Fluid Mechanics

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Lecture - 31

Boundary Layer Theory and Applications

Welcome back to the video course on fluid mechanics. In the last lecture, we were discussing about the boundary layer, boundary formation and how the laminar boundary layer is formed for a flow over a flat plate. Then we have seen how the transitions to turbulent boundary layer and we have also seen fundamental definitions for boundary layer like boundary layer thickness, then displacement thickness, momentum thickness, energy thickness and then related theories.

So today, we will be further discussing first about the turbulent boundary layer and then various theories behind the boundary layer then how we can get some solutions for various parameters as far as boundary layer is concerned.

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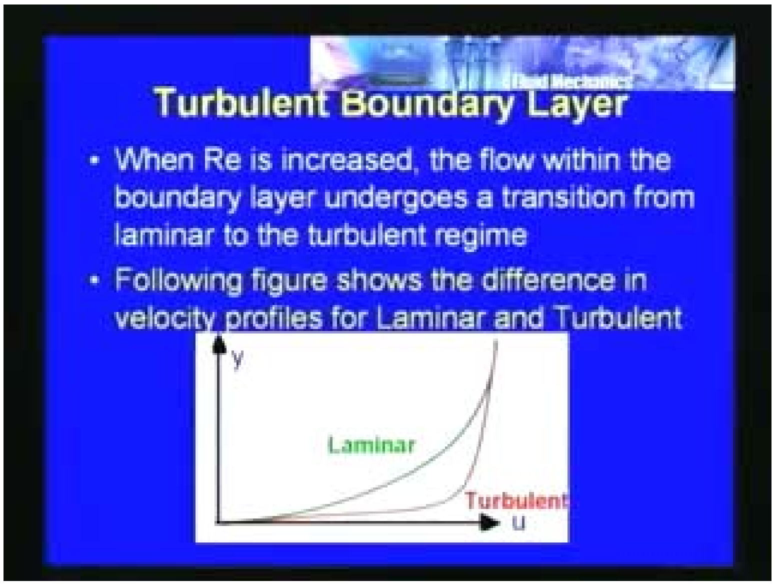

So first, let us now see the turbulent boundary layer. We have already seen how a laminar boundary layer is formed. For example, when we consider flow over a flat plate, we can see that initially, as we have already seen earlier initially the flow will be laminar image and then later the flow become turbulent.

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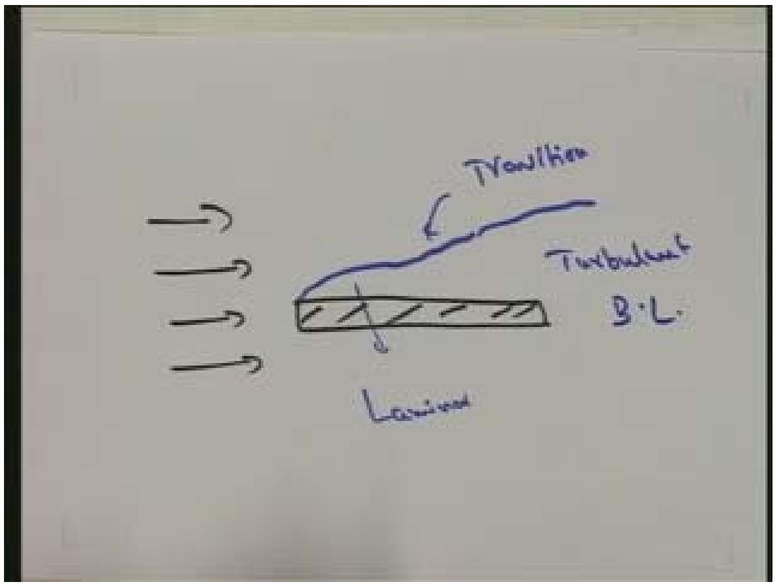

Here where is a flat plate lying here and then free stream flow comes like this and then we have seen how boundary layer is formed like this. Initially it will be laminar and then we can see that there is a transition here and then we have turbulent boundary layer. We have already seen some of the important aspects as far as laminar boundary layer formation is concerned.

Now, we will see some of the important aspects the as for as turbulent boundary layer is concerned.

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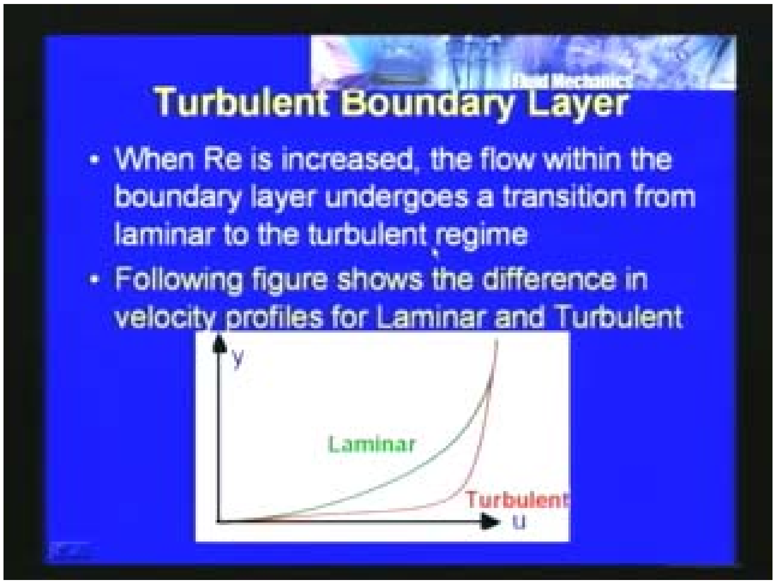

Here now as I mentioned, when the Reynolds number is increased the flow within the boundary layer undergoes a transition from laminar to the turbulent regime. As we have already discussed in the case of flow over a flat plate or flow through a pipe in all these cases, it can happen. Initially, it can be laminar boundary layer formation and then as the Reynolds number increases then turbulent boundary layer formation takes place.

Here as far as the difference between laminar and turbulent boundary layer is concerned, this figure shows the difference in velocity profiles for laminar and turbulent boundary layer. This is just like in the case of a flow over a flat plate as we have already seen. Here this velocity is on the x axis and depth y axis.

Here the laminar boundary layer is concerned, you can see that it is just like in a very say as in the case parabolic variation just like this turbulent, you can see, it is much more in this direction. This is, if you differentiate between the laminar boundary layer and turbulent boundary layer, then with respect to y axis the velocity variation is like this. Now we have already seen how the transition taking place with respect to a flat plate. Initially, when the free stream velocity touches or heats on the leading end of the flat plate, it is say just due to the shear effect and then viscous effect boundary layer is formed and then the transition from laminar to turbulent flow.

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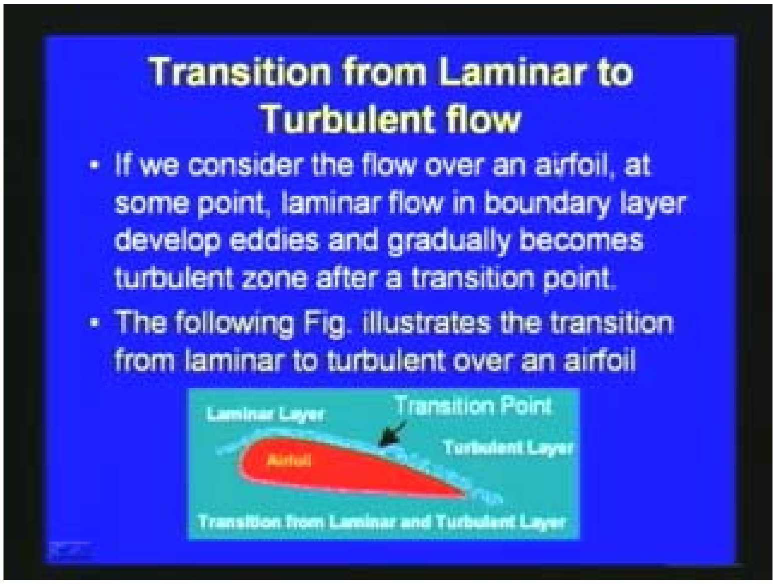

If we consider the flow over an airfoil, so as you can see here an airfoil is considered. We can see that the free stream velocity is coming here and this is the airfoil and then the boundary layer formation takes place. Then we can see that this is the laminar layer and then after sometime, this term will starts. Then this here, we can say identify this is as the transition point and then finally as we have seen in the earlier cases, how the turbulent are created.

That we have discussed in earlier lecture. The boundary layer now transforms to turbulent like this. If you consider the flow over an airfoil at some point, laminar flow in boundary layer develop eddies. As we can see gradually become turbulent zone after a transition point. So this is that transition point as explained in this figure.

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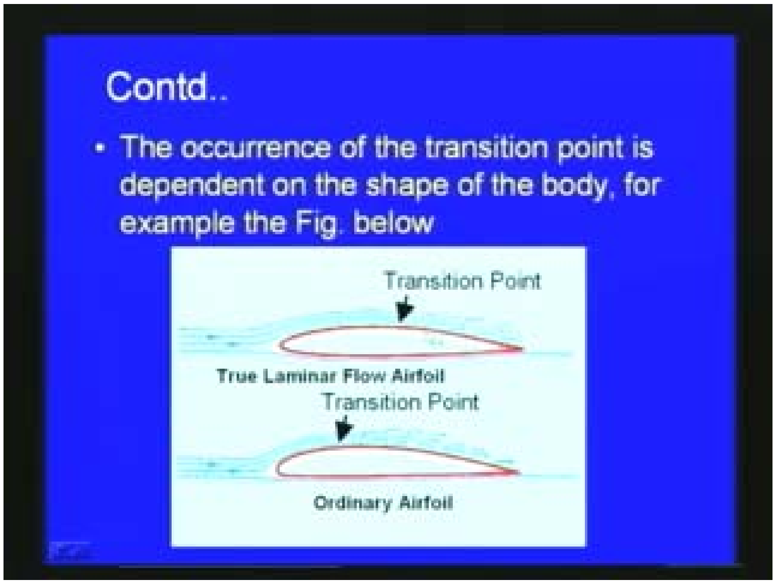

The occurrence of transition point is depending upon various parameters, which depends upon the shape of the body or which the flow takes place. It will be different case with respect to flat plate flow over a flat plate, or flow over an airfoil or flow over a cylinder or a sphere.

This transition that means from laminar to turbulent boundary layer transition will be different at different locations depending upon the shape of the body. Then also the fluid in the velocity, then the viscosity of the fluid, like that many parameters are here. Here in this figure you can see this is the transition point in the case an airfoil. This shape here you can see that the shape of the airfoil is changed from upper figure. Here to this figure the shape of the airfoil is changed.

Here we can see that two laminar flows this is called two laminar flow airfoil. You can see that laminar flow is up to this point. But in this case, you can see the airfoil ordinary airfoil here; we can see the size here is larger. Hence see that the transition point is much before as in the case of or much ahead of the laminar as in the case of laminar flow airfoil. Here the transition point is here, but in the case of ordinary airfoil like this, when the shape changes we can see that the transition point or the changes from laminar to turbulent takes much earlier.

Then you can see this is the transition point. That means this change from laminar boundary layer to turbulent boundary depends upon the shape of the body or which the flow takes place, then the inflow velocity and many other parameters that we have to consider while dealing with these kinds of problem.

Now, depending upon the problem, we can identify when these changes take place, from laminar to turbulent or when this transition starts. In number of experiments, especially the flow over a flat plate shows that critical value, the Reynolds number signifies the change of flow in the boundary layer.

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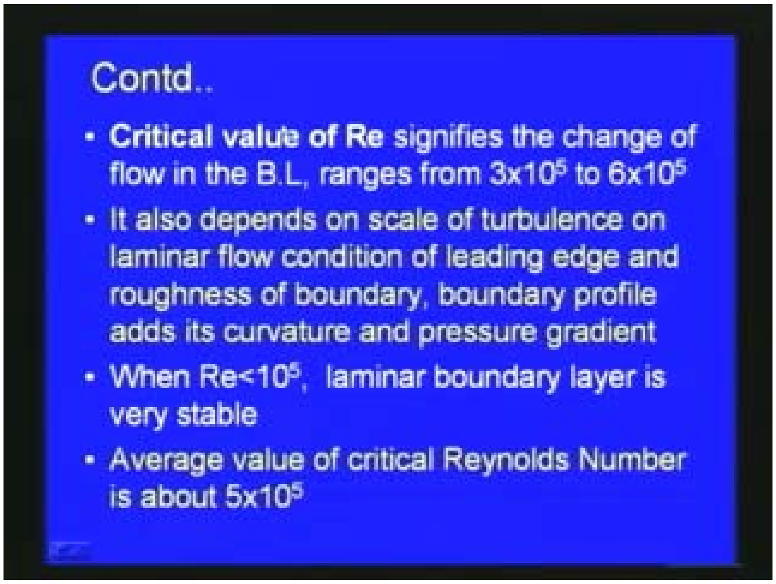

When the Reynolds number changes from particular range to other range, the boundary layer the translation starts and then the laminar boundary becomes turbulent boundary layer. So especially for flat plate n number of experience shows that, this is can be the range of 3 into 10 to the power 5 to 6 into 10 to the power 5. This is the critical value, the Reynolds number where the changes rest of the transitions starts depending upon the various other conditions which can be between 3 into 10 to the power 5 to 6 into 10 to the power 5 Reynolds number. It also depends on scale of turbulence on laminar flow condition of leading edge and roughness of the boundary profile adds it curvature and pressure gradient.

As I mentioned, the boundary layer formation definite to depends upon the nature of the surface over which the flow takes place. If it is more rough then the boundary layer formation is totally different than when it is in smooth surface. So this the critical Reynolds number, where the transition takes place also depends upon the boundary profile and then roughness of the boundary.

But anyway most of the experiments which is conducted, which is reported literature shows that when the Reynolds number is less than 10 to the power 5, most of the time it will be laminar boundary layer. Then you can see that as in the case of laminar flow here, in laminar boundary layer also when the Reynolds number is less than 10 to the power 5, then we can see that the boundary layer is much stable.

So since it is laminar boundary layer it is very stable and we can say an average value of based upon various experiments and reported theory is in the literature we can say that the average and the critical Reynolds number, where this transition starts can be put as 5 into 10 to the power 5 as reported in many of the text books on fluid mechanics.

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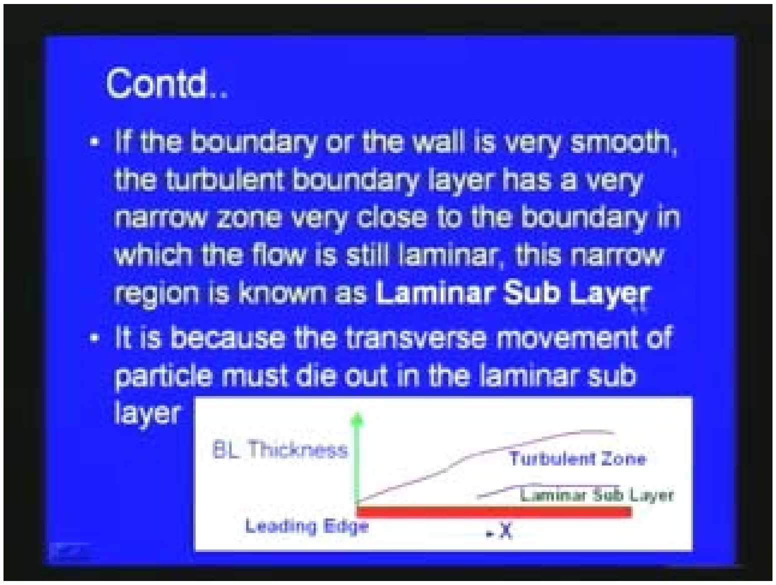

Now, if the boundary of the or the wall is very smooth as I mentioned, the boundary layer formation and also its stability and its development depends upon whether the surface over which the flow takes place is smooth or rough. So in the boundary or the values very smooth the turbulent boundary layer has very narrow as shown, very close to this boundary.

As we can see when the surface is very smooth we will be having very narrow boundary layer. So narrow zone very close to the boundary in which the flow is still laminar. This narrow region is known as laminar, sub layer, even in the case of same wherever the boundary layer is turbulent, if we consider a flat plate as we discussed earlier.

Even after some time the boundary layer can be turbulent, but still just very close to the boundary layer especially, if the plate is very smooth. Then there will be a narrow layer where the boundary layer is still laminar. This is called laminar sub layer. Here you can see that this is the flow over a flat plate. Here the flow is coming in this direction and then the boundary layer thickness you can plated like this.

This is a leading edge of the plate, so here initially as we have discussed here is the boundary layer is laminar and then a transition takes place here and then this is the turbulent zone, turbulent boundary layer zone. Still for this turbulent boundary layer zone also in the case of very smooth plate especially you can see that there will be a small sub layer which is called laminar sub layer.

This laminar sub layer is forming because the transverse movement of the particle must die out in the laminar sub layer. So there will be with respect to the fluid movement there will be transverse movement of the fluid particle also, so that will be dying down with respect to the very near to the flat plate very near to surface. So that is why this laminar sub layer form, now we can say that if we consider the flow over a flat plate as we are discussing here.

Initially there is a laminar boundary layer then, there is a translation boundary layer and then turbulent boundary layer but even for the turbulent boundary layer also we can identify a very small thickness of the laminar sub layer. So this also we have to consider but most the time when we discuss since this effect of this laminar sun layer is very small and that thickness also very small. So generally this laminar sub layer may not be considered depending upon the accuracy. We may not consider this laminar sub layer in most of the calculations.

But if you correct very exact results are required then we may consider the laminar sub layer also. So this turbulent boundary layer as we can see here, so after the transition the boundary layer thickness is increasing. So the turbulent boundary layer thickness increases in the downstream reaction from the leading edge of the plate, as we have already seen in the previous figure here.

From the leading edge the boundary layer thickness keeps on increasing. As far as turbulent boundary layer is concerned, experiment shows that especially for flow over a flat plate, shows that the boundary layer thickness increases in the downstream direction from the leading edge, in proportional to x to the power 0.8. Here you can see that this is x is starting from this location. This increase is with respect to x to the power 0.8 is in the downstream direction from the leading edge, in the proportion to x to the power 0.8, where x is the distance from the leading edge and similarly from laminar flow this boundary layer thickness delta varies in proportion to x to the power 0.5.

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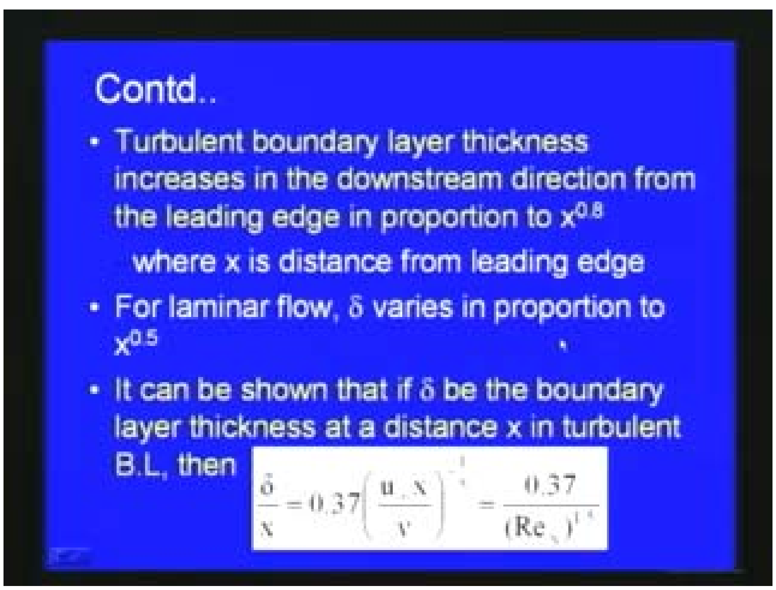

Here we can see this variation generally, it will be x to the power 0.5, x is starting from this location. x to the power 0.5 and this turbulent zone is consigned this variation is portion to x to the power 0.8. So this experimentally we can show.

Also through various experiment and in the literature we can see that if delta is the boundary layer thickness at a distance x, in the case of turbulent boundary layer then we can show that this delta by x is equal to 0.37 $u_{infinitive}$, which is the free stream velocity into x by new the kinematics viscosity to the power minus 1 by 5.

This is equal to 0.37 divided by Re, x to the power 1 by 5 where Re x is the Reynolds number at location x. Since this quantity is with respect to the Reynolds number. The delta by x that means with respect to the boundary layer thickness and x the distance from the reading edge, we can show that this is equal to delta by x is equal to 0.37 divided by Re x to the power 1 by 5.

Here we can see from here this x starts and then delta by x is with respect to this boundary of thickness to the distance from this location to particular location. When we consider Re_x is a Reynolds number at this particular location where we consider the boundary layer thickness. This is as far as flat plate is concerned. The experimental observation by various investigators and then if you consider the boundary layer with respect to the laminar and turbulent, if you plot the boundary layer thickness and then the shear stress, then we can see the variations are plotted in this slide here.

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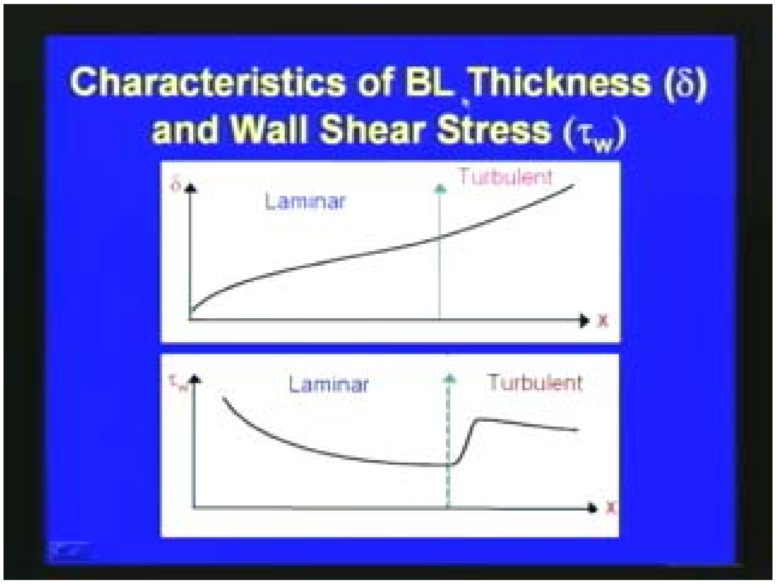

The characteristics of boundary layer thickness delta and wall shear stress tow_w, here this is delta 1 y axis and this direction with respect to x the laminar is concerned. This is the variation boundary layer thickness and turbulent as we have already seen, it is going like this and then if you flat the wall shear stress you can see that it will be laminar is concerned. Initially to the very high and then it will be coming down and then it reaches a minimum point. Then as far as turbulent sent I can it is like increasing and then it will be again decreasing like this. This is generally as far as a flat plate is concerned with respect to wall shear stress, the variation is as far as laminar boundary layer and turbulent boundary layer is concerned the variation is like this. Now this is what we have what we have observed so for is with respect to a flat plate.

As I mentioned, the boundary layer can develop in the case of internal flow like in the case of pipe or external flow in the case of flow or flat plate flow over a sphere or flow over a cylinder, so the boundary layer formations can be in the case of external flow as well as internal flow. So before we discuss various theories and some of the equations as for as boundary layer is consigned we will see how the variation takes place with respect to a pipe flow which is an internal flow so here we can see that this slide shows say the pipe flow the flow is coming from if you consider distance from here.

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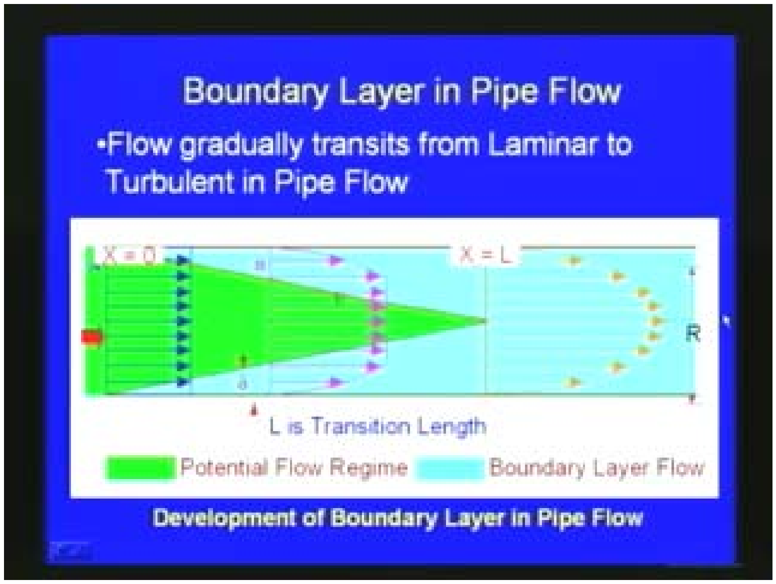

So, let us assume here the potential flow regime so then say with respect to the shearing effect and then the viscous effect you can see that boundary layer will be formed and then say we can here we can say that is the boundary layer flow and after some time we can say it is a fluid direct flow. So flow gradually transits from laminar to turbulent in the case of pipe flow like this so you can see that this is say with respect to the no slip condition of the pipe y so this boundary layer is formed and then you can see that it is becoming a fully developed flow. Initially when the flow is entering is at this location and if you consider as a potential flow regime and then after say we can say it is here this blue color indicates the boundary layer flow and the variation with respect to velocity we can plot like this.

Finally, the flow become fully develop with respect to the boundary layer formation so as in the case of what we have seen earlier in the case of flow over a flat plate very similar way we can explained here for flow through the pipe which is an internal flow, so we can see that as the flow encase the pipe the boundary layer will initially be laminar. So we have seen the flow over a flat plate initially at the leading edge the boundary layer is laminar and then there is a transition and then finally, the boundary layer become turbulent. In a very similar way the case of pipe flow also initially the boundary layer will be laminar and then this will change depending on the ratio of the inertial and viscous forces.

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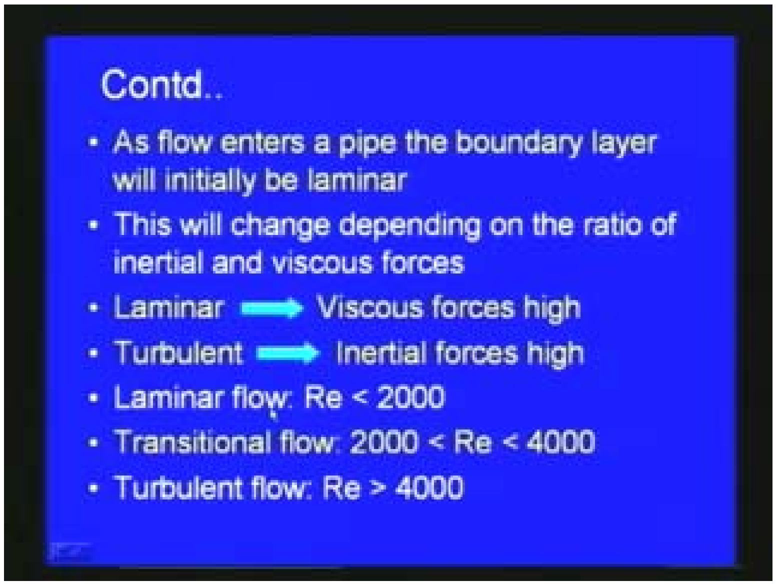

So in the pipe flow is a concerned, say laminar boundary layer to turbulent boundary layer the change place takes depending upon the ratio of the inertial and viscous forces so laminar flow is consigned viscous forces are high, turbulent is concerned in inertial forces are high, so you can see that the flow will be as for as pipe flow is consigned as we discussed earlier the flow will be laminar when the Reynolds number is less than 2000 and transitional flow regime will be between 2000 to 4000 and beyond 4000 Reynolds number the flow will be turbulent. So here we can observe earlier say initially laminar and then turbulent the flow become fully developed or fully turbulent here as you can observe here.

When the Reynolds number approaches for thousand the flow become fully turbulent and then we can say that flow is fully developed in this case for the flow through the pipe so the boundary layer developed and also you see laminar to it is transits to the turbulent. So now we have seen the flow over a flat plate like in the case of a flow external flow and also flow over a airfoil and then now we are seen the flow through pipe which is the internal flow regime. Now we will see we can derive the equations for the boundary layer say with respect to various theories and then we will be discussing how to get expression for various parameters as for as boundary layer flow is concerned. So the initial theories on boundary layer proposed by Prandtl's and then various other scientist like a Blasius say how to solve the boundary layer problems and then try to get expression for velocities shear and other parameters.

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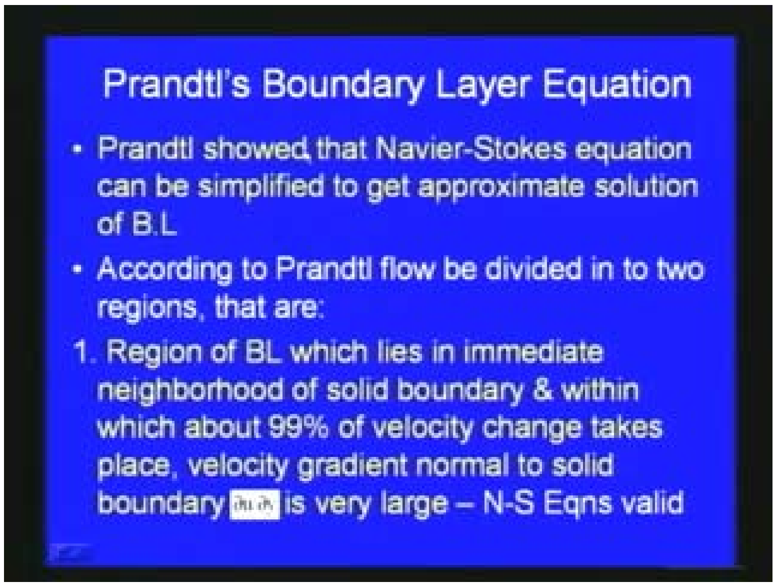

First, we will see here the Prandtl's boundary layer equation and then we discuss how the Blasius got the solutions and see then momentum into equation as for as boundary layer is concerned. So Prandtl's showed that so the Navier-Stokes equation which we are discussed earlier can be simplified to get approximate solution for the boundary layer, so boundary layer we can see that lot of changes takes place and then he try to use the Navier-Stokes equations which we are developed by the end of the nineteenth century for Prandtl's type to use this equation to solve the boundary layer. So according to Prandtl's the flow can be divided into two regimes that is first one is within the boundary layer and then say the above boundary layer.

First one according to Prandtl's the flow is divided into two regimes first regime is regime of boundary layer which lies in immediate neighborhood of solid boundary and within which about 99% of velocity change takes place and velocity gradient is normal to the solid boundary, so here say for example del u by del y is very large where this Navier-Stokes Equations are valid. So we have already seen in the Navier-Stokes Equations. Prandtl's divided the flow into two regions, one region where the boundary layer formation takes place that means we ask for the definition which we have already seen for boundary layer where the velocity change is in 99% up to that thickness is the boundary layer thickness which we defined. So this first regime where this boundary layer thickness is consigned. In that regime say Prandtl's showed that this Navier-Stokes Equations are valid and then we can directly apply the Navier-Stokes equations and then he defined the second region outside the boundary layer which constitute the main body of flowing fluid which has velocities of the order of free stream velocity $u_{infinitive.}$

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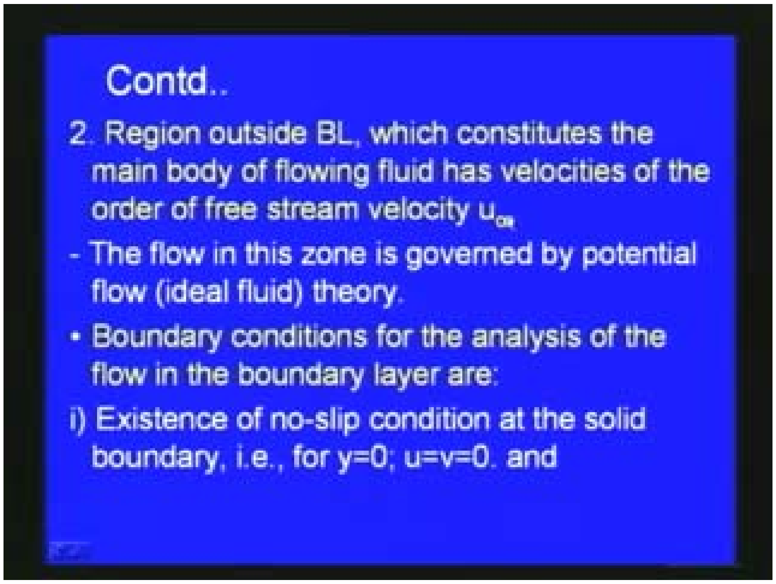

So the free stream velocity is coming and then above that say the boundary layer so consider as separate region which is outside the boundary layer, so this construct the main body of the flowing fluid and which has velocities of the order of free stream velocity $u_{infinitive}$ so flow in this region, this second region the flow in this zone is governed by potential flow or ideal fluid theory so that we can simplify the problem so through this proposition say by this Prandtl's assumption actually the solutions as for as boundary layer consigned has become much easier since the boundary layer is consigned it has to solve using the Navier-Stokes Equations and then he proposed that beyond the boundary layer say we can use the potential flow theory which is much easier to deal with than the Navier-Stokes Equations which we have discussed earlier. So then with respect to this boundary conditions for the analysis of the flow in the boundary layer are say the existence of no slip condition at the boundary that is for y is equal to 0, u is equal to v is equal to 0, so here as we have seen say if we consider the boundary layer flow over a flat plate so here the region 1 as per of the Prandtl's theory is here this is region 2, so here the boundary conditions say when we consider this region 1 and region 2, the boundary conditions here are the existence no slip conditions at the solid boundary.

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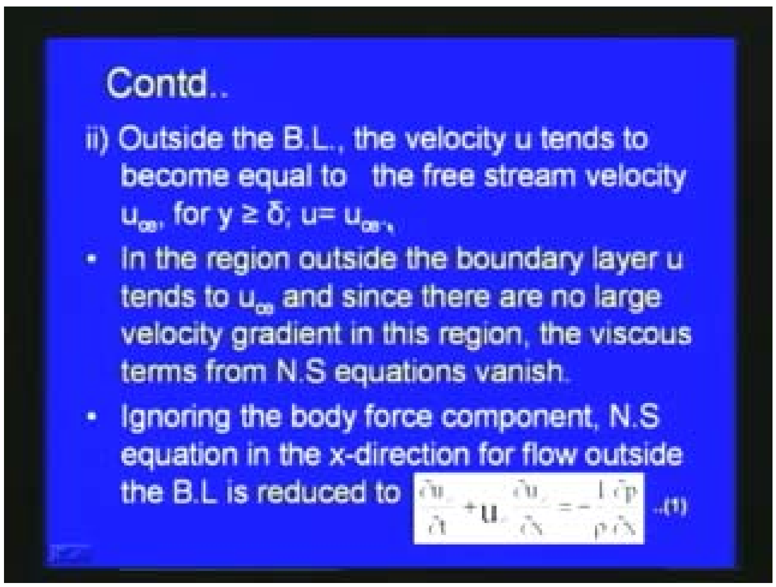

So that, here at this location we can say us is equal to say at y is equal to 0 we can say u is equal to v is equal to 0 so this is one boundary condition and then say outside the boundary layer the velocity u tends become equal to the free stream velocity $u_{infinitive}$ so that we can say that for y greater than equal to delta u is equal to $u_{infinitive.}$ So here when we consider here this boundary layer at this boundary layer location, we can see that at y is greater than or equal to delta for Prandtl's proposed to y is greater than or equal to delta u is equal to $u_{infinitive,}$ which is the this is the $u_{infinitive}$ which is the free stream velocity.

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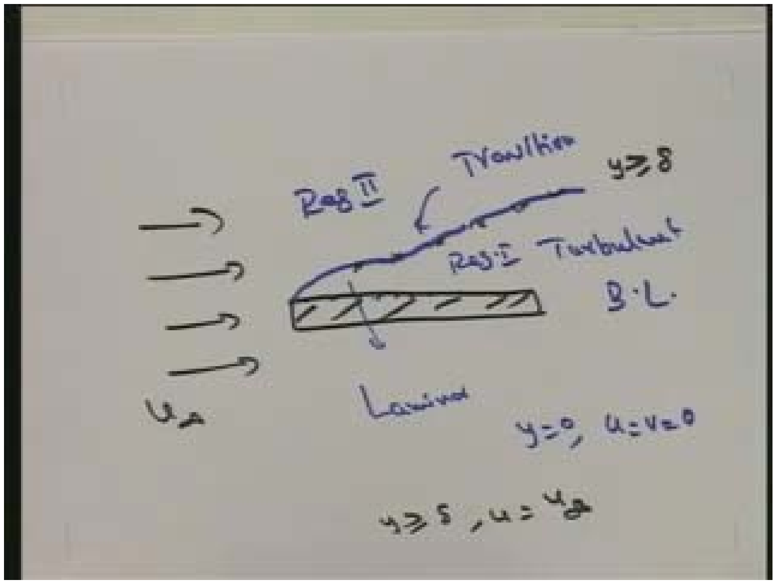

Like this we divided into two regions, region 1 and region 2 and he has put forward the boundary conditions and in this region where the boundary layer formation takes place that region he use the Navier-Stokes Equations and beyond this potential flow theory can be applied and then with respect to this assumption he tried to solve the boundary layer problems. So in the region outside the boundary layer u tends to $u_{infinitive}$ and since there are no large velocity gradient in this region the viscous terms say in the Navier-Stokes Equations vanishes so you can see that outside the boundary layer the viscous term effects are not there so we can easily solve say outside the boundary layer.

So ignoring the body force component, as for as the Navier-Stokes Equations is consigned, we have already seen the Navier-Stokes Equations, so Prandtl's ignored the body force component in the Navier-Stokes Equations and then in the x direction for flow outside the boundary layer is reduced to del $u_{infinitive}$ by del t plus $u_{infinitive}$ del $u_{infinitive}$ by del x is equal to minus rho del p by del x as in equation number 1. So where this $u_{infinitive}$ is the free stream velocity; t is the time; and p is the pressure; rho is the density of the fluid. So this is the after this equation of after ignoring the boundary force component in the Navier-Stokes Equations this equation is valid for flow outside the boundary layer so this is the first equation and then for steady state flow the pressure p is function of x only so earlier equation is with respect to for transient condition. If we consider the steady state then del u by del t is equal to 0, so for steady state flow the pressure p is a function of x only, so del p by del x is equal to we can write dp by dx. Hence from equation 1, we can write $u_{infinitive}$ del u x by del x is equal to minus 1 by rho dp by dx as in equation number 2.

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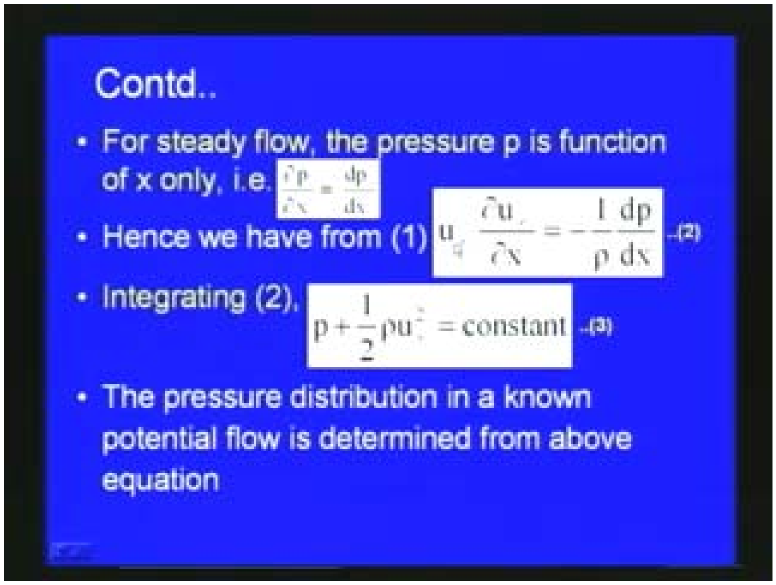

This equation we can see this is equation outside the boundary layer, so finally this is transformed in this form by putting the del p by del x is equal to dp by dx, so $u_{infinitive}$ del u x by del x is equal to minus 1 by rho dp by dx equation number 2. So if you integrate this equation number 2, we can write p plus half rho $u_{infinitive}$ square is equal to a constant as in equation number 3, so this constant we can find out with respect to boundary conditions. The pressure distribution is a known potential flow is determined from the above equations so this equation we can utilize find out the pressure distribution since generally the free stream velocity will be known. Once the free stream velocity is known in the potential flow region outside the boundary layer, we can solve this equation number 3 to get the pressure distribution. This is the condition the region 2 where the Prandtl's consider the flow outside the boundary layer. Then inside the boundary layer now as we discussed inside the boundary layer Prandtl's consider the variations in the flow and then he considered the Navier-Stokes Equations as such.

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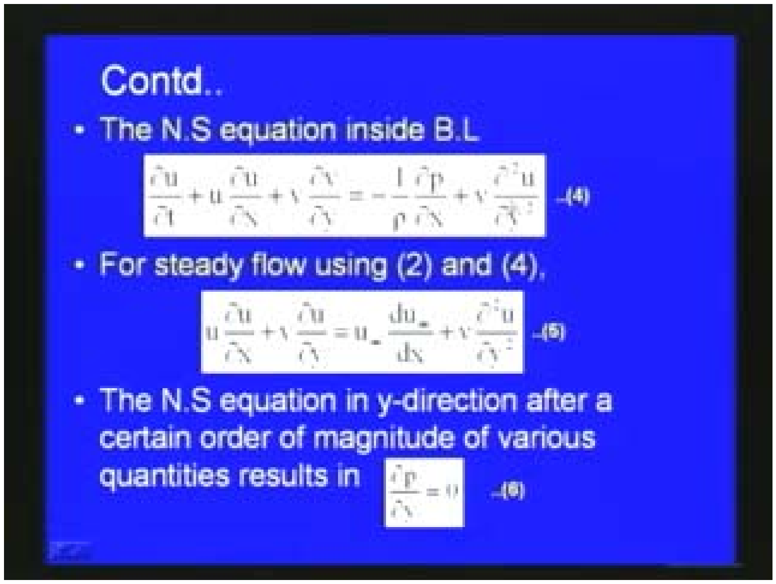

So we can write say if we consider two dimensional flow in Navier-Stokes Equation inside the boundary layer we can write as del u by del t plus u into del u by del x plus v into del u by del y is equal to minus 1 by rho del p by del x plus nu into del square u by del y square where u and v are the velocity in x and y direction; rho is the density; p is the pressure and nu is the kinematic viscosity of the fluid. So this is the equation inside the boundary layer as forwarded by Prandtl. Now if you consider steady flow using equation 2 and 4, so here we have seen the equation number 2 here so for steady state flow using equation 2 and 4 we can write this del p by del x term we can write with respect to the earlier condition as derived in equation number 2, so we can write u into del u by del x plus v into del u by y is equal to $u_{infinitive}$ $du_{infinitive}$ by dx plus nu into del square u by del y square as in equation number 5. The Navier-Stokes Equation in y direction after a certain order of magnitude of various quantities we can show that this del p by del y is equal to 0 as in equation number 6.

Now, the y direction is concerned we have del p by del y is equal to 0. Now in the continuity equation now, the remaining equation is continuity. The continuity equation for two dimension flow we can write us del u by del x plus del u by del y is equal to 0 as in equation number 7. Now these are the equations as for the boundary layer is concerned we have already seen the equations for the potential flow which we consider outside the boundary layer.

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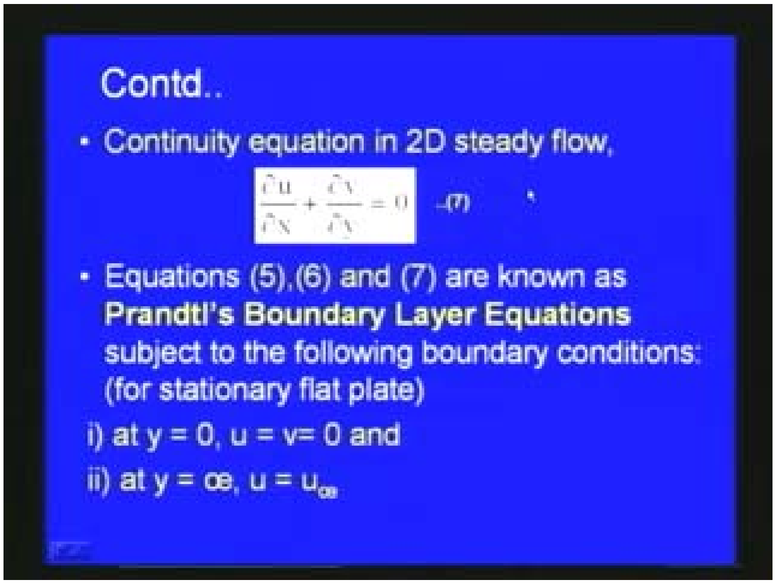

So these equations 5, 6 and 7 are known as the Prandtl's boundary layer equations subject to the following boundary conditions. So this equation number five six and then seven so these equations are called Prandtl's boundary layer equations subject to the following boundary conditions are at y is equal to 0, u is equal to 0, v is equal to 0 and at y is equal to infinitive outside, u is equal to $u_{infinitive,}$ so y is equal to delta also as we have seen here at this location here, this is say here u is equal to $u_{infinitive,}$ of course outside the boundary layer it is free stream velocity but here in this location also say we can write at y is greater than or equal to delta u is equal to $u_{infinitive.}$ These are the boundary conditions as for as the Prandtl's boundary layer equations are concerned. So equations 5, 6, 7 and the boundary condition as we discussed this together we can used to solve the boundary layer problems. Based upon this, these equations the boundary layer equations are put forwarded by Prandtl's at the beginning of the 20th century and then this as a lot to derive the approximations as for as the velocity variations pressure variations and shear variation in the boundary layer and many people try to solve these equations and in the one of the important solution put forwarded by Blasius is discussed here, so the Blasius solution for laminar boundary layer flow over a flat plate. In 1908, Blasius derived the simplified solution as for as flow over a flat plate which is for most of the time a typical problem which people consider to get solution from these kinds of problems.

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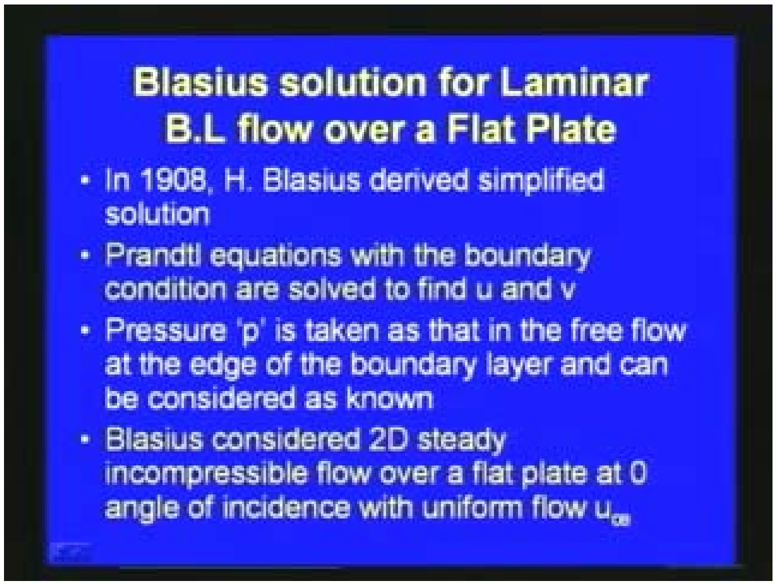

The Prandtl's equation with the boundary conditions are solved here in terms of u and v, so Blasius also tried to solve the Prandtl's equations but here again Blasius considered some more approximation so here the pressure p is taken as that in the free flow at the edge of the boundary layer and according to Blasius this can be considered as known and then Blasius consider two dimensional steady incompressible flow over a flat plate at 0 angle of incidence with uniform flow velocity of $u_{\text{infinitive}}$.

These are the various conditions which are considered by Blasius and then according to Blasius, since the plate is flat and has a negligible thickness and is uniform so according to Blasius this pressure gradient del p by del x in the x direction that means in this direction this is here, the x direction is here, so this is x direction, so del p by del x as per according to Blasius this must vanish as for as flow over a flat plate is concerned. So according to the then Blasius reformulated the Prandtl's boundary layer equations and then he wrote for the boundary layer the Prandtl's boundary layer equations where changing in this way, so you as written u into del u by del x plus v into del u by del y is equal to nu into del square u by del y square.

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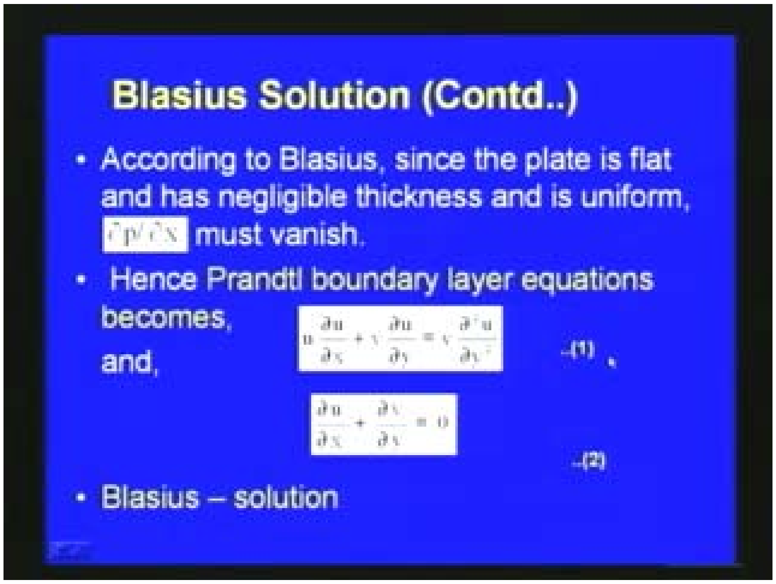

Blasius neglected this del p by del x so that he got this equation number 1 u into del u by del x plus v into del u by del y is equal to nu into del square u by del y square and then with this he used the continuity equation del u by del x plus del v by del y is equal to 0, so these are the equations used by Blasius to get a solution equations number 1 and 2. So the Blasius solutions which is developed in 1908, he used to say this equation which we have seen here the Blasius the approximated the Prandtl's boundary layer equations and we have seen the equations. He tried to derive an analytical solution or a approximate solution for this, the Prandtl's modified equations. So he used the stream function sie by putting u is equal to del sie by del y and v is equal to minus del sie by del x, so he used the stream function approach here. He put forward u is equal to del sie by del y and v is equal to minus del sie by del x, so he derive the for various parameters by putting this u is equal to del sie by del y and v is equal to del sie minus del sie by del x in the equations equation number 1 and 2 and then he used to various transformations and I am not going to your details of this derivation forward by Blasius this derivations are given standard text books like fluid mechanics by R. A. Granger.

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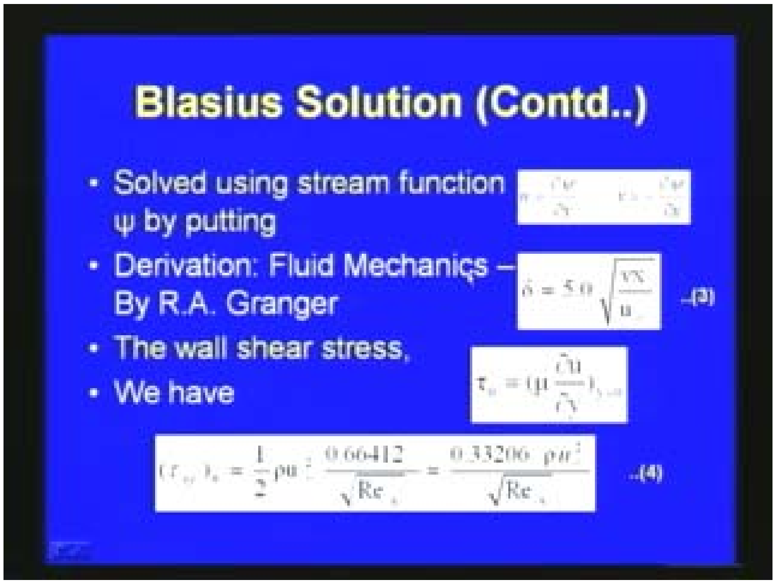

Finally, here we discuss the various results puts forward by Blasius. So Blasius showed that the boundary layer thickness delta for the case which we considered here the flow over a flat plate he show that is delta is equal to 5 into square root of mu x by $u_{infinitive}$ where nu is the is the kinematic viscosity, so Blasius go delta is equal to 5 into square root of nu x by $u_{infinitive}$ as in equation number 3 and then he tried to get the solution for the wall shear stress. Wall shear stress can be written as tow_0 is equal to mu del u by del y at y is equal to 0, so you after substitutions he showed tow xy at 0 is equal to half rho $u_{infinitive}$ square into 0.66412 divided by square root of Re_x where Re_x is the Reynolds number at the particular location as we consider earlier from the leading and other plate so that is equal to 0.33206 rho $u_{infinitive}$ square by square root of Re_x where $u_{infinitive}$ is the free stream velocity rho is the fluid density Re_x is the Reynolds number at that particular locations. This solution which is obtained by Blasius say he also experimentally verified this so almost it was matching with respect to the experimental observations and then he derived the local skin friction or dimensionless skin friction coefficient as C_f is equal to tow_{xy} at say 0 will divide by half rho u square that is equal to 0.664 divide by square root of $Re_{x.}$

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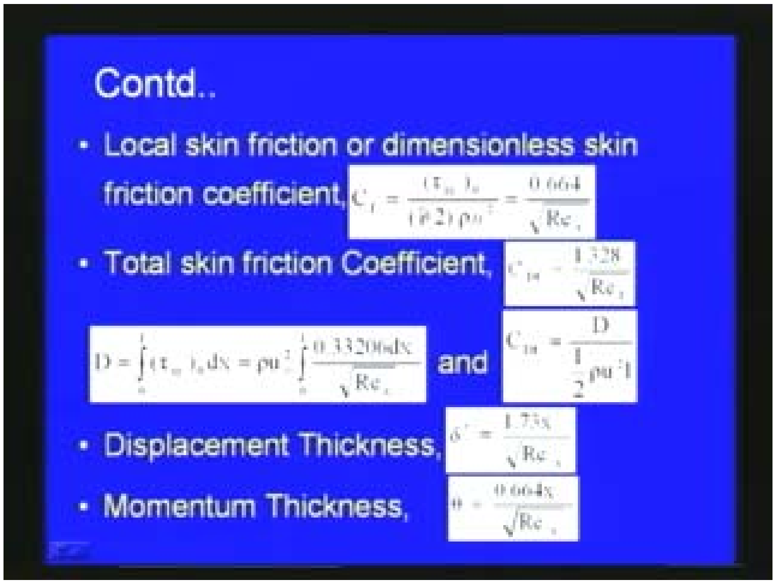

The total skin friction coefficient as derived by the Blasius is C_{Df} is equal to 1.238 divided by square root of Re_l, where Re_l is the Reynolds number at the trailing edge of the flat plate which we considered or this is equal to C_{Df} is equal to D by half rho u square l, so here the D is the total drag is concerned where D is equal to integral 0 to l tow$_{xy}$ at 0, dx is equal to rho u 0 infinitive square integral 0 to l, 0.33206 dx by square root of Re_x and then finally Blasius got the explicit expression for displacement thickness and momentum thickness. So he derived the displacement thickness as delta star is equal to 1.73 x by square root of Re_x and theta is equal to 0.664 x divided by square root of Re_x so like this Blasius derived expression for the boundary layer thickness then displacement thickness and then the momentum thickness. Using these expressions we can easily get other parameters like the velocity variations and also we can get the shear variation as we have seen in the previous slide. So like this say Blasius sold the boundary layer equations proposed by Prandtl and then by certain approximations.

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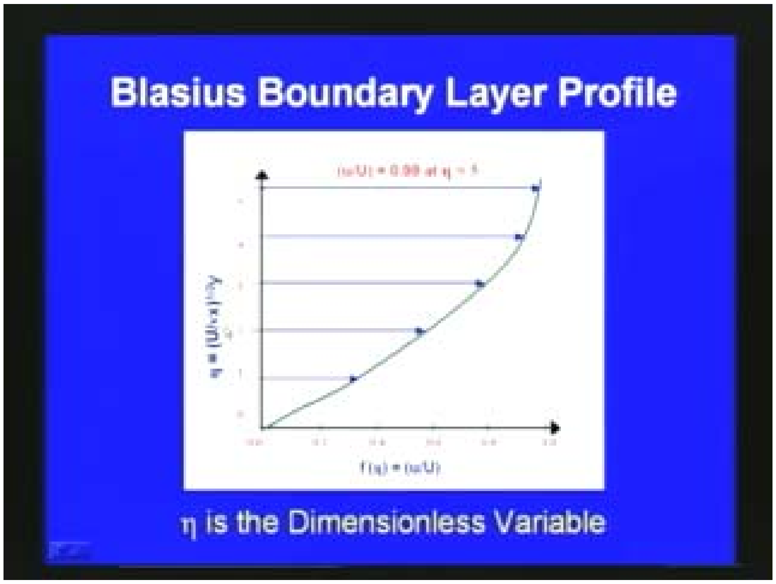

If you plot the Blasius boundary layer profile you can see that, we can plot c u by u on the x axis and here u by mu x to the power half into y if you put this eta here, so this variation as proposed by Blasius will be like this at u by u is equal to 0.99 at eta is equal to 5 variation will be like this so this shows as the Blasius boundary layer profile as for as the flat plate which he consider here for this case. So like this the say the boundary layer problems were solved by Blasius by approximating the Prandtl's boundary layer equations. So we have seen the boundary layer equation proposed by Prandtl's and then also we have seen the proposed solution by Blasius and then another important equations or another important theory for which we can seen in the boundary layer literature is momentum integral equations. So here the boundary layer flow analysis based on the momentum principle derived by Von-Karman by integrating the Prandtl's boundary layer equations.

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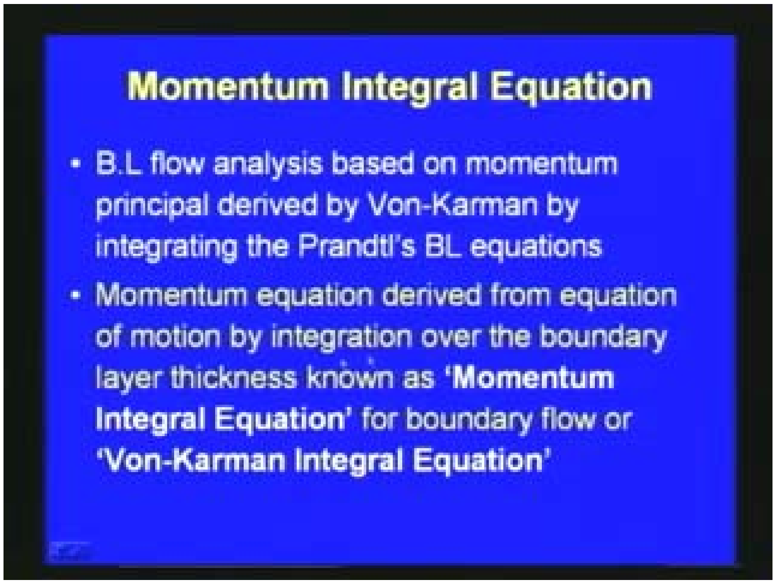

Again the Von-Karman use the Prandtl's boundary layer equations and then we try to integrate it so that you got an equation called momentum integral equations. So these momentum equations derived from equation of motion by integration over the boundary layer thickness known as momentum integral equation for boundary layer flow or this also called Von-Karman integral equations. So basically Von-Karman the Prandtl's boundary layer equations and then he tried to integrate from 0 to delta say for example if you consider the flow over a flat plate as we have already seen here, so here what Von-Karman tried to integrate with respect to 0 to, so this is 0 and then delta, so between the boundary layer thickness he try to integrate the Prandtl's boundary layer equations and then these equations are called the momentum integral equations or also called Von-Karman integral equation as it was derived by Von-Karman. So for this purpose the control volume used by Von-Karman is say here, we can see that this is the flat plate and free stream velocity is coming here and then he considered a control volume like this so this is the boundary layer which is formed here, so the variation from 0 to like this.

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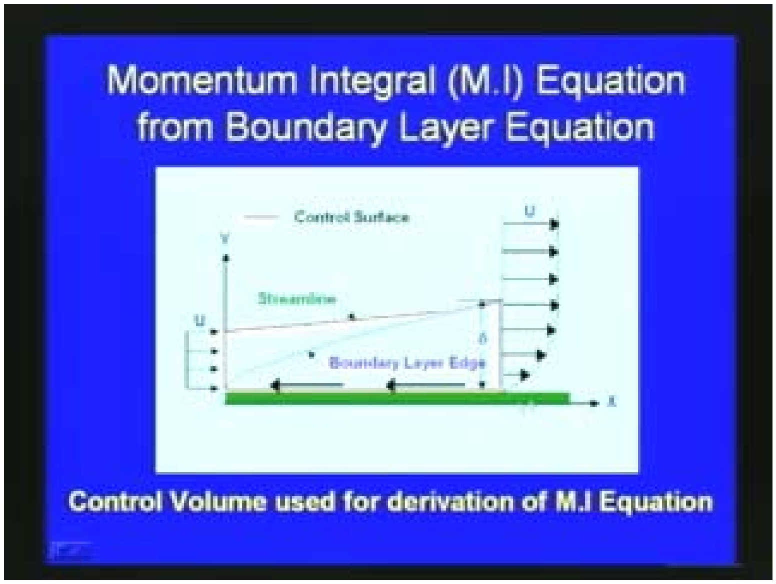

Von-Karman considered a control volume like this and then he tried to integrate. So with respect to this figure here we see that the integrations between 0 to delta so that within the boundary layer. So boundary layer equations which derived by which are derived by Prandtl are here.

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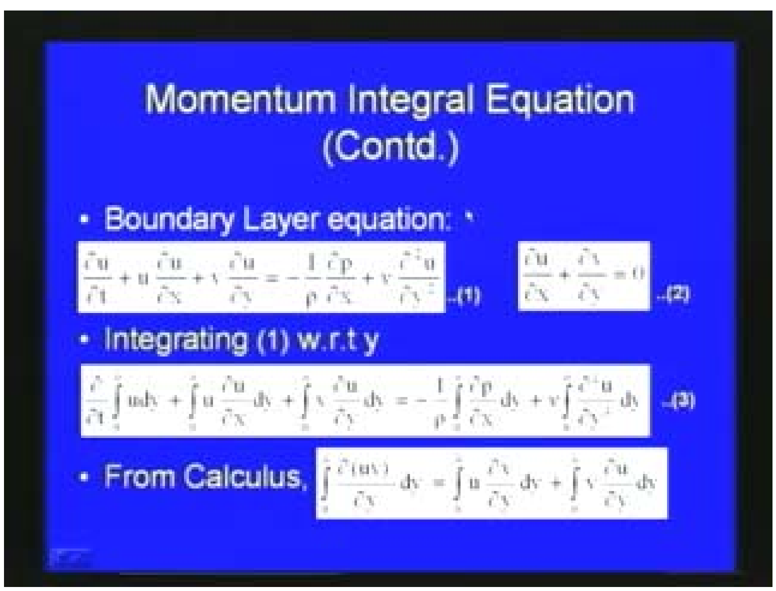

If you consider unsteady state or transient flow then the equations are del u by del t plus u into del u by del x plus v into del u by del y is equal to minus 1 by rho del p by del x plus nu into del square u by del y square that is equation number 1 and then the continuity equation del u by del x plus del u by del y is equal to 0. So Von-Karman tried to integrate this equation number 1 with respect to y with respect to the depth as we can see with this is the y direction, so with respect to this only Von-Karman tried to integrate. After integration with in the limit of 0 to delta we can write del by del t integral 0 to delta u dy plus integral 0 to delta u del u by del x dy plus integral 0 to delta v del u by del y dy is equal to minus 1 by rho integral 0 to delta del p by del x dy plus nu into integral 0 to delta del square u by del y square dy equation number 3.

From calculus now, this equation we can see this is Navier-Stokes Equations in two dimension force, it is complicated equation, so we have to use certain techniques to say the momentum integral equation as proposed by Von-Karman. So from calculus we can write this integral 0 to delta del uv by del y dy can be written as this is equal to integral 0 to delta u del v by del y dy plus integral 0 to delta v del u by del y dy, so this is coming from calculus and now hence we can write integral 0 to delta v del u by del y dy is equal to say u v within the limits of 0 to delta minus integral 0 to delta u del u by del y dy as written in equation number 4 here.

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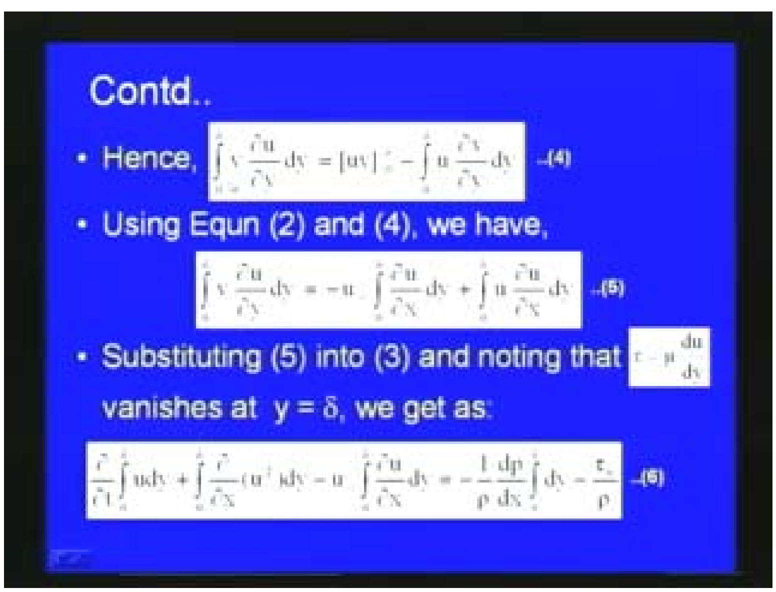

Here if you use equation number 2 which is the continuity equation. This equation now we will use this continuity equation here in this equation number 4 so that we can write this term integral 0 to delta v del u by del y dy is equal to minus $u_{infinitive,}$ so the u at delta that means at the edge of the boundary layer is $u_{infinitive,}$ so minus $u_{infinitive}$ which is the free stream velocity, integral 0 to delta del u by del x dy so this v at this location we can write as with respect del u by del x dy from the continuity equation plus integral 0 to delta u del u by del x dy as in equation number 5. Here now this approximation is put in this earlier equation number 3 here, so if you put here substituting 5 into equation number 3 and noting that so here we use the Newton's law viscosity.

So tow is equal to mu du by dy, so and noting that tow is equal to mu du by dy vanishes at y is equal to delta we get as we can write this equation number 3 as del by del t integral 0 to delta u dy plus integral 0 to delta del by del x u square dy minus $u_{infinitive}$ integral 0 to delta del u by del x dy is equal to minus 1 by rho dp by dx integral 0 to delta dy minus tow_0 by rho. Here rho is the fluid density tow_0 is the wall shear stress um at the bottom of the plate and the u and v are the velocity in x and y directions and delta is the boundary layer thickness. So now this equation number here to get a solution not a actually solution actually, he derived this momentum in the equation.

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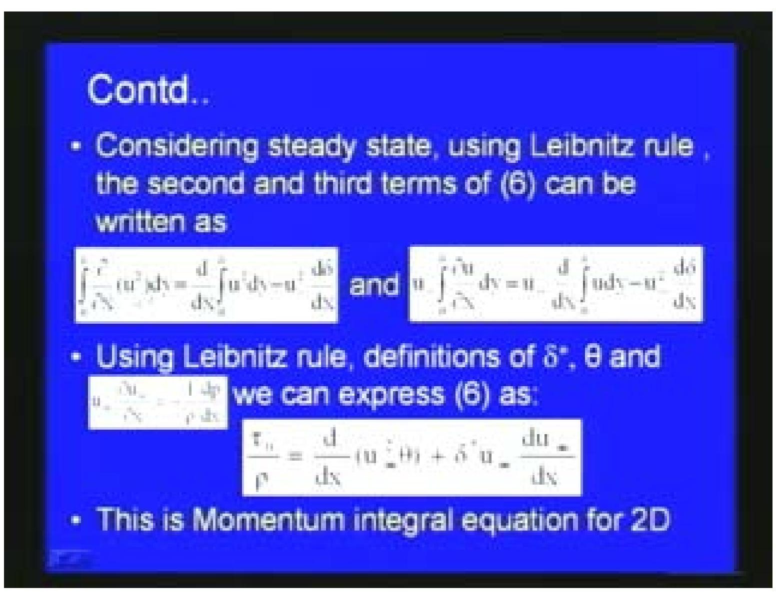

He considered the steady state problem, so he use the Leibnitz rule here and then second and third terms of equation 6 we can write here second and third terms here, this second and third term Von-Karman wrote like this integral 0 to delta del by del x so u square dy is equal to d by dx of integral 0 to delta u square dy minus $u_{infinitive}$ square d delta by dx and he wrote $u_{infinitive}$ integral 0 to delta del u by del x dy is equal to $u_{infinitive}$ d by dx of integral 0 to delta u dy minus $u_{infinitive}$ square d delta by del x. Now using Leibnitz rule and then using the definitions of delta star, the displacement thickness theta the momentum thickness and $u_{infinitive}$ del $u_{infinitive}$ by del x is equal to minus 1 by rho dp by dx, finally, Von-Karman derived the momentum in the equation.

He derived this equation as tow_0 by rho is equal to d by dx so $u_{infinitive}$ square theta plus delta star $u_{infinitive}$ $d_{infinitive}$ by $du_{infinitive}$ by dx. This equation is called momentum integral equation in two dimensions. So this was derived by Von-Karman he used the Prandtl's boundary layer equations and then he used the some transformations and Leibnitz rule and actually this equation is for steady state conditions and he used the various assumptions and also he used the fundamental definitions for the displacement thickness and momentum thickness and finally, Von-Karman got the boundary integral equation as tow_0 by rho is equal to d by dx so $u_{infinitive}$ square theta plus delta star $u_{infinitive}$ $d_{infinitive}$ by dx. So this equation is called momentum integral equations or Von-Karman momentum integral equations. Now this we can see that Von-Karman did not get solution for the problem but he transformed the Prandtl's boundary layer equation to another form of the equation called momentum integral equations.

Again this momentum integral equations, our purpose here is to get a solutions either in terms of velocity or shear stress or in terms of boundary layer thickness or displacement thickness etcetera, but what based upon the Prandtl's equations what Von-Karman got is the momentum integral equation which is an equation connecting the wall shear stress then the free stream velocity momentum thickness, the displacement thickness and the variation of the free stream velocity with respect to x. So this is the expression which is which has obtained by Von-Karman by using the Prandtl's boundary layer equations.

Now to get the solutions we have to say explicitly, now this equation is quite implicit it is having various terms like displacement thickness tow_0 then momentum thickness and the um Leibnitz derivations, the free stream velocity and its gradient, it is not explicitly in a solution, so we have to use say some assumptions or if you know some measure values so say boundary layer thickness or the velocity or the wall shear stress or the displacement thickness then we can find out other parameters, if one or two of the parameters are known then the remaining parameters can be found. So the momentum equation is called general and holds the laminar and turbulent flow, so the equations are valid for laminar as well as turbulent flow and provide velocity distribution u x y is known. So the velocity variation x and y directions are known then we can say this generalize equation momentum equations we can it holds for laminar as well as turbulent flow.

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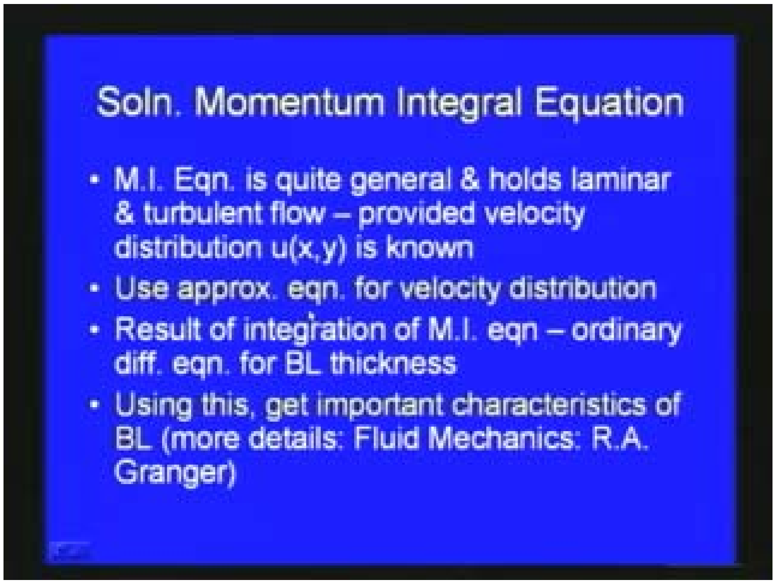

So to get a solution we say generally either we can start with approximation for the velocity distribution and then try to find out the boundary layer thickness or the momentum thickness or the delta star the displacement thickness or we can start with this parameters delta star delta and then come back to the velocity. So the result of integration of momentum integral equation ordinary differential equation for boundary layer thickness, so using this we can get important characteristics of boundary layer. You can see in various text books of fluid mechanics with just like in the case of Granger and others you can see how this momentum integral equation is used to we can various parameters as for as boundary layer is concerned. This is general expression as for as correcting various parameters, so generally what we can do here, as shown in this figure.

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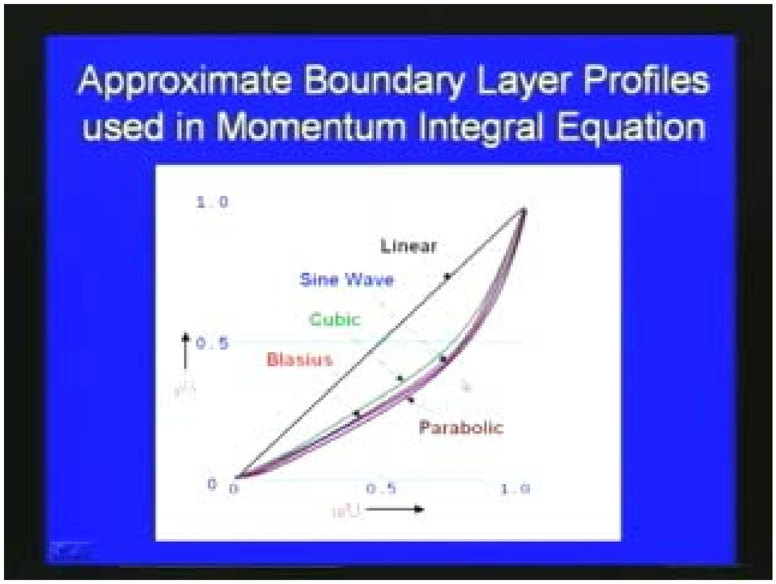

Here this figure shows the boundary layer profiles which we can use so the velocity u by u $u_{infinitive}$ with respect to y by delta, we can use various approximations like whether it is linear or we can use the what is proposed by Blasius or we can use the cubic variations or we can use the parabolic variations etc and then we can assume this velocity variations and then we can substitute that to the to the equations the momentum integral equations, so that we can find out the other parameters. This is the general way of solving this kinds of the problem using the momentum integral equations and then the pressure distribution in the boundary layer is concerned outside the boundary layer in the main flow from the Prandtl's equations we have already seen, u infinitive $du_{infinitive}$ by dx is equal to minus 1 by rho dp by dx here pressure does not vary in y direction.

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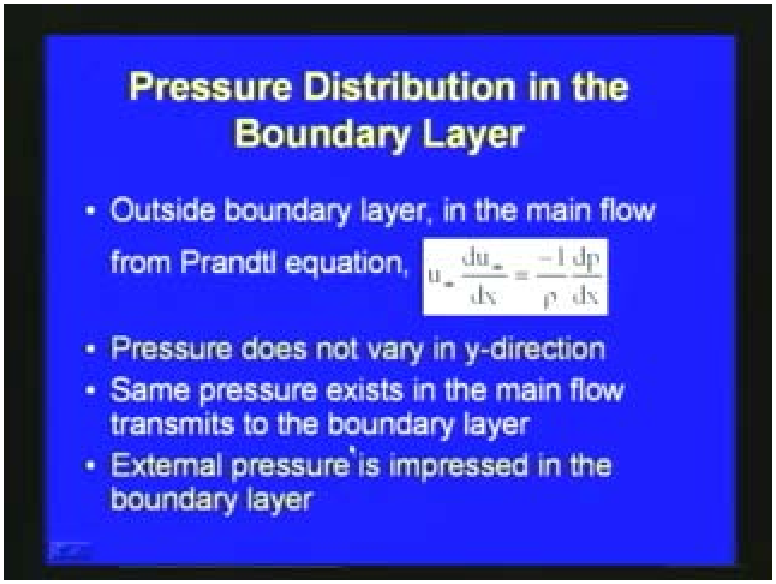

Same pressure exists in the main flow and transmits to the boundary layer. So we can say that external pressure is impressed in the boundary layer. As far as pressure distribution in the boundary layer is concerned, we can use these expressions and then we can see with respect this assumption here we can say that external pressure is impressed in the boundary layer and the pressure does not as far as this assumption is concerned the pressure does not vary in the y direction. So here we can see if say the flow surrounding a cylinder we can see the free stream velocity and the pressure variations the stream lines are rotates the flow around a circular cylinder stream lines are shown here and the also the pressure variation, the pressure distribution around a circular cylinder is plotted here for various conditions, so here P_0 plus half rho u square and here P_0 minus half rho u square here P_0 minus rho u square P_0 minus 3 by 2 rho u square the capital U is actually the free stream velocity.

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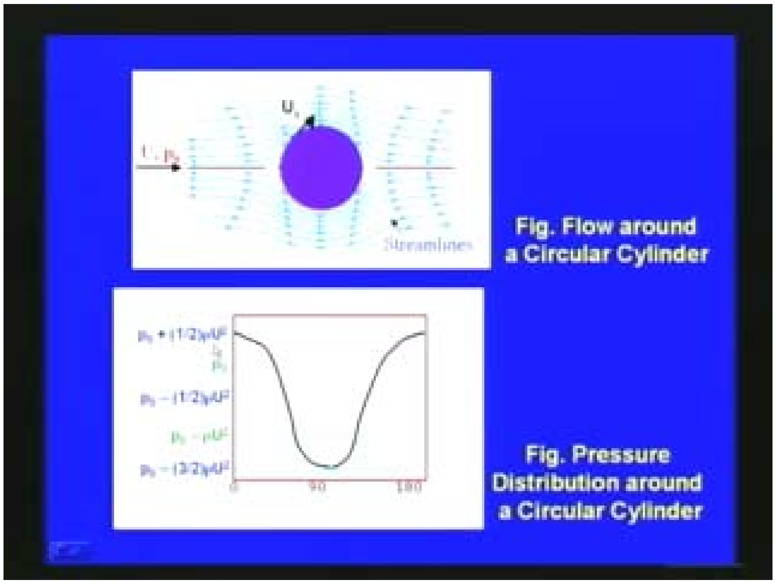

As in the earlier notation, it was U infinitive so here capital U is the free stream velocity. We can flat the pressure distribution also. Now, with respect to what we have accept here, what we have discussed so far, is the Prandtl's discovered the boundary layer equations and then various people tried to solve it by various assumptions including the momentum integral equations and then the pressure variation also can be found by using the various expressions