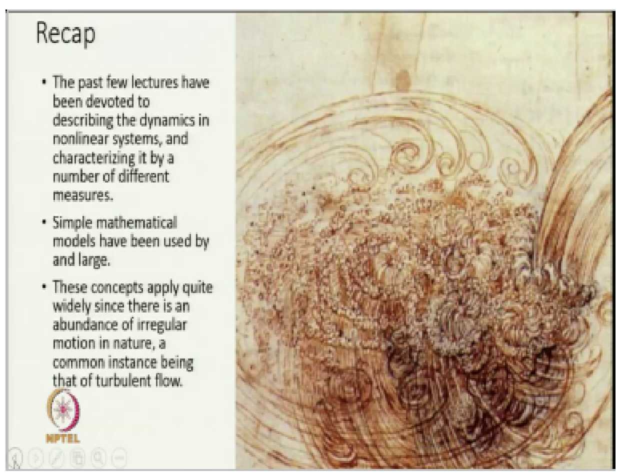
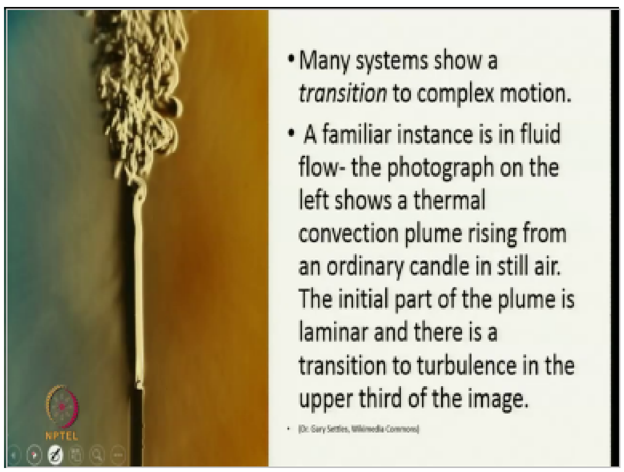
Introductory Nonlinear Dynamics
Prof. Ramakrishna Ramaswamy
Department of Chemistry
Indian Institute of Technology, Delhi
Lecture - 12
Chaos in Flows - The Lorenz and Rossler Systems.

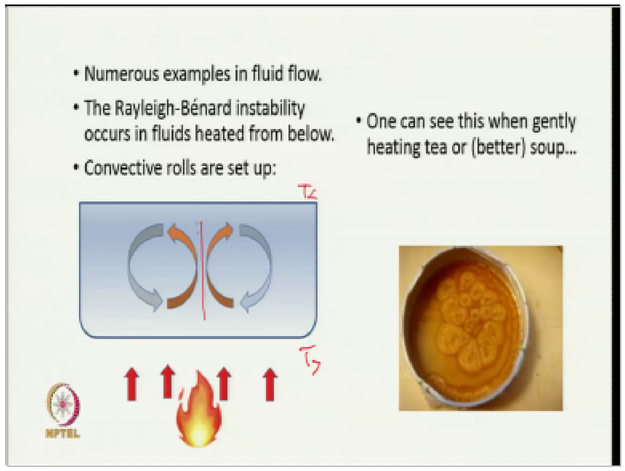
Hello. Today we are going to be talking about Chaos in Flows and particularly about the Lorenz and the Rosslers Systems which are very archetypical chaotic systems. In the last week, weeks rather we have been worrying about how to describe dynamics in non-linear systems, we worrying a lot about how to characterize it using a variety of different measures, Lyapunov exponents, fractal dimensions and so on. And, by enlarge we have used very simple mathematical models to illustrate the concepts that have been introduced. (Refer Slide Time: 01:01)



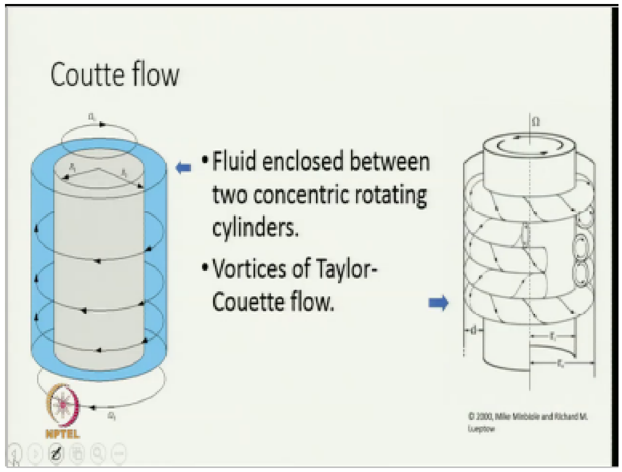
These concepts it turns out apply quite widely because in nature there is an abundance of irregular motion. Common examples of irregular motion include turbulent flows such as rivers or you know the roads of Delhi during of the rain and whenever you have a lot of water that has to move from one place to another, the flow can be extremely complicated. The image that you see over here on the right of the screen is from an illustration by Leonardo da Vinci. And, it dates back to a few hundred years ago and he was one of the first people to describe what turbulent motion looks like. And, you can see over here that as he has observed very keenly, turbulent motion seems to have a lot of water or fluid that is moving around in some kind of a circular fashion. The image that you see on your screen to the right is from an illustration by Leonardo da Vinci. It dates back to the 1600s and Da Vinci describes what turbulent, the structure of turbulent motion. If you have a fluid where the whether be liquid in this particular case is moving around in a sort of a crazy fashion, you have these very interesting; you have this interesting motion, where there are walls you know movement of these large scale motion of the fluid in a circular fashion over here. And this then feeds into a smaller ball and these have got sort of an interesting vorticity structure over here these smaller walls and so on and then, you have these tiny little walls all over the place. Now, this image of turbulence is still a major challenge to most to theoretical physicists. In fact, it is supposed to be one of the large unsolved problems in physics. (Refer Slide Time: 03:20)



Now, even simpler if you light a candle and switch it off, you find that this candle if you know if you have got the conditions exactly correct, the smoke from the candle flows upwards and then at a certain point there is an instability and it starts becoming turbulent. This is an example of what is termed as transition to turbulence and many systems have this namely you have a regime where the motion is smooth. This entire regime the motion is very smooth and then there is an instability of some kind and then it becomes turbulent. So, this is a thermal convection plume arising from an ordinary candle in air. (Refer Slide Time: 04:19)



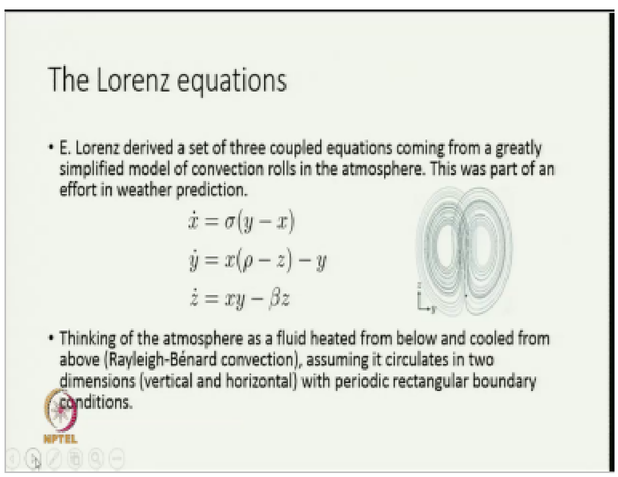
Examples like this are very very common as are a whole lot of other examples in fluid flow. Every morning when you make a cup of tea or actually when you make a cup of soup, we witness the Rayleigh Benard instability. And, this is an instability that happens when fluids are heated from one side and like typically as shown over here, you have a fluid being heated from below and it is open to the air up here. So, the temperature on top is lower than the temperature over here the temperature below. So, there is a temperature gradient and because of the temperature gradient what happens in the fluid is that convective roles are set up. The fluid rises from here and now since it has to since it cannot go further up, it goes to the surface of the fluid and then it comes down, so that you set up these convective flows of motion in a fluid. You can actually see this rather nicely when you heat soup. A soup has got a little higher viscosity, a higher density than ordinary water. So, when you are making one of these packaged soups for example and boiling it, you find that these convective roles are set up and the image of these convective roles on the surface turns out to be these hexagonal patterns that you are that are visible in this particular cup. These are so common that one finds many many such images on the internet and I have taken this particular image from precisely the internet. (Refer Slide Time: 06:12)



More interesting in many ways still worthy of a lot of study in instability which is common in fluids and again it was found you know in a lot of everyday types of flows is the so-called Couette flow. Now this Couette flow occurs when you have got a fluid in case between two concentric rotating cylinders. So, the image on the left over here shows this geometry. You have got one cylinder of radius R1 and that is rotating at a angular velocity of omega 1. The outer cylinder has got radius R2 and that is rotating at the angular velocity of omega 2. Fluid is enclosed between these two cylinders and as you rotate them as the angular as the difference in the angular velocity is changing, you find phenomena dynamical phenomena that occurs because of what is known as the Taylor Couette in the Taylor Instability. So, you have got this Couette flow geometry and what happens as the angular velocity difference increases is that again convective rolls are a set up and this time these rows occur as sort of toroidal flows between these two walls. In most experiments it is quite common to keep the inner cylinder rotating while the outer cylinder is fixed. So, this is a typical geometry that you study this system in and when you do that you find that these convective rolls are set up like little vortices between the inner and the outer walls and depending on the distance between the inner and the outer walls you setup vortices of different patterns. This whole problem is extremely interesting and as I said it is still subject to a lot of study. Now, one example where we find this the Coutte flow that gets used in a practical situation is in South India where the where one you know the in order to make dough for dosa or idli, this is the exact geometry that is used where you have the stationary outer cylinder and you have an inner grinder which moves around. And, when the velocity of this inner grinder increases, you can see the fluid which is in this particular case a slurry of rice and water and so on. This slurry moves around and you can see the Taylor instability with the fluid flowing in a vortex circle. So, these are both practical examples of the Rayleigh Bernard Instability as well as the Taylor Coutte Flow. (Refer Slide Time: 09:52)

Both these occur in sort of not in common enough everyday instances. Here is an example of the Couette flow in a laboratory as it happens in this case at JNU and let me show you an image which comes out of YouTube on this one ok. (Refer Slide Time: 10:14)

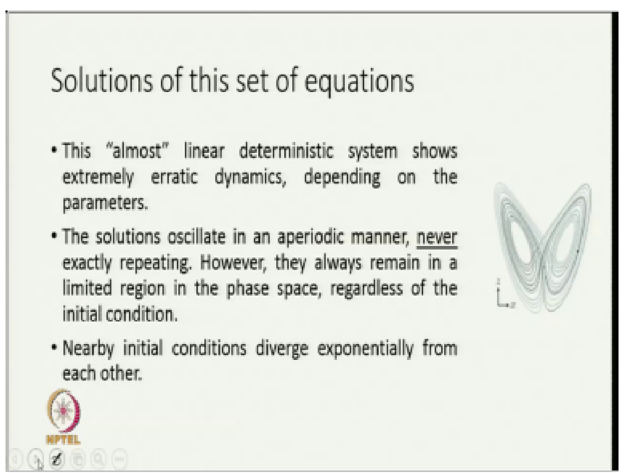
So, here is an example of the Taylor Coutte Instability which has been carried out in a laboratory at JNU. The apparatus consists as you can see over here of plexiglass outer cylinder and a rather dark inner cylinder. The fluid enclosed between these two cylinders is the glycol solution and there is a fluid called chloroscope that has been added over here to make the flow visible. What you will see in this in this little demonstration and this can be seen on YouTube search for Siddharth Krishnamurti. His name you will find that first the flow is featureless for very very low velocities and then as the as the inner cylinder which is being rotated as the velocity increases, there is an instability and these vortices are set up and you can see the image of the vortices as bands. So, let me just show it to you over here. You can see here the Taylor Instability and you can see the bands. Now these bands are going to increase in velocity and there is going to be further instabilities that happen as you can see over here developing a different kinds of instabilities and as you keep increasing the velocity of the inner cylinder, you see that the outer cylinder gets increasingly more complex eventually becoming quite turbulent towards the very end of this particular video, alright. (Refer Slide Time: 12:21)



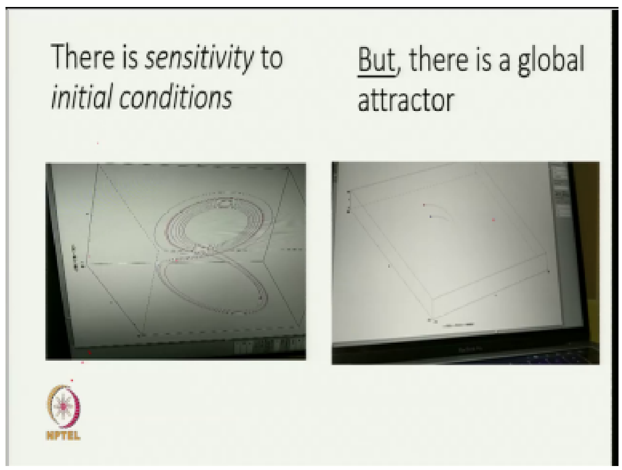
So, now to return to the discussion here based on a study of the Navier Stokes equation I am trying to think of the atmosphere as a convective system which is heated from below that is the surface of the earth and you have got a free surface on top high up. In the 19, early 1960s Ed Lorenz derived a set of three coupled equations which come from a very very simplified model of convection rolls in the atmosphere. The motivation was to try to predict the weather of course. And this was this is sort of part of the lower of Coutte theory as to how he came about these equations and how he undertook to numerically simulate them. The important point of a here is that this set of equations comes to us from the Navier Stokes equation through a very controlled approximation and it has his extremely elegant and very simple form where there are three variables; x, y and z and the it is a set of ordinary differential equations.

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\rho - z) - y$$
$$\dot{z} = xy - \beta z$$

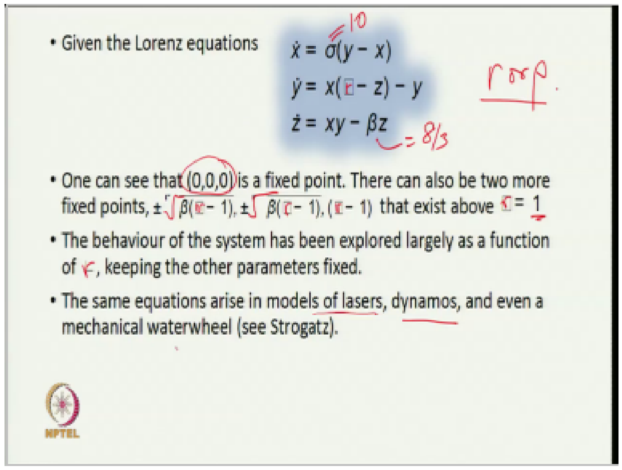
So, and it is almost linear. There are only two non-linear terms here which are which there in the 2nd and the 3d equations and the solutions of these equations have provided enough material for us to study in the last 60 years and perhaps even and there will be a much more to discover in these equations. Now the motivation over here was to think of the atmosphere as a fluid which was heated from below and cooled from above. So, it is the Rayleigh Benard problem in some very abstract sense and an assumption was made that it circulates in two dimensions namely the vertical and the horizontal with periodic rectangular boundary conditions and so on. Now, the interesting thing about these equations is that the solutions look something like. So, the orbits are in three dimensional space in x, y and z and if you look at them in projection, these seem to just be confined to some lower space. (Refer Slide Time: 14:57)



So, this almost linear deterministic systems shows as I will show you in a moment extremely erratic dynamics which depends on the values of the parameters. The solutions oscillate in a fashion over here. You can see that the orbit is going around these two branches of what looks like a butterflys wings. And as you rotate this image of this object, this is the famous Lorenz attractor. You can see that there are two different segments of this attractor and the orbit seems to go in an erratic manner from one side to the other nevertheless the orbit always seems to remain in the same general area. Lawrence's great observation was that nearby initial conditions diverge exponentially rapidly from each other. (Refer Slide Time: 16:04)



And one can see this in a simulation to show sensitivity to initial conditions. Let us start the orbit from a point over here. In fact, let us start two orbits from a point very close to one another and follow them as the equations are integrated as you will see over here. And, I hope you can see it. The orbits are moving together for some period of time and at a certain point, they move now on to different branches of this particular attractor and once they have moved onto different branches, these orbits are going to be as far from each other as the system will allow. The reason that they do not ever move very far from each other is that there is a global attractor and this global attractor can be seen over here where I have started with two very different initial conditions nevertheless as time goes on the orbit eventually fill up the same region of space. So regardless of where you start in phase space, your solutions eventually land up on this part of the on onto this part of the phase space and the orbit moves on this object which is the so-called attractor. (Refer Slide Time: 17:40)



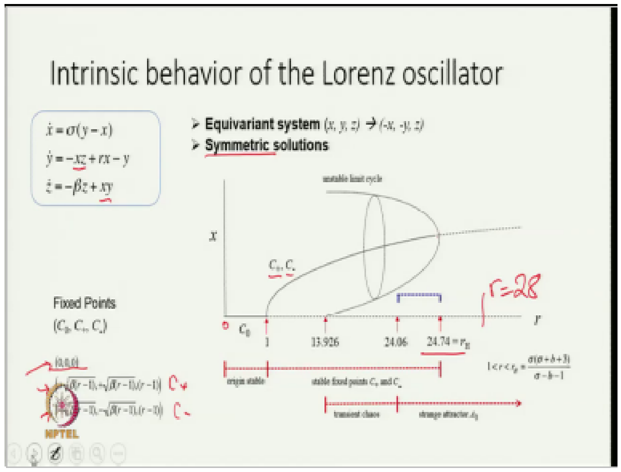
Now given the Lorenz equations which are

$$\dot{x} = \sigma(y - x)$$

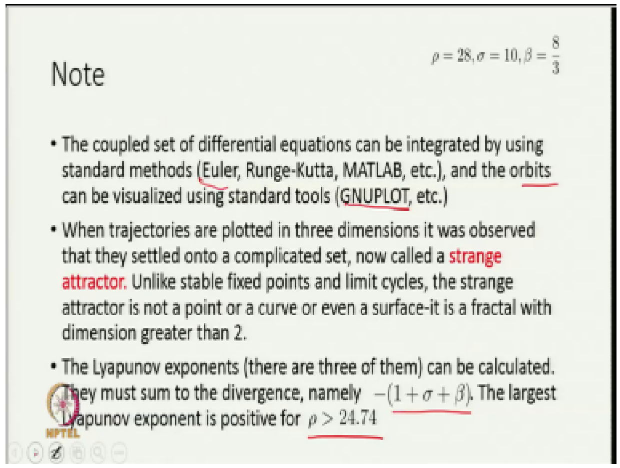
$$\dot{y} = x(\rho - z) - y$$

$$\dot{z} = xy - \beta z$$

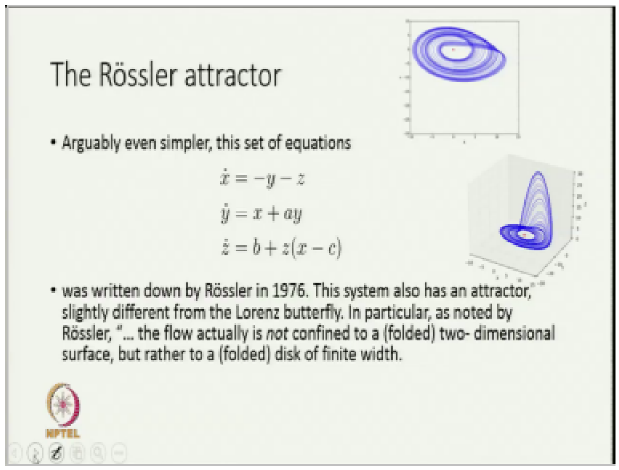
and these are the 3 Lorenz equations. You can immediately see that x is equal to 0, y is equal to 0 and z is equal to 0 leaves us a fixed point of this system. So, the origin is a fixed point, but there are also two other fixed points and these two fixed points happen for $\pm\sqrt{\beta(r-1)}, \pm\sqrt{\beta(r-1)}, (r-1)$ and these will only exist above the value of $r = 1$. Now, r is the Reynolds number which you know in the flow would have be in the Reynolds number it here. It turns out to be just a parameter and people have studied this system for a variety of different parameter values mostly as a function of r keeping these other parameters fixed where as so-called canonical use of the parameter where sigma is equal to 10 and beta is equal to 8 by 3 for reasons that that Lawrence has described and he is very very readable and beautiful brilliant paper that started this field of chaos as it happens. The same equations arise in models of lasers in dynamos and even a mechanical waterwheel and a sand wheel. Similar equations seem to come up and these are discussed in the book by Strogatz. (Refer Slide Time: 19:43)



Now, the Lorenz system has extremely interesting behavior and that is one of the reasons that is contributed to it is sort of importance and vitality in this field. The equations themselves as I have told you earlier are almost linear. There are these two non-linear terms over here and, but otherwise it would have been a linear system. It has a beautiful symmetry which is that x y and z. If you change x to minus x, y to minus y keeping z the same, the system looks identical. So, no matter if you have a particular solution, a solution with negative values of x and y and the same value of z would also be a valid solution. So, that is the symmetry in this particular system. I have already pointed out that we have these two, these two types of fixed points. One is the origin and then these two fixed points because the square root has the plus or minus sign. Now these will exist only for r bigger than 1. So, if I were to draw a bifurcation diagram and this is only a sort of a sketch of what the system complexity of this system is like between r is equal to 0 and r is equal to 1. There is only a single fixed point at r is equal to 1. You can easily show that this fixed point becomes unstable, but these two fixed points C plus and C minus that is this is C plus and this is C minus, these two fixed points become stable. Now these two fixed points are stable up to a certain point over here which is which is where bifurcation happens. This is a Hopf bifurcation which we have not discussed in this course so far but it may be discussed in the forthcoming lectures or perhaps in the next course. And after this Hopf bifurcation, these fixed points also become unstable. Lorenz studied this system for the value of r is equal to 28 which is somewhere over here say when neither the fixed point at 0 is stable nor are these fixed points C plus and C minus. They are also not stable. (Refer Slide Time: 22:29)



What he observed therefore was rather what you have observed was a system which had interesting behavior that you could observe by solving the set of coupled differential equations and you can integrate them yourself. And, I would urge all of you to do this using standard software like MATLAB or if you write programs by we can write those you know for these equations which are really so elementary and simple. Write down a code in either using the Euler method or the Runge Kutta method or what have you and these orbits can be visualized using very standard tools you know like GNU Plot which you have in Linux and so on. Now as you have already seen some of those examples of orbits and you can verify for yourself when you plot these trajectories in three dimensions, they settle onto a complicated set and this is nowadays it is termed a strange attractor and one can show at least that you have this kind of complicated behavior because there is the fixed points are unstable. There are no periodic orbits and so on and this object is a limiting object, but it is not a curve. It is not even a point, it is not a point for sure. It is not a simple closed curve and it is not even a surface. As it happens, it is a fractal and it has a dimension a little greater than 2 and therefore, because it is a fractal and motion on it does not repeat, you have you have an example of an attractor on which the dynamics is chaotic and that is why it is termed a strange attractor. The Lyapunov exponents for this system we have already been introduced to the idea of a Lyapunov exponent, but since this system is in three dimensions there are three of them. So, the sum of the Lyapunov exponents is a large negative number and this says how rapidly face place shrinks, but the largest of the Lyapunov exponents is positive showing that you have chaos and you can show that it has to be positive for sufficiently large value of rho. So, the system can be studied and there is a lot of very rich behavior that you can discover about the Lorenz system and as I have said I would argue to do that just to see that these concepts are not that these concepts can also be explored by you. (Refer Slide Time: 25:23)



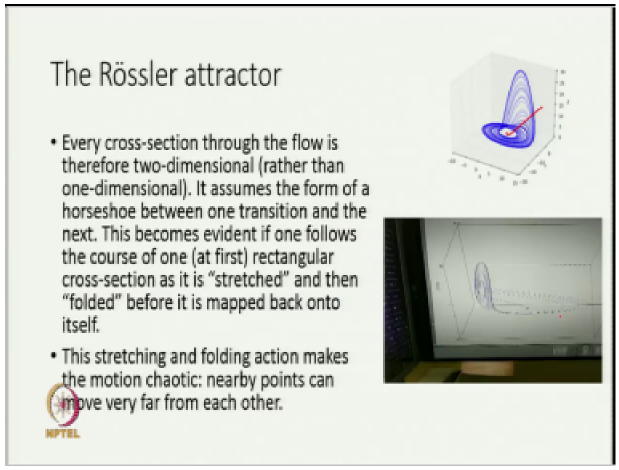
A similar set of equations in some sense a little simpler is provided by Rossler attractor.

$$\dot{x} = -y - z$$

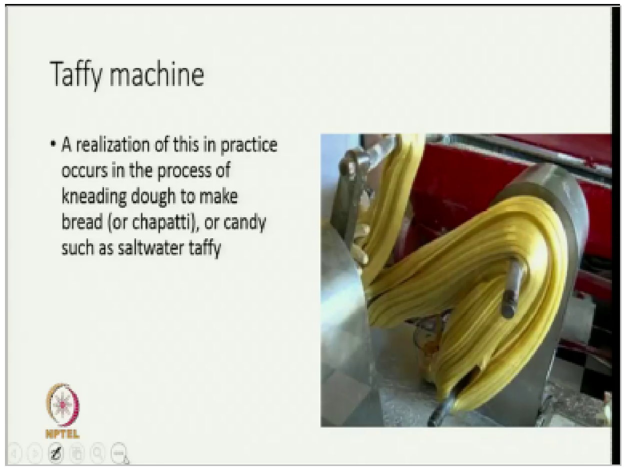
$$\dot{y} = x + ay$$

$$\dot{z} = b + z(x - c)$$

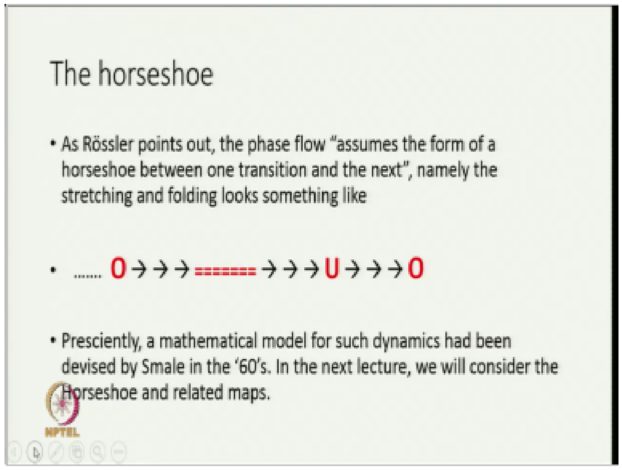
This was first written down by Otto Rossler in the mid 1970s and this is even simpler in the sense that there is only a single non-linear term orbits of this system look like. So, again they do not repeat themselves. They are in three dimensions and part of the orbit is very flat almost in the x y plane and then it shoots off into the z direction and then comes down again. This system as it turns out also has an attractor as you can see over here and this is different of course from the Rossler, a butterfly the sorry the Lorenz butterfly. The Lorenz butterfly has two branches because there are two fixed points which unstable at that at that moment over here. There is only a single fixed point which is unstable. Rossler pointed out that the flow is actually not confined to a two dimensional surface, but to a folded disk of finite width. (Refer Slide Time: 26:51)



And we can see an image of this folded disk over here. In the Rossler attractor every cross section through the flow is two dimensional rather than one dimensional. You see if it was just if you were to take a cross section by cutting the Rossler attractor if it was a simple surface, then that cross section should have been just a one dimensional curve or one dimensional object. Rossler noted that this assumes the form of a horseshoe between one transition and the next and this becomes evident if one follows the course of one rectangular cross section as it is stretched and then folded before it is mapped back onto itself, ok. So, let me just show you this orbit of this system. As you see over here the orbit is you know it is first almost two dimensional over here. Then it goes in and comes back and it is this orbit repeatedly which folds into itself stretching and folding which makes this motion chaotic. This stretching and folding in some somewhere where Rossler mentions his motivation comes from a Taffy machine. I am going to show that to you in a moment, but basically nearby points can move very far from each other if you take phase space and stretch it and fold it and stretch it and fold it repeatedly. (Refer Slide Time: 28:38)



The Taffy machine was described by Rossler as the inspiration for some of this of this development. So, let me just demonstrate that to you in a moment ok. So, this is taken out of Youtube and you can see the reference up there. Let me just play it and you can see the way in which this Taffy where different layers of this system, ok. So, this is a finite system and you have the this is an image of the phase space as it is getting stretched and folded and stretched and folded and stretched and folded unlike our Rossler system. Phase space volumes are not actually decreasing over here because the phase space volume is conserved clearly because we are talking about Taffy, but on the other hand the stretching and folding that happens in this system is precisely what the manipulations of phase space are what is being done to the phase space or how phase space is manipulated in a chaotic dynamical system. A common example of this stretching and folding happens in your kneading dough to either make bread or to make chapatis. So, again this is a very practical way in which one sees this kind of dynamics in everyday life. (Refer Slide Time: 30:27)



Now, the horseshoe idea that Rossler pointed out was basically to say that what happens to the phase flow is that you take this phase; sorry you take the phase flow. Let us take you take a volume in phase space stretch it out. So, you know this is the Taffy as it was pulled out, then it is folded back onto itself and then squished up again. So if you just keep on doing this, then any two points that were over here will suddenly find themselves really far from one another and once they are far from one another, they could just go. They could have very very different histories as it happens in the 1960s. A mathematical model for such dynamics had been constructed by Stephen Smale very famous mathematician who actually was awarded the Fields medal. So, Smale had described dynamics of this kind with a construction very much like the horseshoe and this is the this is our best idea of how chaotic motion happens in such systems when phase space is stretched folded and put back into the same region, into the same general region. In the next lecture we will consider both the horseshoe map of Smale as well as related maps they deal with the manner in which phase space evolves.