Introductory Nonlinear Dynamics
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Lecture - 13
The Baker and Horseshoe Maps.

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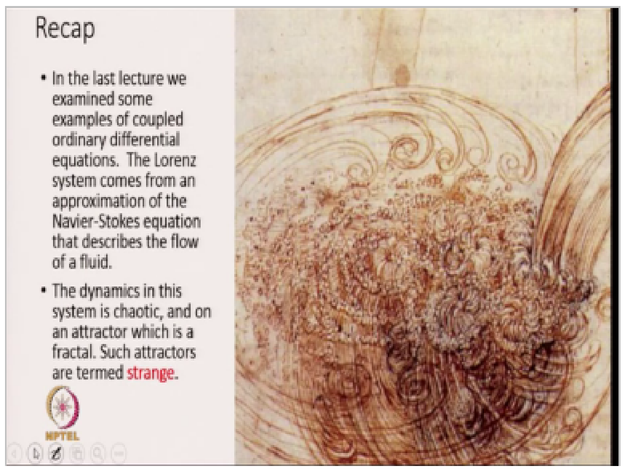
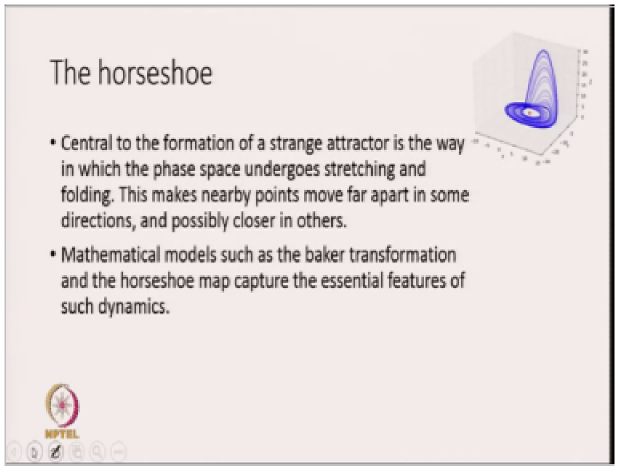
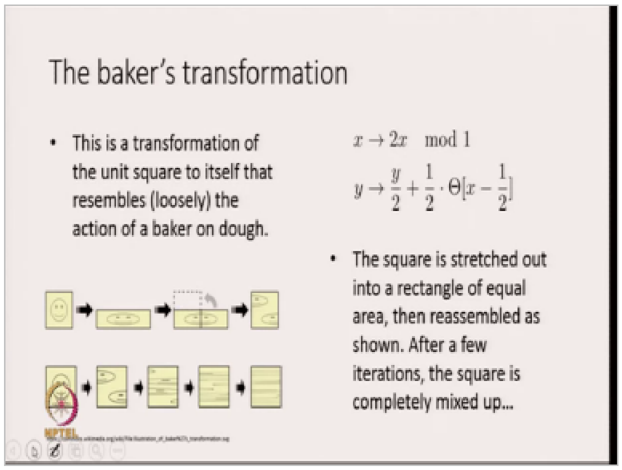


Today we will continue our discussion of flows and chaos in flows and so on via a discussion of two very paradigmatic examples of The Baker and The Horseshoe Maps. Recall that in the last lecture, we looked at some examples of coupled ordinary differential equations which come from a truncation or a simplification of the Naiver-Stokes equation. The Lorenz system which is being so important in the development of this subject, the motion in the Lorenz system as we saw was circulatory, it was a periodic, it was chaotic, and it was on a fractal attractor. Now, if the attractor is fractal and the motion is chaotic such an attractor is called a strange attractor. (Refer Slide Time: 01:17)



And what we saw is that there are examples of strange attractors not just in the Lorenz system, but also we saw in the Rossler and the Rossler which is sort of imaged over here in the upper right hand corner; the way in which the motion goes on the Rossler attractor is that it goes circulatory for a little while and then it goes up into the third dimension and then it again circulates in the x and y plane and so on. Central to the formation of a strange attractor is the way in which phase space undergo stretching and folding. We saw simple examples of that in the image of the taffy machine but the way in which phase space dynamics occurs in a in this in this kind of a dynamical system is that volumes get bent over and then stretched apart so that nearby points move can move very far from each other and even though they may move closer in other directions. There are two or there are a few important models two of which we will discuss today and these are known respectively as the baker transformation and the horseshoe map. (Refer Slide Time: 02:43)

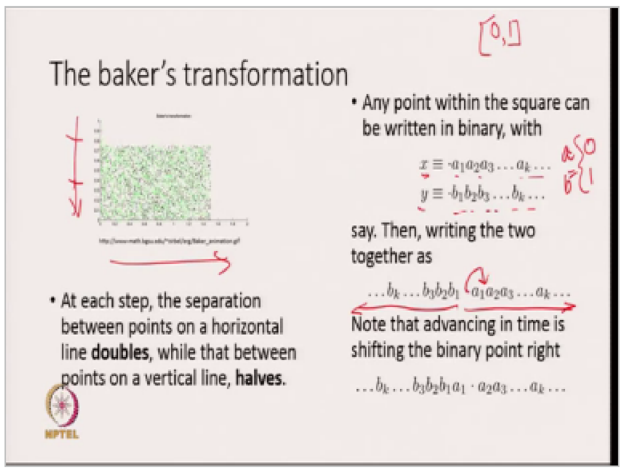


So, the baker transformation is a; it is a transformation of a square onto itself and it loosely resembles the action of a baker as he or she would be kneading dough. So, or to make a more familiar example in our context how you how one mixes the dough for chapatis or for to make a naan or something like that ok. So, the action of this baker action is the following; you start with a unit square, you stretch it out into a rectangle of half the height and twice the length so that the area is the same, then you cut the rectangle into two and put the second half of this rectangle onto the first, so, that you get back a square. Through this transformation you can see that the area of the square has not changed. After one transformation you can see that the square is such that those two points let us say, the two eyes of this little figure over here which were close by in the inertial in as we started out, after one iteration of this or one transfer one application of this transformation one eye is on the lower part of the square and the other eye is on the upper part. If you do it again you can see now that things have got stretched out in this x direction and the two eyes so to speak are at two different locations over here. You would apply it once more and now you have got a completely messed up figure, but the area has remained the same. The actual map itself written algebraically is the following;

$$x \to 2xmod1$$

$$y \to \frac{y}{2} + \frac{1}{2}\Theta[x - \frac{1}{2}]$$

that was the first part of this transformation. In the second part of the transformation we apply the mod one operation so, all these points are not no longer bigger than 1 they are taken back to this side and you add one half depending on where x was. So, if you just sit down with a piece of paper and work it out, you can see that this particular mapping is exactly what the baker transformation is all right. So, after a few iterations the square is completely mixed up and this is a model which we can analyze in some more detail. (Refer Slide Time: 05:38)



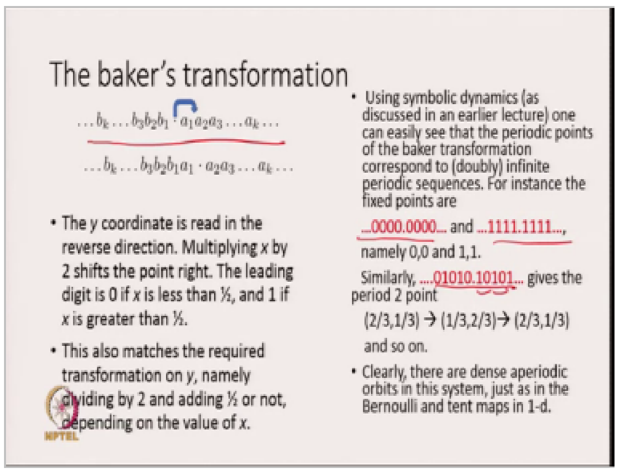
So, here it here is an very nice animation that I found on the net and you can see what is happening over here. You start with the inertial square and well let me just come back to it in a moment, you had a square which started out with one side red and one side green and one side black and then as you keep applying it, you find that there are successively more strips of green and black until finally, all the points are completely mixed up and you can see that the action is exactly as it was on that smiley face that we started with. At each step now the separation between points on the horizontal lines this doubles, because we are stretching out by a factor of 2 and points on a vertical line their distance decreases and in fact, it becomes a half. So, points in the vertical direction come closer together and in the horizontal direction they spread far apart. Now you can actually do much more with this particular map. Any point in [0,1] can be written in binary expansion as we have done in various examples in the past. So, if I were to write the x coordinate as

$$x = .a_1 a_2 a_3 ... a_k ...$$

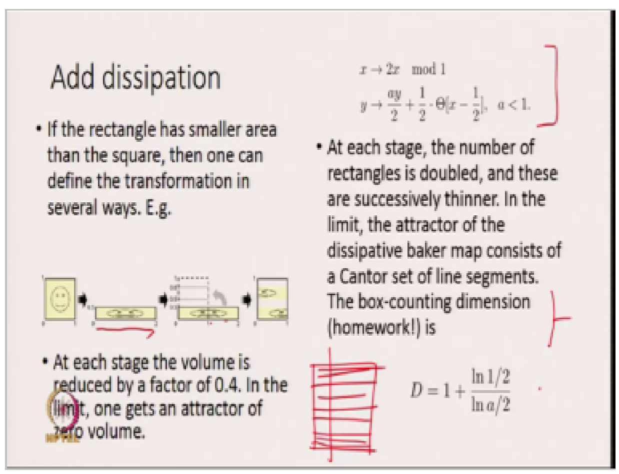
all the way the sequence as long as it is required to infinity and if I write the y coordinate as

$$4y = .b_1 b_2 b_3 ... b_k ...$$

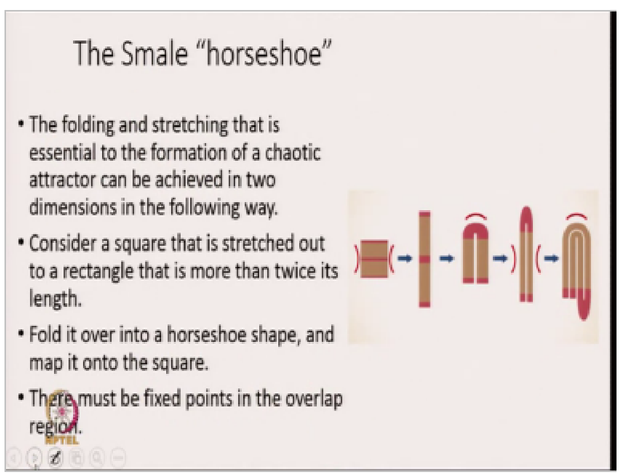
also in binary so, the symbols a and b are either 0 or 1 right. So, all the as and bs are 0 and 1 and this is a unique index of each point x , y. Now if I should write both of them together, the x coordinate in the forward direction and the y coordinate in reverse direction note that advancing in time is the same as taking this point and shifting it one space to the to the right. So, if I start with a_1, a_2, a_3 etcetera and b_1, b_2, b_3 this way after one iteration the x coordinate becomes $a_2 a_3$ all the way up till a_k, whereas the y coordinate becomes $a_1 b_1 b_2 b_3$ etcetera going down this way. (Refer Slide Time: 08:38)



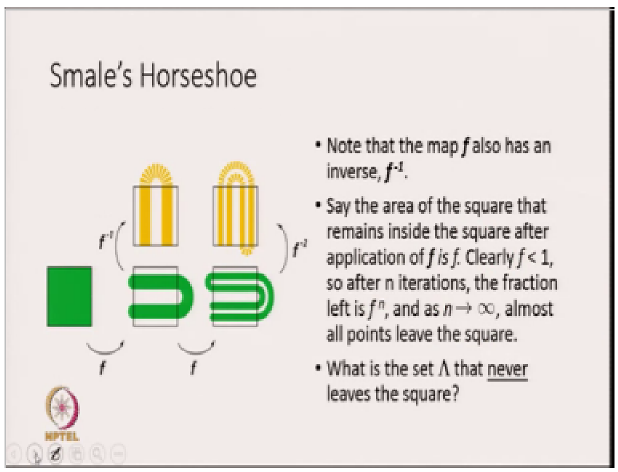
Now, this is a very nice feature of this particular map and you can see why this happens, since we are reading the y coordinate in the reverse direction multiplying x by two shifts the point to the right there is no surprise in that, because we know how we have done this before in other examples. Now, what about y? You see if a 1 is 0 then the x coordinate is less than a half. If a 1 is 1 the x coordinate is bigger than a half. Now we also know that if the x coordinate was less than a half, then the y coordinate only gets halved whereas, if the x coordinate was bigger than a half, then the y coordinate gets a halved and you have to add 1 halve to that. That is the point of this algebraic representation over here, y goes to y by 2 plus 1 half and that depends on whether x was bigger or less than half all right. So, again a piece of paper and just working it out and you can convince yourself that this is exactly equivalent to the action of the baker transformation. You just move your binary point one step to the right to go forward in time. Now this means that we can use symbolic dynamics very effectively in this system, just as we have in earlier cases and all the points on the square can be written down in this binary representation and as since the dynamics forward in time just go you know is moving the point to the right, going backwards in time is moving the point to the left and you can immediately infer that there are two fixed points that is the sequence 0 0 0 and 1 1 1 1 1 etcetera because and they are the end points of the square. If you take this periodic sequence over here, you get a period 2 point as you would expect because this periodic sequence is a periodic is a symbol sequence of length 2.There are you know all the things that we have discussed in the case of the Bernoulli map are going to be true over here; there are dense a periodic orbits, there are many different kinds of periodic orbits, there is a there are through all the interesting kind of symbolic dynamics can be done in this system. (Refer Slide Time: 11:33)



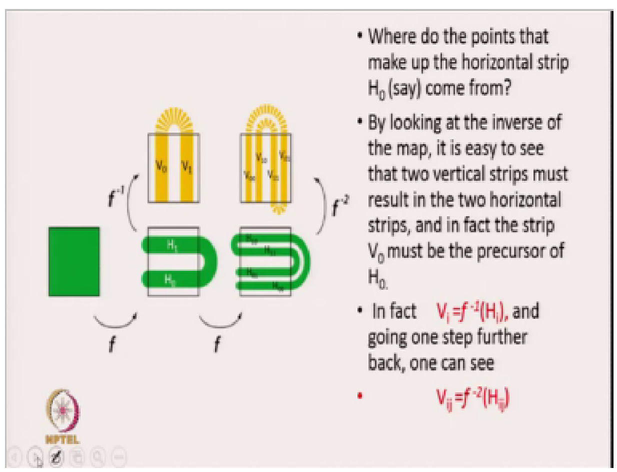
If you were to add dissipation and the way in which we would do that is to multiply let us say the you still go you increase the x coordinate by a factor of 2 but instead of instead of halving the original square, you multiply it by a factor which is bigger than sorry which is less than a half. So, in this particular example over here, this square is stretched out by a factor of 2 but the height is reduced to 0.3 of the original height so, the area now is 0.6 rather than 1. And when you cut the second half of it and put it back on top you take care to put it in the upper half. So, now, you have the square with 2 gaps in it essentially. So, at each stage the volume is reduced by a factor of 0.4 and if you keep doing this successively, you will get an attractor essentially of 0 volume. At each stage therefore, I mean this is the algebraic version of the mapping and you just start getting two rectangles at stage 1, 4 at stage 2, double that at stage three, double that at stage four and so on and so forth. And each of these are going to be getting thinner and thinner because they are now getting multiplied by this factor a to the power k. In the limit the attractor of this dissipative baker map is actually a set of lines if I may just draw it over here finally, you will find a whole set of lines that are there which are just covering this square completely. If you examine it carefully it is a cantor set in one direction and of course, the line and the other direction and in the next homework you will see you will have to calculate the box counting dimension of this and show that it is this particular quantity. So, the point of looking at the baker map is that if you have got a conservative baker map, we know that the dynamics is as this can be very very complicated, it can be just like the Bernoulli system or the tent map or any of those chaotic examples that we have looked at. It is however, area preserving. So, there are no attractors. If you add some dissipation you get an attractor and the price that one pays for it is that the volume of this attractive is of course, 0 and we are not able to do the symbolic dynamics in quite the same way. Nevertheless, is this is an important example. (Refer Slide Time: 14:40)



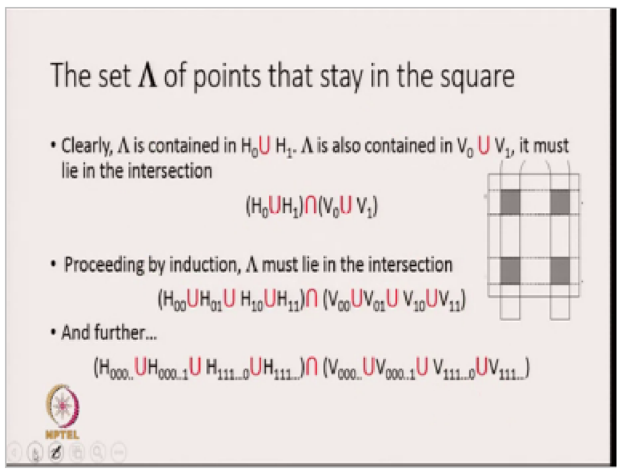
Proceeding some of these ideas was a very important notion that was introduced by the mathematician Stephen Smale and it goes under the name of Smales horseshoe. This is a good example of how stretching and folding can be can be looked at mathematically and this stretching and folding as we have been emphasizing is essential to the formation of chaotic attractors. So, this is a model in two dimensions and again one starts with a square. Now, this square is stretched out into a rectangle that is longer than the original square was I mean it is more than twice the length of the original square and it is shrunk down by a factor. Some of these details are not that important, but the basic the basic idea is the following; that you start with a square, stretch it out, fold it into a u shape as I have shown over here this is the horseshoe shape and then place it back onto the square right. And we ask for the overlap between the original square and the horseshoe and then one keeps on doing that over and over again as one can see over here. Since you have taken the original square elongated it, shrunk it in one other dimension and turned it around and put it back it is interesting to ask whether there are any fixed points or any points that remain inside the square after this operation has been done and clearly, because this is a map which is on to itself there must be a fixed point somewhere. (Refer Slide Time: 16:41)



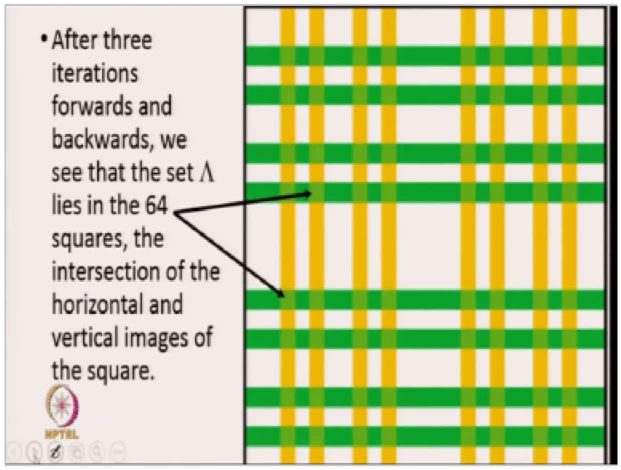
So, here is a slightly better image of the Smales horseshoe. So, here is the original square all coloured green, it is pulled out stretched into a u turn folded into a u and put back onto the original square and here is the act action of f again. So, this is pulled out twice and again turned around and now you can see how this at the action of this of the mapping is when applied twice. Note that this map also has an inverse because the way in which it was stretched out now you can imagine going backwards in time or one step back and there must have been 2 vertical strip strips that when you stretch them out and folded them again would give you exactly this image over here and here is the map backwards in time two steps it will now consist of vertical strips over here. So, let us just take a step back and note that after I have done the action of the map once I get 2 horizontal rectangles over here; when I apply the map again, I get 4 horizontal rectangles over here. The pre-image of these 2 horizontal rectangles if you like are these 2 vertical rectangles over here in yellow and then 2 steps back from here will give me 4 vertical steps and so on. Now if the area of the square that remains inside the square after application of f is lowercase f and the is this quantity f over here you can see that actually f has to be less than 1, because there is a smaller area that is remaining in the inside the square. So, after n iterations the fraction that would be left is f to the n and as n goes to infinity all the points essentially leave the square. So, we can ask the question is the are there points that never least leave the square and we call that set lambda. (Refer Slide Time: 19:16)



Now, if you should ask the detailed question by giving notation over here so, this is the horizontal strip it is 0, this is the horizontal strip H 1. After two steps this is H 0 0 this is 0 1 1 1 and 1 0 and the reason for these subscripts you can easily figure out that when I stretch out this original square and fold it around and put it back, the origin of so, this lower part which is H 0 this part comes from H 0 and therefore, H 0 0 this part will come from H 1 and hence H 0 1 and in the same way back you can figure out since it has been folded over, this is H 1 0 and this is H 1 1. The same logic a little more pencil and paper work at home and you can convince yourself that I can give this the notation V 0 and V 1 for the vertical strips 0 and 1 and their histories will go back in time like so, so, that this plus this essentially come from here and this and this come up from the other one over there all right. Now where do the points that make up the horizontal strip H 0 come from? You can look at the inverse of the map and you can see that this must essentially come from V 0 because when I stretch out this to a longest into a longer rectangle and then flip it over all the points from here essentially goes straight into H 0. Likewise, all the points from V 1 will go into H 1. As a matter of fact, I can write introduce some notation and say that v sub i is just f inverse of H sub i and going back another step you see that vij is the map twice backwards on H i j. (Refer Slide Time: 21:49)



So, the set of points that will stay in the square forever can now be deduced by construction. Now, at stage 1 itself, the points that will stay in the future must either be an H_0 or H_1 they must have come from either V_0 or V_1 and therefore, they must lie in this intersection $(H_0 \cup H_1) \cap (V_0 \cup V_1)$. One step further back, they must lie in the intersection of this union and this union namely two steps back on both sides. Now, I can go back three steps or four steps or n steps and therefore, I will find some long union of strips over here, horizontal strips and long union of a vertical strips and their intersection tell me that this set lambda must stay inside this particular intersection. So, here is the intersection for the 3 for 3 iterations. So, you find that there are 64 such squares that lie at the intersection of the green and the yellow. So, all these light coloured squares that we see over here that is where the set lambda must lie. (Refer Slide Time: 23:04)



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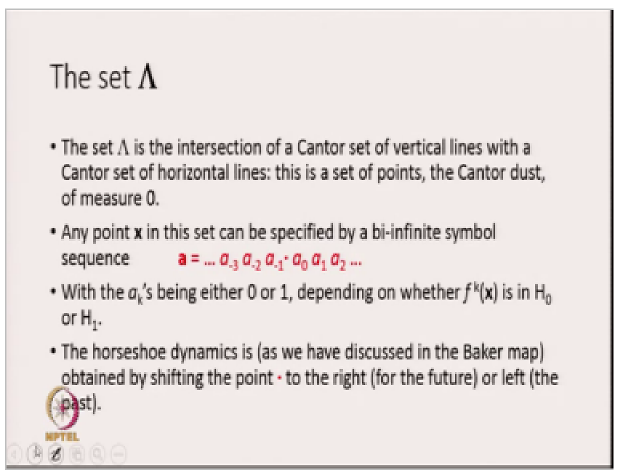
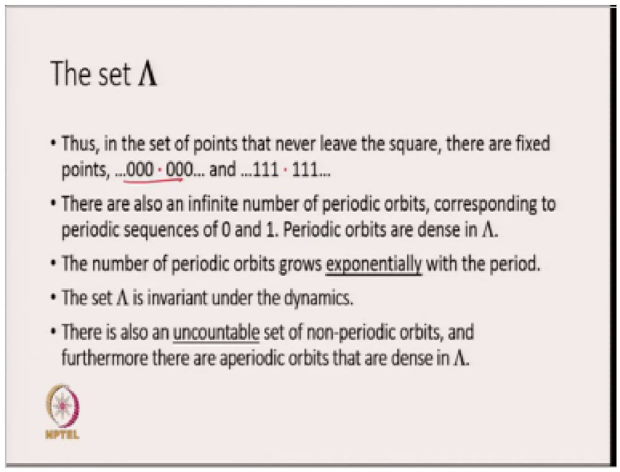
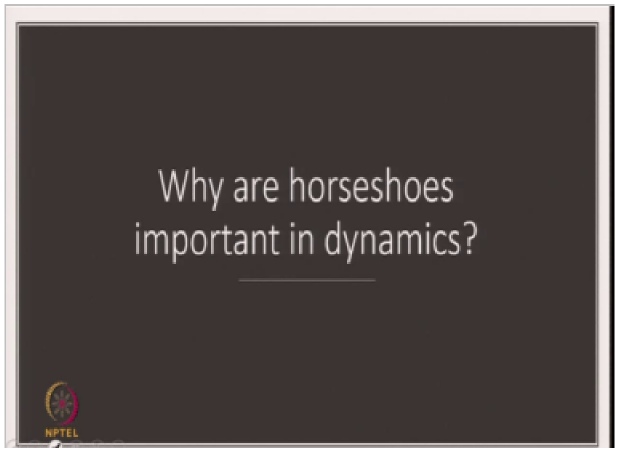


And the set itself as you can deduce now is the intersection of a cantor set of vertical lines and a cantor set of horizontal lines and this is just a set of points which is a cantor dust. Itself it has measure 0, but it is an attractor and any point in this sequence can clearly be specified by a by infinite symbol sequence $a_0a_1a_2$ etcetera $a_{-1}a_{-2}a_{-3}$ etcetera with the a_ks are either 0 or 1 depending on whether $f^k(x)$ is in H_0 or in H_1. So, depending on whether it comes into the horizontal H_0 or H_1 that is the number of you know so, that tells you whether these as are either 0 or 1. The same kind of symbolic dynamics can be done and we have discussed this in the baker map just now. So, we keep shifting the point to the right for the future or the left for the past and one can see that the dynamics is going to be really complicated, but we can describe it. (Refer Slide Time: 24:56)



There are fixed points (0,0,0) and (1,1,1). There are an infinite number of periodic orbits which correspond to periodic sequences of 0 and 1. As a matter of fact periodic orbits are dense in lambda. The number of periodic orbits will grow exponentially with the period, it is just the number of sequences of two symbols that you can write of length n. Ah this is an invariant set so, once you are in this set you just you know under the dynamics you just keep circulating inside this set and there are an uncountable number of non-periodic orbits and furthermore there are a periodic orbits that are dense in lambda. So, this dynamics is extremely complicated, but there you have a very simple model the horseshoe which gives you this dynamics. (Refer Slide Time: 25:58)



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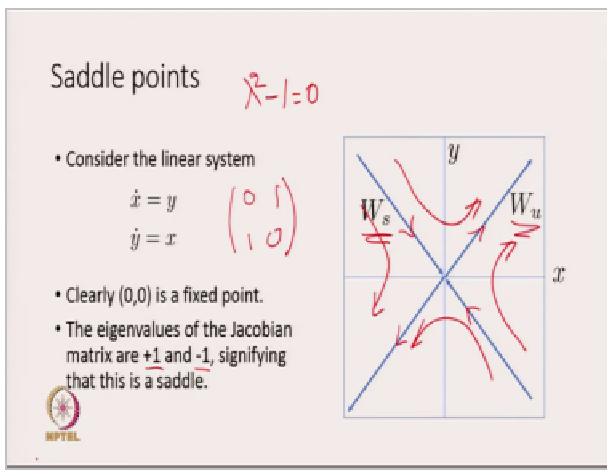
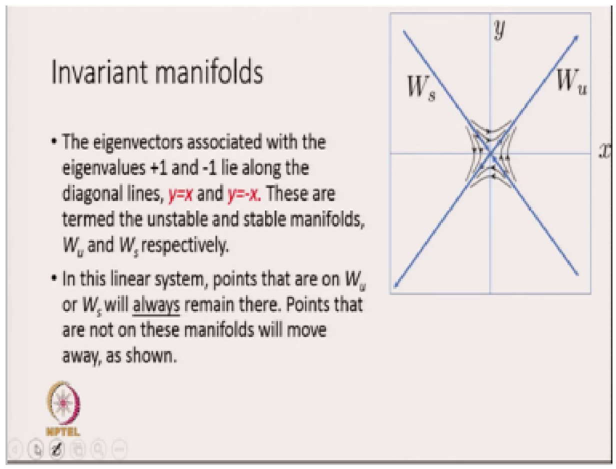


So, why are horseshoes important in the dynamics? The simple answer is that horseshoes are important because they occur all over the place and to see that lets consider a simple linear system, well let us introduce the context via the simple linear system

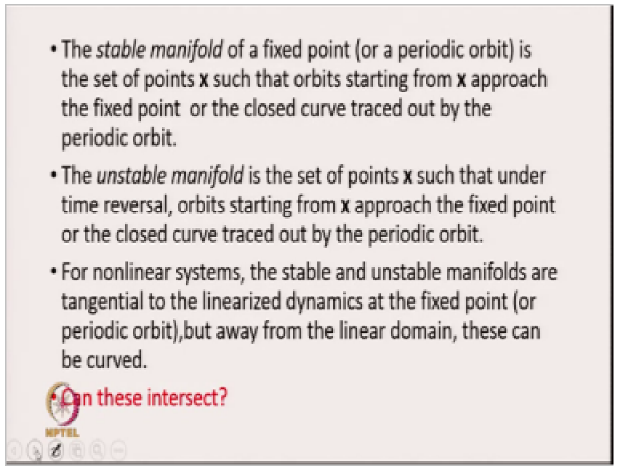
$$\dot{x} = y$$

$$\dot{y} = x$$

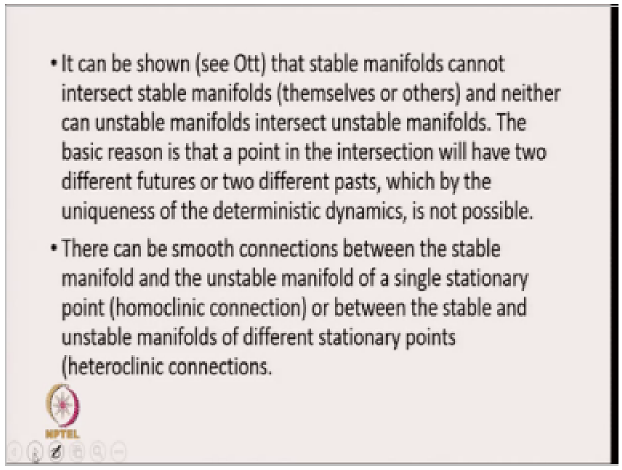
. Now clearly (0,0) is a fixed point the Jacobian is 0 1 1 0 and the eigenvalues of this characteristic equation lambda squared minus 1 is equal to 0. They are plus 1 and minus 1 signifying that this fixed point the origin is a saddle. So, this saddle this is the inward or this inward direction or the stable part of the saddle. This is the unstable direction so, the unstable the unstable direct unstable manifold the unstable direction and trajectories are clearly moving away like so; except of course, on these directions on the eigenvectors. (Refer Slide Time: 27:23)



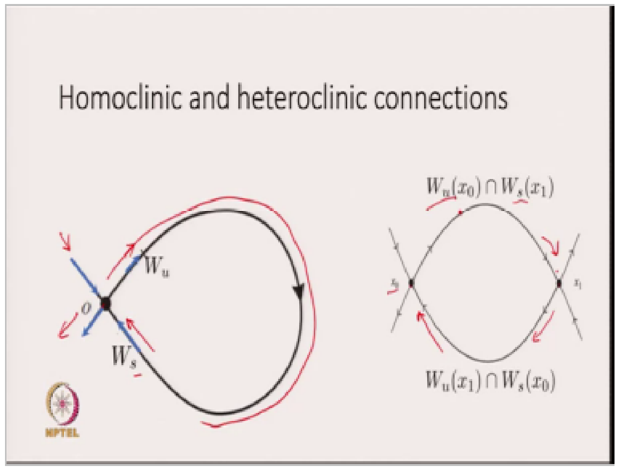
So, the eigenvectors which are associated with the eigenvalues plus 1 and minus 1, they will lie along the diagonal lines $y = x$ and $y = -x$. Now, these we term the stable and the unstable manifolds respectively and give them the subscripts u and s. Now in this linear system points that are on W_u or W_s will always remain on these manifolds. Points which are not there like I already indicated if a point started from here then it will flow in this direction but then it will eventually move out whereas, a point which is on this manifold has to stay on this manifold and eventually reach the fixed point at any you know it infinitely further in the future. (Refer Slide Time: 28:27)



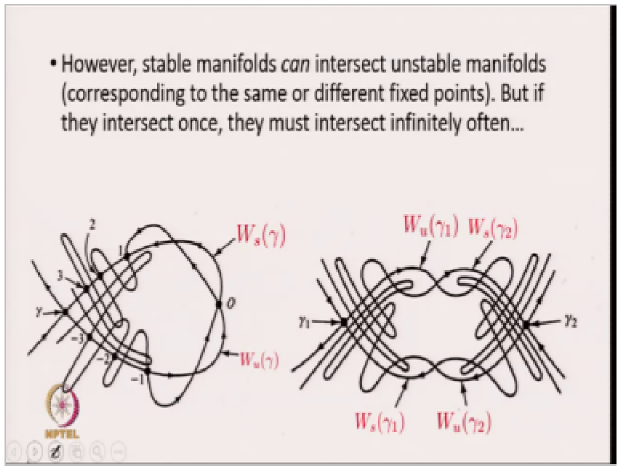
This allows us to define the stable manifold of a fixed point or in fact, for a periodic orbit, is the set of points x such that orbits which start from x approach the fixed point or this periodic orbit as T goes to infinity conversely. The unstable manifold is the set of points x such that under time reversal orbits starting from x will approach the fixed point or the closed curve traced out by the periodic orbit. Or to put it in other words points on the unstable manifold in the infinite past were at the fixed point, points on the stable manifold in the infinite future go to the fixed point. Now, for non-linear systems these stable and unstable manifolds are not the straight lines that we see over here, because they are only straight very close to the fixed point away from the fixed point in a non-linear system the manifolds can actually be curved and when they are curved one can ask other questions of them because they are no longer constrained to be always orthogonal to one another. So, the question one would ask is for non-linear systems the stable and unstable manifolds are tangential to the linearized dynamics near the fixed point. But away from this linear domain, they these stable and unstable manifolds are curved or can be curved and if they are curved the question that arises is can they intersect. (Refer Slide Time: 30:29)



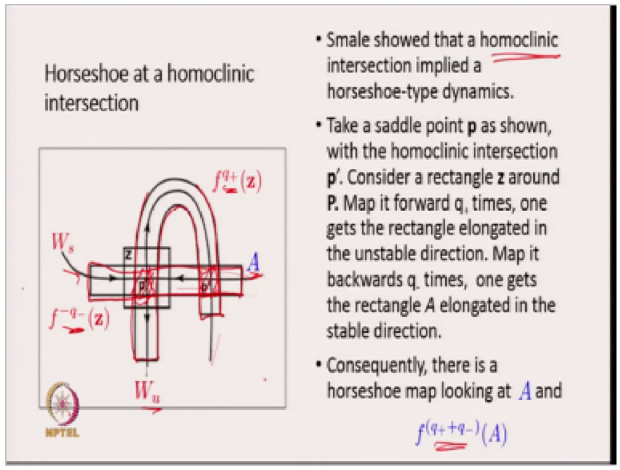
Now it can be shown, and you should see the textbook for a discussion for a very nice discussion of this that stable manifolds cannot intersect stable manifolds either themselves or others and unstable manifolds cannot intersect unstable manifolds either themselves or others and the basic reason is that the dynamics is unique in the phase space and a point at the intersection will have two different futures if it is on two and two stable manifolds, then it has to go to two different fixed points or conversely come from two different fixed points or two different parts. And since you cannot have two different futures or two different parts this is not possible, because the deterministic dynamics is unique on in the phase space. They can; however, be smooth connections between the stable and the unstable manifolds of a single stationary point and this is called a homoclinic connection or between the stable and unstable manifolds of different stationary points which are known as hetero clinic connections and we will see what they look like. (Refer Slide Time: 31:51)



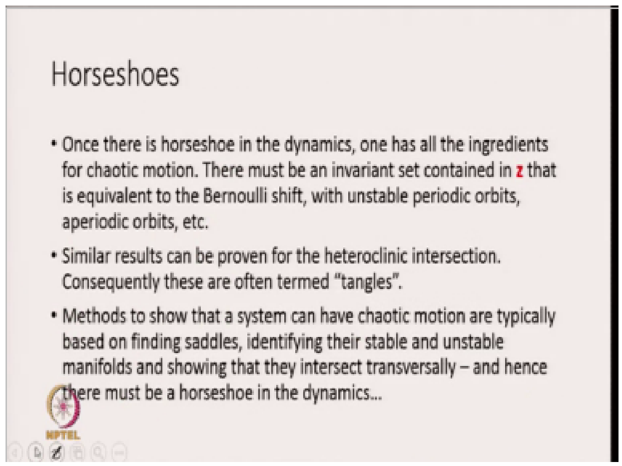
So, here is a point o with which is in a non-linear system, this is the stable direction so, here is your stable manifold and here is the unstable manifold and you can see that the unstable manifold turns around, bends around and then smoothly joins the stable manifold. So, a point which in the infinite past was at the fixed point we will also come back to the fixed point in the infinite future. So, this is your homo clinic connection and in the heteroclinic case, here is your point x_0. The unstable manifold of x_0 smoothly joins the stable manifold of x_1 and the stable unstable manifold ofx_1 smoothly joins the stable manifold of x_0 and there is no contradiction over here because a point over here in the infinite past was at as the infinite future is at x_1 and this is actually a very common kind of an example these are separate traces which separate different kinds of motion. So, hetero clinic connections and homo clinic connections are actually quite common. (Refer Slide Time: 33:17)



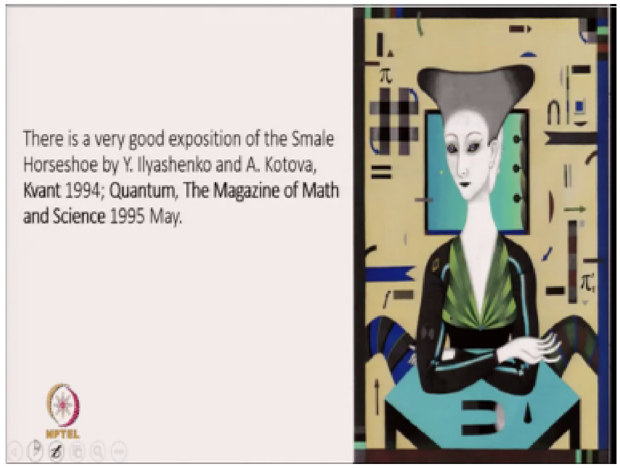
However, stable manifolds can intersect unstable manifolds right corresponding to the same or different fixed points, but it comes with a caveat. If they intersect once they must intersect infinitely often. Here is an example of a homoclinic intersection. So, here is the point gamma and the stable manifold of gamma is this direction, well here you can see the stable manifold of gamma and the unstable manifold of gamma. So, the unstable direction is moving on this side and the stable direction is moving this side. Now, o is a point of intersection of the stable and the unstable manifolds, but now you see that o lies both at the fixed point in the infinite past as well as in the infinite future. Imagine that it gets translated in time by one step, if it now reaches this point over here after one step it is again at the intersection of the stable manifold and the unstable manifold. If it is now at the intersection of the stable and the unstable manifold again then its image must also lie on the stable manifold and its image and its image and so on and the same argument goes true for the past. So, now we are stuck with the fact that every point of intersection its image must also lie on the stable manifold and must have line on the unstable manifold and this since this will only reach the fixed point in infinite time, every intersection has got to be repeated infinitely often. The same argument goes for the case of the hetero clinic intersection. Again, if they intersect once they must intersect infinitely often. These figures are taken from horse book and there you can see the discussion based on that. (Refer Slide Time: 35:38)



Now, the importance of the homo clinic or the hetero clinic intersection is in this observation of Smale that a homo clinic intersection or a hetero clinic intersection implies a horseshoe-type dynamics. So, to see this let us look at the fixed point p and the homo clinic intersection p prime over here. So, if I have got p and this is my unstable manifold W u marked over here, this is the stable direction W s right and I have just moved it out to make the horseshoe more evident. Now, as one maps this, take a square z that is that is marked out also over here as time progresses, z will get stretched out along the unstable direction. It will get pulled out into a longer shape along the unstable direction and let us say that after some amount of iteration q plus it becomes this particular rectangle that you see over here like so. The, in reverse time this rectangle z is going to get stretched out along the stable manifold because the stable manifold is defined as all those points that were that in the infinite future are going to be in the at the fixed point. So, in the infinite past they were really stretched out along the along the stable direction. So, in after q minus steps in the past, the image of z is this long rectangle over here which I have written down as A. Now, if you look at A you can see clearly that there is a horseshoe map which is sort of you know leaping out over out of the screen at you, because if I take a and map it q minus plus q plus times that is if I map it so many times then this long rectangle goes into this horseshoe shaped rectangle and I have got these two points of intersection that have come back to the original rectangle. So, yeah so that is that is the basic argument of Smale that any homoclinic intersection implies that there is a horseshoe type dynamics and once I have got this horseshoe dynamics for A after so, many steps then I know that in these two regions I must have points which are periodic with any arbitrary period there must be dense a periodic orbits and so on and so forth. (Refer Slide Time: 38:59)



Because once there is a horseshoe in the dynamics one has all the ingredients for chaotic motion, there must be invariant sets contained in z that then these are equivalent to the Bernoulli shift, there must be unstable periodic orbits and so on. Similarly one can come up with the same argument for the hetero clinic intersection and as a matter of fact given the kind of image that these have because the motion is so complicated, these are often called homo clinic or heteroclinic tangles. Now, methods to show that a system can have chaotic motion in the first place are typically based on finding saddles in the phase space, identifying their stable and unstable manifolds and then showing that these stable and unstable manifolds instead of joining smoothly, they intersect transversely and therefore, there must be a horseshoe in the dynamics and because there is a horseshoe there must be chaos and so on and so forth. (Refer Slide Time: 40:10)



For those of you who may be interested in following some of these arguments with a little more detail, there is a very nice exposition of the Smale Horseshoe in this in an article in and now sadly defunct magazine called Kvant it usually published in the Soviet in then Soviet union Russia by it is by Y. Ilyashenko and A. Kotova this was translated into English and in again is sadly short lived magazine called Quantum. If you search for it on the net you will find it in 1995 the May issue ok. These are separated by this illustration is from the quantum magazine; that is it.