are getting total n number of equation here. Now, next is the derivative with respect to lambda j ok. So, if you take derivative with respect to lambda j, so in that case, I am getting G j X, Y and which is equal to small g j X plus y j square equal to 0. So, how many equations we are getting? So, we are getting total m equation. So, number of equation is m here because that is the number of inequality constraints.

And the third one the derivative with respect to y j. So, y j is also variable here. So, with respect to y j and if you take derivative, then we are getting twice lambda j y j equal to 0. So, these are all necessary conditions. So, this equations we are getting from the that necessary condition of optimality ok. So, that we have already discussed in the last class and I think you are quite comfortable with this equation now.

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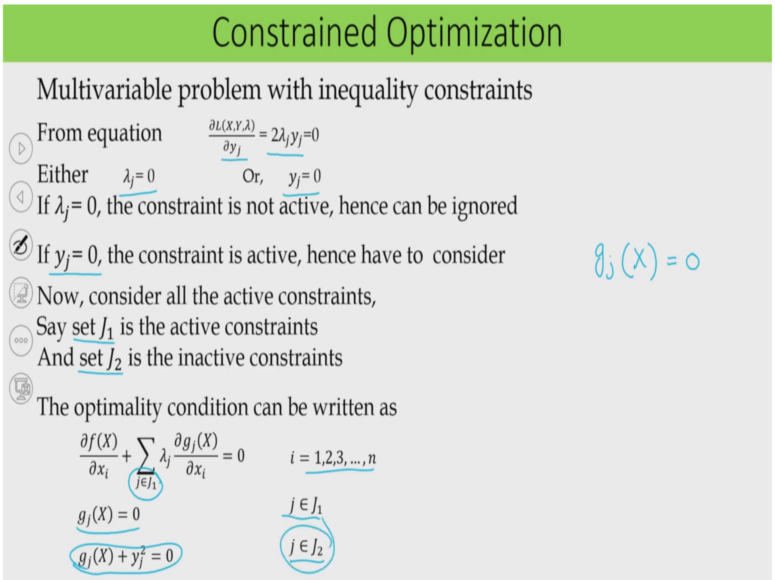

Now, let us take the last equation. What is this equation? That derivative with respect to y j and which is equal to twice lambda j y j equal to 0. So, from this equation, what we can get actually? So, either that lambda j equal to 0 ok or y j equal to 0. What does it mean? So, either lambda j if lambda j is 0, then twice lambda j y j equal to 0.

So, from this equation, so we have two condition that lambda j can be 0 or y j can be 0 ok. So, if lambda j is 0, the constraint is not active; hence, can be ignored. If lambda j is 0 ok, so if lambda j is 0; that means, y has some value. So, if y is not 0, so in that case, the constraint is basically not an active constraint.

So, that is not actually the solution is not on that particular constraint. So, therefore, we can ignore those ignore those constraint. So, if y j is 0, what is it means that g j X is equal to 0 ok, so that is 0. So, what does it mean? That if g j X is 0, so in that case; that means, the constraint is active ok, the solution is on that particular the solution must satisfy that constraint ok. So, it will be that solution will be on that particular by your constraint. So, therefore, now what we can do basically? So, now, we can consider all these suppose if I consider all these active constraints. So, what I do doing here?.

Suppose, we are we have a set of J 1 ok; J 1 is the active constraint. So, we have total m number of constraints. So, we have total m number of constraint and out of m, if I say that J 1 is the set of active constraint and J 2 and set J 2 that is J 2 J 2 is the inactive constraint. So, what we are trying to do? That out of this m constraint, so if I know that these are the constraint which is active constraint and these are the constant which are not active constant.

So, I am putting in that particular set that J 1 is the set of active constraint and J 2 is the set of inactive constraint ok. So, if I know tha, then I can write my optimization problem something like that that I can write it the necessary condition something like that is that del f by del x i and summation of. So, now, it is basically I am only considering the set of this thing set of J 1 basically.

So, which are active and basically. So, this is I am getting i equal to 1 to n and similarly, that g j X equal to 0 and this is for your active constraint and we have the n inactive constraint that is g j X plus y j equals 0. So, for this constraint, the lambda is equal to your 0. So, these are inactive constraints. So, these are inactive constraint ok. So, these are inactive constraint ok.

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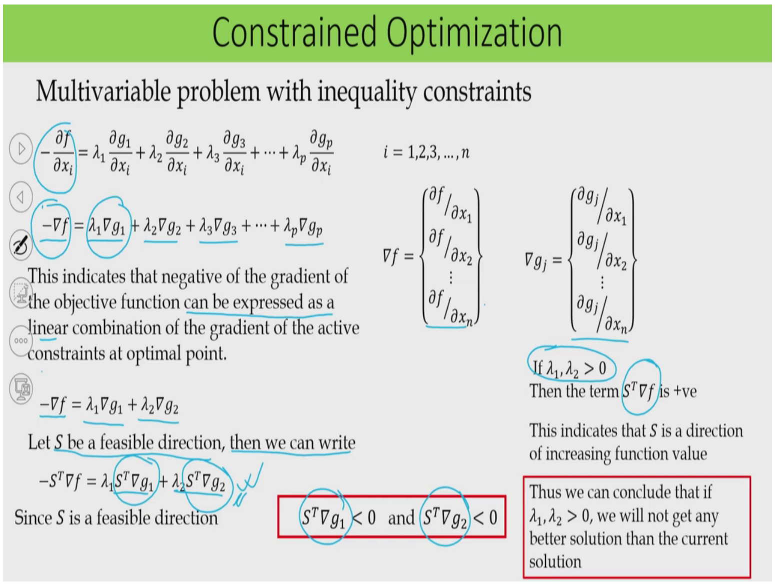

So, if I consider that only the active constraint ok, so if we consider the active constraint, then if you look at that the first necessary condition. So, what is this necessary condition? That del f by del X plus that is lambda j del g by del x and basically it is a summation and which is equal to 0 and this constraint, I can write it like this ok. So, I am just taking del f by del x i on the other hand side of the equation. So, it is it become negative now.

So, negative of del f by del x i equal to lambda 1 del g 1 by del x i plus lambda 2 del g 2 by del x i plus lambda 3 del g 3 by del x i; so, I am writing up to p. So, this is up to your p, I am writing and basically this p is basically the total number of active constraint ok; that is active constraint. And now, in this case, the variable is n, so I can write it like that.

So, in that case, what I am getting now? That is minus del f which is equal to lambda del g 1 lambda 1 del g 1 plus lambda 2 del g 2 lambda 3 del g 3 plus lambda p del g p ok. So, del f is I have defined here, what is del f and your del g ok. So, del g that I have defined.

So, now, what does it mean basically? So, it is saying that this indicate that negative of the gradient of the objective function. So, minus del f is the gradient of the objective function at optimal point; at optimal point that this is negative of the gradient of the objective function can be expressed as a linear combination linear combination of linear combination of the gradient of the active constraint at optimal point.

So, this indicates that negative of the gradient of the objective function can be expressed as a linear combination of the gradient of the active constraint at optimal point. So, optimal point what how what we can; what we can express? The negative of the gradient, so negative of the gradient of the objective function is nothing but the linear combination of the gradient of the constrained functions ok. So, this I can write it.

Now, let us consider two constraints; just to explain the concept, so let us consider two constraint; that means, we have only two constraint that is g 1 and g 2. So, if I consider, then negative of constraint that is minus del f is equal to lambda 1 del g 1 and lambda 2 del g 2.

So, I am expressing it like that. Now, let us take a so this is basically at a particular solution point ok. So, me this is an suppose this is an at optimal solution, so this point as an optimal your point, at optimal point that minus del f is equal to lambda 1 del lambda 1 del g 1 plus lambda 2 del g 2.

Now, let us see if I take a direction S ok. So, let S be a direct a feasible direction, then we can write it like this. So, if I take a direction from this point basically, so where we have actually calculate the gradient of the objective function as well as the constraint function, so if I write it, if I multiply it by the direction. So, I am getting that S transpose del f equal to lambda 1 S transpose del g plus lambda 2 S transpose del g 2 ok. So, I am getting this ok. So, now, this is a feasible direction.

Feasible direction means again you please you recall that the constraint is the constraints are inequality constraint and this is less than equality type constraint ok, that is less than equality type constraint and the function is the function is the minimization function. So, we are considering a minimization function here. Now, what does it mean basically? Suppose, if it is a feasible direction that this value should be less than 0 ok and this value also should be less than 0.

So, I will explain with a figure in my next slide. So, this S transpose g 1 should be less than equal to 0 and S transpose g 2 should be less than equal to 0 ok, since S is a feasible direction ok. So, the S is a feasible direction. So, therefore, that S 1 S T g 1 is less than 0 and S T g 2 is less than 0 ok.

Now, if we look at this particular equation, if you look at this particular equation, if the lambda 1 and lambda 2 both are positive ok, greater than 0, then what will happen? Then, your S T del f will be positive ok. So, if lambda 1 and lambda 2, so you just look at this particular equation and if because the S transpose del g 1 and S transpose del g 2 is negative.

So, therefore, if lambda 1 and lambda 2 are positive, so in that case, the S transpose del f is positive. So, what does it mean basically? If S transpose del f is positive, so in that case that indicates that S is a direction of increasing function value. So, what does it mean? That S transpose del f is positive; that means, if you go along that direction, your function value will increase that is not a decent direction ok.

So, therefore, the whatever you are at the current point and from here, if you are going towards the direction S, your function value will increase. So, you are not getting any improved solution. Therefore, that must be the optimal solution ok. So, then what is the condition? Thus, we can conclude that if lambda 1 and lambda 2 are positive, then we will not get any better solution than the current solution.

So, if lambda 1 and lambda 2, in this case if lambda 1 and lambda 2 both are positive, in that case if you go if you are not getting any improved solution, then the current solution. So, current solution is the optimal solution in that case. I hope this is clear.

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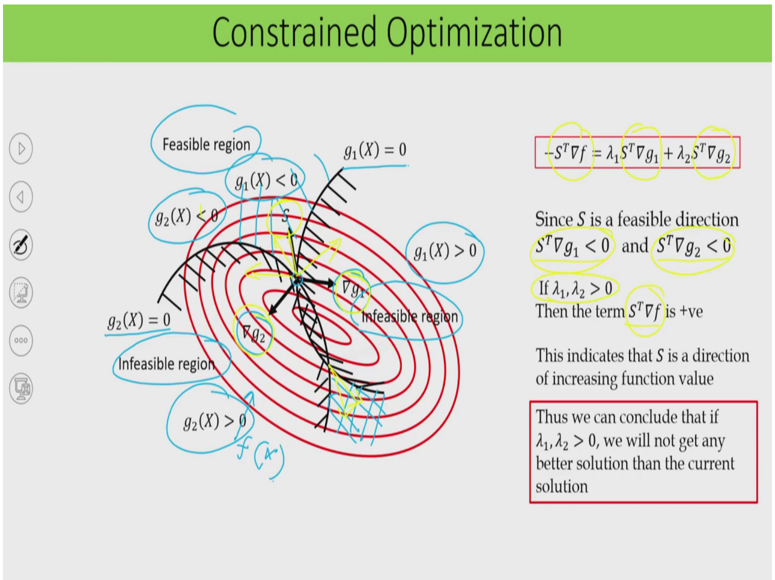

So, let us look at this yeah, I would like to explain whatever we have discussed using this figure. So, here, this is the function; suppose this is the function f X ok. So, this is the function f X ok. So, I have written. So, we have two constraint that is your g 1 X equal to 0 and g 2 X equal to 0.

So, this constraint as I said I as I said earlier. So, these two constraints are active constraints. So, we are not considering inactive constraint, only active constraint. So, what does the active constraint mean? That basically the solution is on that particular constraint.

So, we have two constraints here the g 1 equal to 0 and g 2 equal to 0 and so, therefore, this is a less than equality type constraint. So, therefore, this is the reason where g 1 is less than 0 and g 2 is less than 0 and this is the reason basically where g 1 is greater than 0 and g 2 is greater than 0. So, therefore, this is your infeasible region; that means, any solution in this particular region will be in infeasible because that will not satisfy the constraint.

But, however, if a solution is somewhere here, so somewhere here, in the in this side basically, in this side or in this side basically, so that is a feasible solution ok. So, this is feasible reason and this is also feasible reason ok. So, now, question is that suppose you are somewhere here ok; so, this is the point you are right now ok and these are active constraints.

So, the solution is somewhere here or somewhere here in this case. Now, question is that if I go along this direction that is del g 1; that means, the direction is means this is towards the increasing function value; that means, constraint value and similarly, in this direction also this is the del g 2 direction and this is del g 1 direction. Now, what I am doing here? So, I am taking I am taking another direction. So, I am taking another direction that is your S direction.

So, now, S direction is the feasible direction. So, if you go along del g direction that is not feasible direction, you are going towards the infeasible region. Similarly, if you are going along del g 2 direction that is also not feasible region. So, any direction if you are going along this direction or along this direction, these are all infeasible you are going towards the infeasible region; but this is somewhat a direction maybe I can go this way or that way or maybe if you are here, so I can go this way or this way.

So, in that case that will be a your you are going towards the feasible region. Now, if you take any direction from here ok, so suppose as we have taken the S direction. So, in that case, so we are writing that S transpose del f which is equal to lambda 1 S transpose del g 1 and plus lambda 2 S transpose del g 2.

So, because that is a feasible direction ok; feasible direction, that means, yours going towards the mean suppose if you go along that direction, your function means constraint function value will reduce and therefore, that S T del g 1 is less than 0 and S T del g 2 is less than 0. So, if you are going along this direction, so this direction that function value constraint function value will reduce.

So, therefore, this must be less than 0 and this must be less than 0 ok. So, this is 0. So, then if lambda 1 and lambda 2 are positive; that means, greater than 0, then S transpose del f is positive in that case. So, therefore, if what does it mean? If it is positive ok, so that means, if you are going along that direction, your objective function value will increase. So, you are not getting any improved solution, if you are going along the direction as basically.

So, therefore, from here we can conclude that the lambda 1 and lambda 2 should be greater than 0 at optimal point ok. So, at optimal point, the lambda 1 and lambda 2 should be greater than your 0. So, in this case, we have only two your two constraints; so, we have only lambda 1 and lambda 2. But if you have multiple constraints, so you are getting that suppose you have m constraint active constraint in that case, you are getting m that lambda 1, lambda 2 up to lambda m and all of them should be positive ok. So, at optimal solution.

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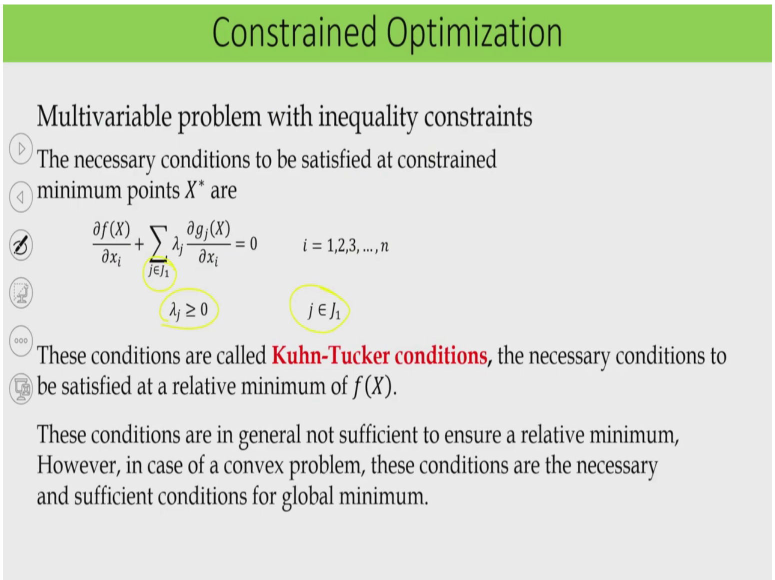

So, therefore, I can write the your necessary condition like this the necessary condition to be satisfied at constrained minimization point X star. So, what is the necessary condition? Necessary condition is this one. So, this is the del f by del x i plus this is for the active constraint that lambda j del j del j by del x i equal to 0 and then, the lambda j should be greater than 0 ok.

So, this is for the active constraint ok. So, for active constraint, so we are getting another condition that is lambda j should be greater than or it should be greater than 0 for the active constraint. So, therefore, these conditions are called Kuhn-Tucker condition, that is the necessary condition to be satisfied at a relative minima of f X ok.

So, these are this condition has to be satisfsied. So, these conditions are in general, not sufficient to ensure a relative minima. These conditions are not the sufficient condition, but this is the necessary condition as I said. So, these conditions are in general not sufficient to ensure a relative minima. However, in case of convex problem; so, already we have discussed what is convex problem.

In case of convex problem, this conditions are necessary and sufficient conditions for global minima ok. So, in case of convex problem, so we can say whether our problem is a convex or non convex problem. In case of convex problem that the then if the Kuhn-Tucker condition are satisfied, so in that case, I can say that whatever solution you are getting that will be a global minima ok. So, global minimum.

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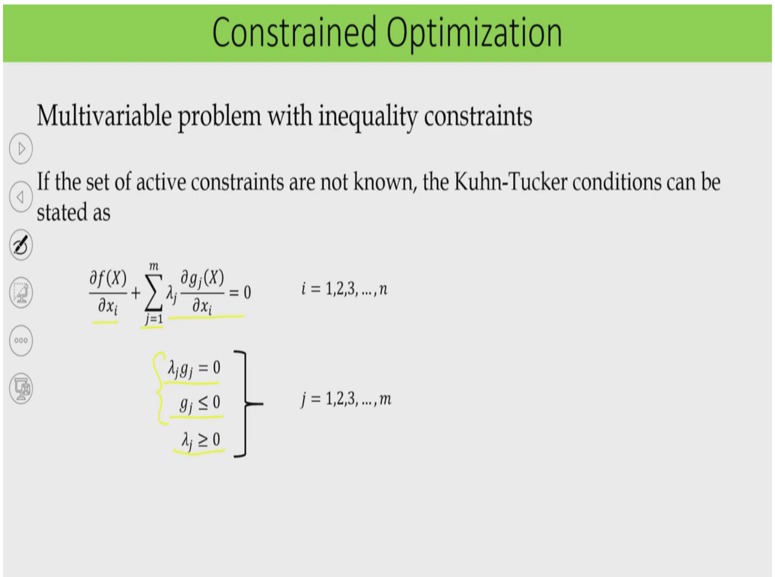

As we have discussed that we have divided the total constraint, the total inequality constraint in two sets ok that is the set 1 these are active constraint that is said J basically J 1 that are all active constraint and J 2 inactive constraint. But question is that for a particular problem I may not be aware actually that which are active and which are not active basically.

So, in that case, if the set of active constraints are not known the Kuhn-Tucker condition can be stated as del f by del x i that so this is j equal to 1 to m del lambda j del g by del x i equal to 0, then lambda j g j equal to 0, then g j is less than 0 and then, lambda j is greater than 0 ok.

So, if it is not known, then you have to do to this additional this additional constraint, you have to this sorry this additional condition you have to put. So, if the set of active constraints are not known, then you have to put this to additional condition that is lambda j g j equal to 0 and g j less than equal to 0 ok.

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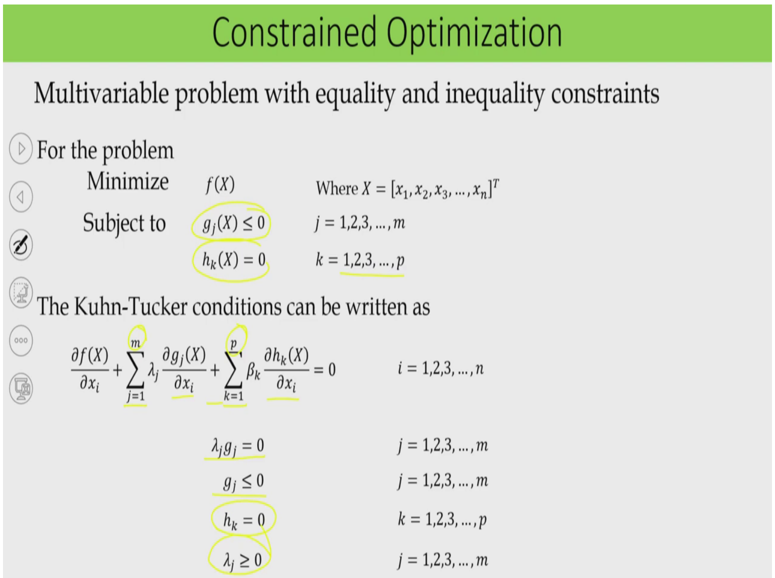

Now, if you have a problem, if you have also equality constraints. So, we have discussed that inequality constraint and if you have equality constraints, suppose in this case, so I have equality constraint that is k equal to 1 to p. So, p number of equality constraint along with the inequality constraint.

So, in that case, I can write the Kuhn-Tucker condition yeah like this. So, here this is del f by del x i, then j equal to 1 to m. So, this is your inequality constraint lambda j del g by del x i plus the k equal to 1 to p, these are the equality constraint beta k del h k by del x i which is equal to 0.

And then, lambda j g j equal to 0, g j is less than equal to 0 and h k that is the equality constraint that is equal to 0 and lambda should be positive ok. So, lambda should be positive. So, that is basically, so in case of the problem with equality and inequality constraint. So, I can use this condition. I can use this condition and basically these are known as the Kuhn-Tucker condition for the problem having equality and inequality constraints.

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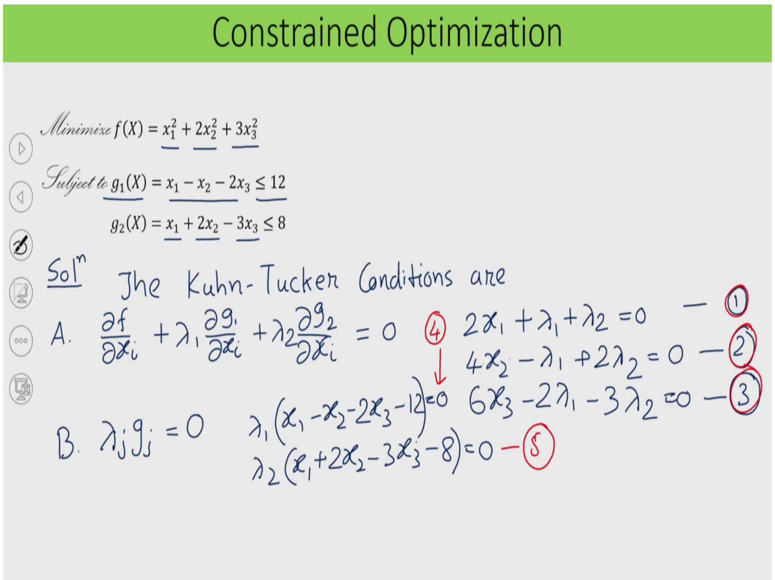

Let us solve this example problem. So, I will explain how we can apply Kuhn-Tucker conditions to find out the optimal solution of this particular problem ok. So, this is a minimization problem. So, minimize f X equal to x 1 square plus twice x 2 square plus thrice x 3 square and we have two constraint here subject to g 1 X which is x 1 minus x 2 minus twice x 3 is less than equal to 12 and g 2 equal to x 1 plus twice x 2 minus thrice x 3 less than equal to 8 ok.

So, let us solve this problem using the Kuhn-Tucker conditions. So, what we can do here? So, let us see let us solve it. So, the Kuhn-Tucker conditions; so let us write what is the first Kuhn-Tucker condition? That is del f by del x i plus lambda 1 del g 1 by del x i plus lambda 2 del g 2 by del x i this is equal to 0 ok.

So, if we apply this condition, then we will get; we will get this equations that is twice x 1 plus lambda 1 plus lambda 2 equal to 0. Then, the next equation, we are getting 4 x 2 minus lambda 1 plus 2 lambda 2 0 and the third equation, we are getting 6 x 3 minus twice lambda 1 minus 3 lambda 2 equal to 0. So, just put this is 1, this is 2, this is 3. Now, if I apply the second condition that is lambda j g j is equal to 0.

So, if you apply the second condition, so I will be getting lambda 1 x 1 x 2 twice x 3 minus 12 equal to 0 and lambda 2 x 1 plus twice x 2 thrice x 3 minus 8 equal to 0. So, I am putting this is equation 4 and this is equation 5. So, this is 1, 2, 3, 4, 5.

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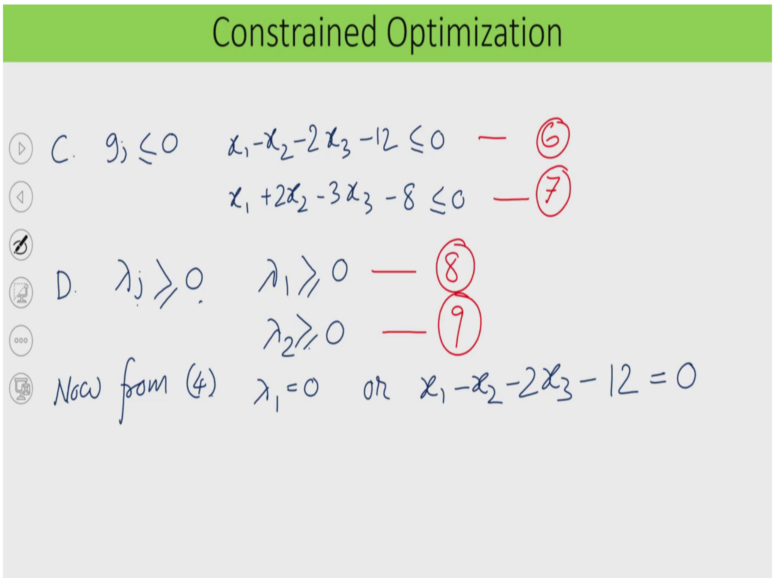

Now, if you apply the third condition that is g j is less than 0 less than equal to 0. So, then, I am getting that x 1 minus x 2 minus twice x 3 minus 12 is less than 0 and x 1 plus twice x 2 minus thrice x 3 minus 8 less than equal to 0. So, this is 6 and this is 7. Now, if you apply the

D condition that next that is lambda j should be greater than equal to 0, equal to 0. So, I am getting lambda 1 greater than equal to 0 and lambda 2 minus 0. So, this is 8 and this is 9.

Now, from 4, that is the fourth equation, so we may get either lambda 1 equal to 0 or x 1 minus x 2 minus twice x 3 minus 12 is equal to 0 ok. So, from equation 4, so we are getting two conditions; either lambda 1 is 0 or x 1 minus x 2 minus twice x 3 minus 12 is 0. So, let us take a condition that lambda 1 equal to 0.

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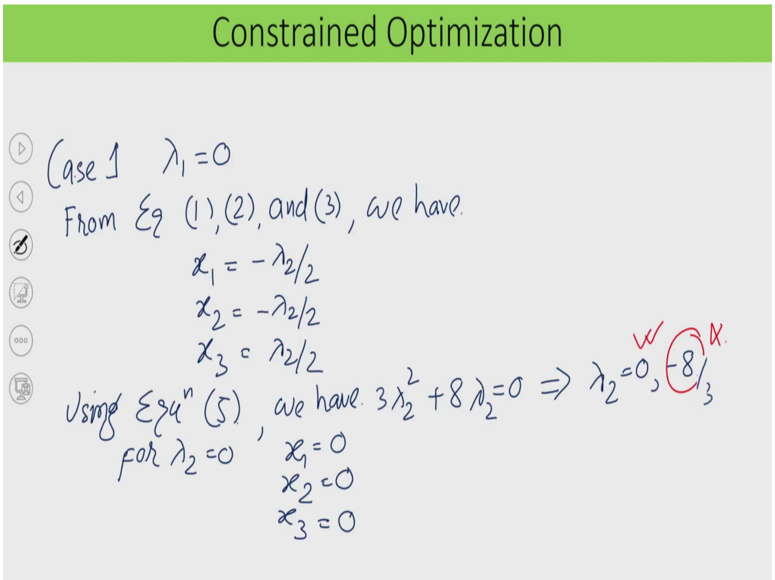

So, I am taking case 1 that lambda 1 equal to 0. So, in that case, from equation 1, 2 and 3 we have x 1 equal to lambda 2 by 2 x 2 equal to minus lambda 2 by 2 and x 3 equal to lambda 2 by 2. Now, from equation 5, using equation 5, we have that 3 lambda 2 square plus 8 lambda equal to 0.

So, if I put this value x 1, x 2 and x 3 in equation 5, so I am getting that lambda 2 square plus 8 lambda 2 equal to 0 and if I solve it, so I will be getting lambda 2 equal to 0 or minus 8 by 3. So, it is not possible because the lambda should be positive. So, this is not acceptable. So, therefore, the only solution is your lambda 2 equal to 0.

So, if lambda 2 is equal to 0, so in that case, the x 1 x 1 equal to 0. If lambda 2 equal to 0, so for lambda 2 equal to 0, x 1 equal to 0, x 2 equal to 0, x 3 equal to 0 ok. So, now, this solution; that means, x 1 equal to 0, x 2 equal to 0 and x 3 equal to 0, they are satisfying all these Kuhn-Tucker conditions ok. So, this solution is satisfying all this Kuhn-Tucker solution. So, this is one of the solution.

So, let us see the case 2 ok; whether we are getting any other solution and which is better than this solution. So, we can evaluate we can evaluate case 2 and basically, we can check that one.

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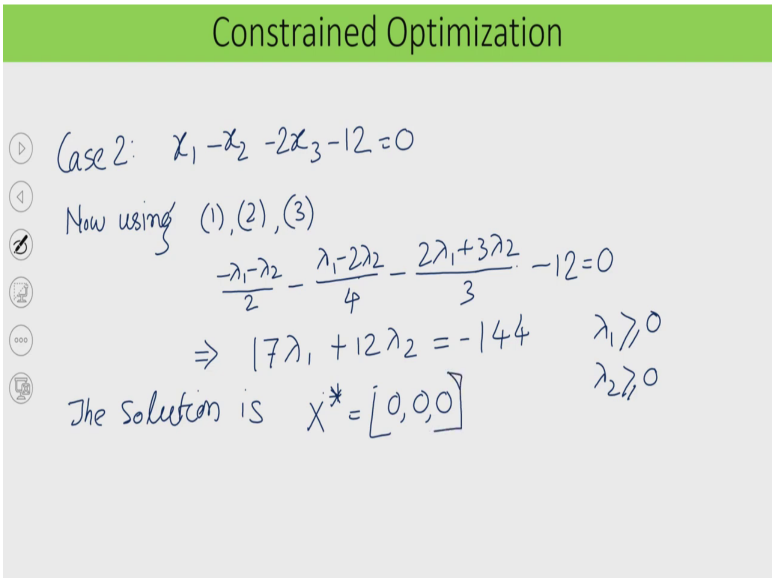

Let us see case 2 ok. Case 2 is that x 1 minus x 2 minus twice x 3 minus 12 is equal to 0 ok. Now, using now using one equation 1, 2 and 3 ok. So, from equation 1, 2 and 3, I am finding x 1, x 2 and x 3 and if I put in this equation, so what I am; what I am getting? That I am getting this equation that is I am getting that minus lambda 1 minus lambda 2.

So, this is from equation 1, I am getting x 1 equal to minus lambda 1 minus lambda 2 by 2; then minus from equation 2, I am getting that is your lambda 1 minus twice lambda 2 and divided by 4 and from the equation 3, I am getting this is twice lambda 1 plus thrice lambda 2 divided by 3 minus 12 equal to 0 ok.

So, if I simplify this equation, so I will be getting if I simplify it, I will be getting 17 lambda 1 plus 12 lambda 2 is equal to minus 1, 144 ok. So, now, if I see in this equation that 17 lambda 1, 17 lambda 1 plus 12 lambda 2 is equal to 144. So, therefore, this lambda 1 and lambda 2 cannot be positive ok. So, because the value should be equal to 144. So, therefore, this is not possible.

So, and as per the equation 8 and 9; as for equation 8 and 9, lambda 1 and lambda 2 should be lambda 1 should be positive greater than 0 and lambda 2 should be greater than 0. But here the lambda 1 and lambda 2 cannot be greater than 0. So, therefore, this is not a feasible solution. So, it is not satisfying the Kuhn-Tucker condition.

So, then, what is the possible solution that does? Then, thus, the solution is the solution; solution is that is X star X star is 0 0 0 ok. So, this is the solution of this particular problem and this solution x x equal to x 1 equal to 0 x 2 equal to 0 and x 3 equal to 0, it is satisfying all your Kuhn-Tucker conditions including that lambda 1 and lambda 2 are greater than 0. So, therefore, this is the possible solution and the solution of this particular problem is x 1; sorry x 1 equal to 0, x 2 equal to 0 and x 3 equal to 0 ok. So, let us stop here.

So, today, we have discussed the problem with inequality constraint. So, in the last class, we have discussed the problem with equality constraint. So, we have derived the Lagrange function and from Lagrange function, I can apply the necessary and sufficient condition and today, we have discussed the problem with inequality constraints.

So, we are getting the Kuhn-Tucker conditions; but this Kuhn-Tucker conditions are necessary condition, but not the sufficient condition of optimality. So, but for the convex problem, this is basically the this conditions are the sufficient and necessary condition for global optima. So, you will get the global optimal solution, if the solution is satisfying this Kuhn-Tucker condition ok.

So, in the next class, we will discuss some other constraint handling method basically. So, we will discuss about the penalty approach ok. So, using penalty approach, I can handle constraint for equality constraint as well as inequality constraints. So, that will be discussed in the next class.

Thank you.