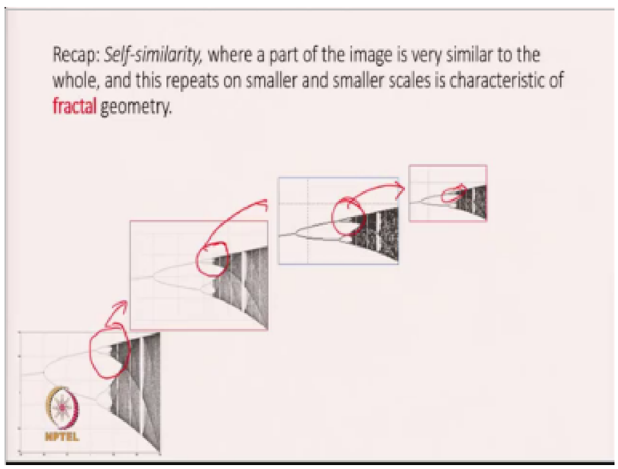
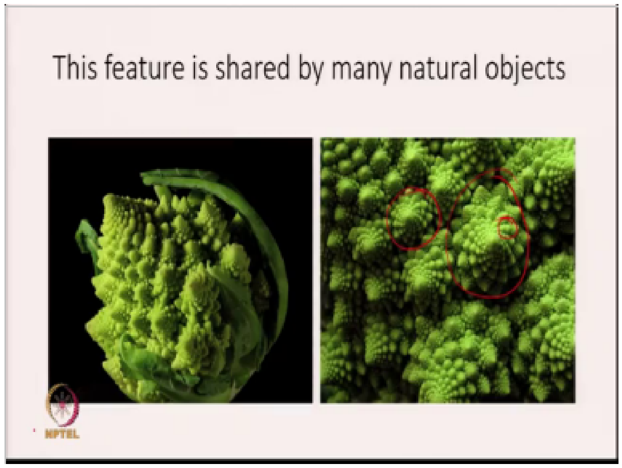
**Introductory Nonlinear Dynamics**
**Prof. Ramakrishna Ramaswamy**
**Department of Chemistry**
**Indian Institute of Technology, Delhi**
Lecture 11
Fractals

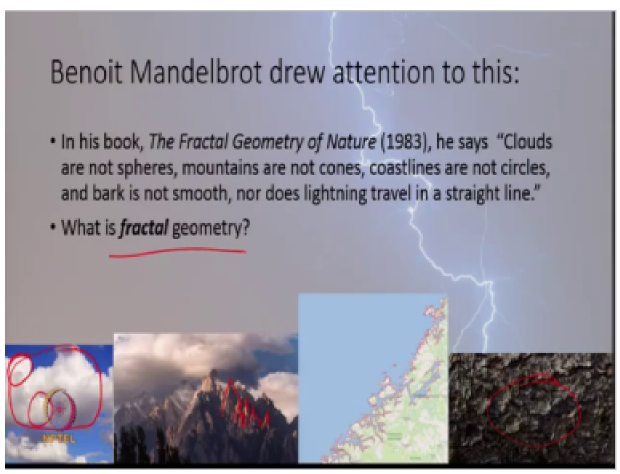
Hello. This week we are going to be looking at characterizing chaos and as a first step it like to discuss Fractals today. Recall that in the last lecture, we were looking at the bifurcation diagram for the logistic map and towards the end I pointed out one curious feature of this bifurcation diagram. (Refer Slide Time: 00:48)



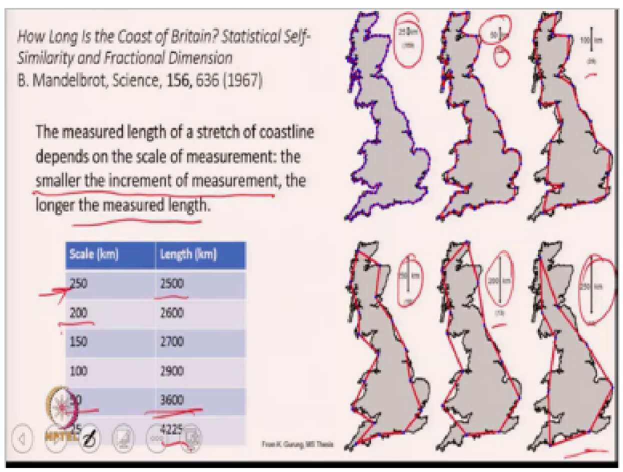

Namely that when you consider the whole bifurcation diagram which is shown here in the extreme left of your screen, then a small part of the bifurcation diagram when expanded looked pretty much like the original diagram itself. And when you took a small part of that and expanded it, it looked pretty much like the old bifurcation diagram itself and you could then take a small part of that and expand it and so on and so on ad infinitum This kind of geometry is described these days by what is termed fractals and the fact that a certain structure in the object occurs on smaller and smaller scales is characteristic of fractal geometry. (Refer Slide Time: 01:53)




Many natural objects also share this feature; spectacular example is that of the head of Chinese broccoli, when you look at the broccoli head you can notice that it is consisting of many little nodules over here. And when you expand it you can see that, the structure of each of these nodules is extremely intricate, but if you focus on one of them you can see that each of these looks pretty much like the whole nodule itself; and the whole structure is repeated endlessly in this broccoli. (Refer Slide Time: 02:48)

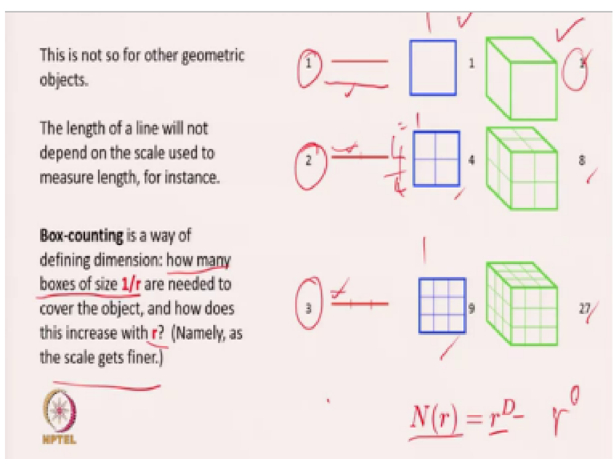



Many natural objects have this feature, if not precisely statistically they have this kind of feature. And the first person to draw our attention collectively to, this is mathematician Benoit Mandelbrot who from the fifties onwards was pointing out that, standard geometry is not enough to describe a whole variety of natural objects. In a very influential book, the fractal geometry of nature in 1983 he says clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line. He was drawing attention to the fact that, when modeling something like a cloud to a first approximation a cloud is like a sphere; but as you can see this cloud is far from spherical and even the sphere has little features on it on smaller and smaller scales. A first approximation to a mountain might be a cone, but this mountain as you can see has structure, there are other mountain, the other little cones, on top of cones and so on and so forth. So, if you really wanted to describe what a mountain look like it would not be by a simple cone. The coastline of a country or the border of any particular country is not always very smooth and in particular a coastline cannot be approximated by a circle. The bark of a tree is not smooth, it is not some smooth surface; but it has a lot of structure on it as you can see in this image that I have over here. Mandelbrot pointed out, that in order to describe these objects we need to turn to fractal geometry. (Refer Slide Time: 04:50)

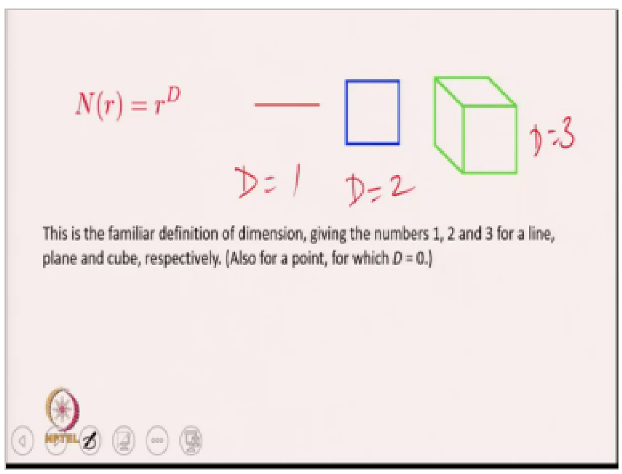



And he posed this in a paper in 1967 asking, how long is the Coast of Britain? He was drawing attention to the fact that, if you take any complicated curve such as a coastline, the length of that curve depends on the scale at which you measure. As shown over here, there is a map of Britain with successively smaller measuring units being applied, at the extreme right corner over here in the bottom right corner; if I use a scale of 250 kilometers, then in 10 with 10 measurements or 10 steps of length 250 each, I would have you know completely covered the coastline at that scale. If I went to a smaller scale

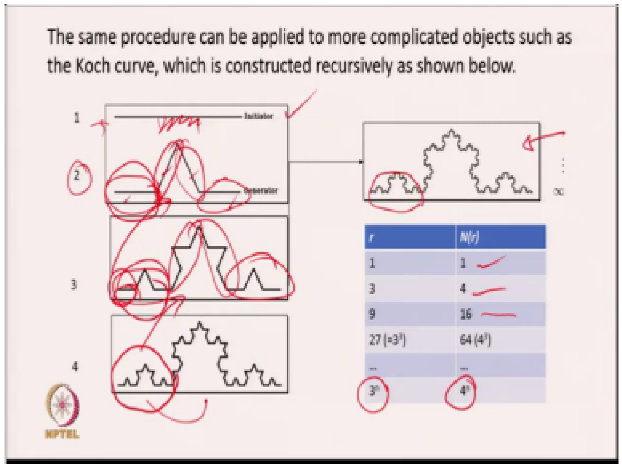
200 kilometers, I need 13 such measurements to do that. If I have a 150 kilometers I need 18; If I need as finer grid a 100 kilometers I will need 29; with 50 kilometers I am at 72 and at 25 kilometers I met a 169 and it is of course is possible to do it at finer scales, is just not easy to show it over here. Notice one thing, that if I use a measuring rod of length 250 kilometers, my estimate of the coastline is as you can see visually very crude and the total length is 2500 kilometers. If I use 200 I will get a length estimate of 2600 kilometers, by the time I go to 50 kilometers my length is up to 3600; but my measurement of the coastline is actually getting better and better; at 25 kilometers the total length is 4225 kilometers, quite different from the initial estimate almost a factor of 2, is the same coastline it is just that my measurement apparatus has gotten finer and finer. So, the smaller the measurement apparatus are the smaller the increment of measurement, the longer is the measured length; but more importantly this kind of property, that the length or some measurement depends on these on the unit on which you are measuring it is not shared by other geometric objects. (Refer Slide Time: 07:42)



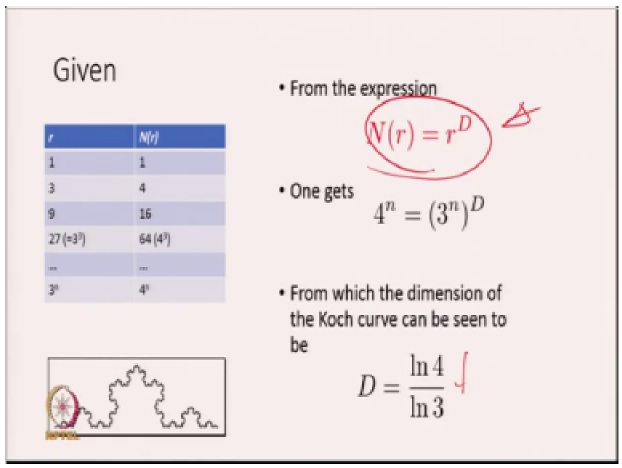

For instance the length of a line whether I measure it with 1 foot or with 6 inches or with 4 inches, the length is always 1 foot. It is 1 unit, 1 measurement unit here, 2 over here and 3 over here, but the answer is the same. Likewise, were I to calculate the area this is 1 square unit; but if I used a quarter of the length or you know a half the length, so it becomes a quarter of the area sorry. So, if I use a quarter of the area I find 4 units over there; but 4 times 1 by 4 is also equal to 1. And likewise if I use a 3rd over here, then my unit square is 1 by 9; but I have got 9 of them, so again the area is 1. The same argument holds to show that, if I measure a cube with length 1 with a box of length 1 or one half or one third my answer for the finer volume is the same, which is that the volume is 1. This is actually at the heart of what we think of as dimension in the following way. If I ask how many boxes of size 1 by r are needed to cover the object, and how many and how does this number increase with r? Because as r increases the size of the boxes gets smaller, namely the scale gets finer, how does this change? So, if I see this line over here I need 1 box of length 1, I need 2 boxes of length one half, I need 3 boxes of length one third, for the square over here for the plane I need 1 box of which has side length 1, I need 4 boxes of length half; and 9 of a length one third and for the cube the answers are 1, 8 and 27. (Refer Slide Time: 10:14)




So, now, if I note that $N(r) = r^D$ . It is easy to see that for the line I will find that D is equal to 1 for the square over here, the plane I will find that D is equal to 2 and finally, for the cube D is equal to 3 as you can verify easily just looking at this little example over here. So, this is the familiar definition that we have of dimension, and it gives these numbers 1, 2 and 3 for familiar everyday objects in 1, 2 or 3 dimensions. Also as it happens, if you take a point which has dimension equal to 0, the number of boxes that is required to cover a point, a single point is always one regardless of how big the box is. So, it will go as r to the power 0 which is 1 and that tells you that the dimension of a point is, in fact, 0. For long time mathematicians have been wondering about object, so which you give get slightly non trivial answers using the same procedure. (Refer Slide Time: 11:29)

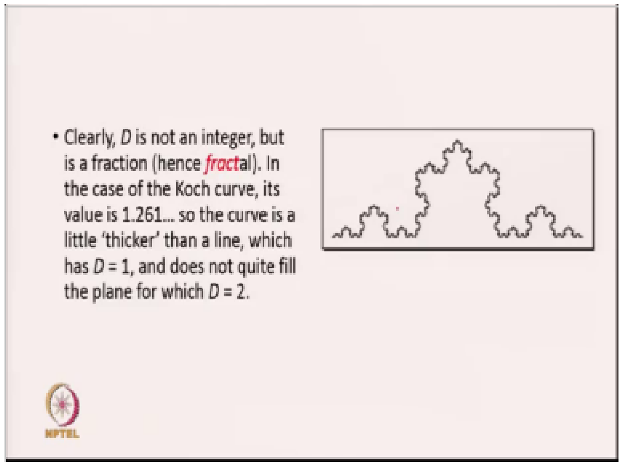



Here is a curve that was described by the mathematician Von Koch curve and when you apply the same procedure that we have outlined to such an object, we find that the answers are more interesting. So, the way in which this curve is constructed is as follows; you start with the line known as the initiator over here. So, you remove the middle one third, the one middle one third is removed and you put in a little hat on that middle one third, where the lengths of the sides of the hat are the same as one third. So, at stage 1; this generator figure over here says that you take a line segment and replace it with this hat function. At the next step what you do is you take this line function replace it with this hat function, this line function replace it with the hat function; this one with the same hat function, and this one also with the same hat function. And now you can see how this procedure works. At the next stage you take every line segment and replace it with this hat function and so on and so forth. So, the curve gets more and more complicated and in the limit this curve that you get is the Koch curve or the Koch curve depending on how you like to pronounce it. The scales get finer and finer of course, but note that this curve has this property that a part of this curve when you expand it, if you blew this up it looks pretty much like the curve at one previous version. And if you blow this up it looks exactly like the curve at the previous direct in a previous stage and as a matter of fact, if you blew up the Koch curve you just took a third effect and you blew it up it would be the same curve itself. What is the box counting dimension of this object? At the initiator stage, if I take a box of length 1, I need 1 box to cover this entire object. At the next stage, what I need is a box of a length 1 by 3 one third; and if I take a box of length one third, I am going to need 4 boxes to cover that as you can see 1, 2, 3 and 4. At the next stage I am going to be down by a factor of 3. So, I am going to need a box of size 1 over 9; and now I am going to need 16 boxes. It is not difficult to do the math, at the next stage you are down by another factor of 3 and you are going to require 4 times more boxes. And it is easy to convince yourself, that no matter how small it is the boxes are of going to be of size 1 over 3 to the n and you are going to need 4 to the n such boxes. (Refer Slide Time: 14:57)

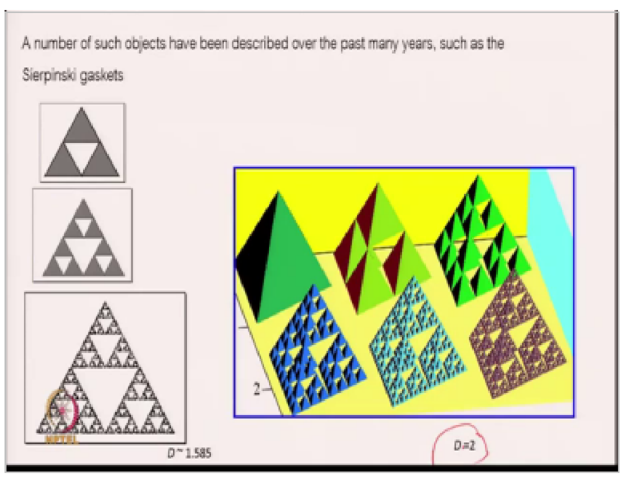



| r | N(r) |
|---|---|
| 1 | 1 |
| 3 | 4 |
| 9 | 16 |
| 27 (=3³) | 64 (4³) |
| ... | ... |
| $3^n$ | $4^n$ |

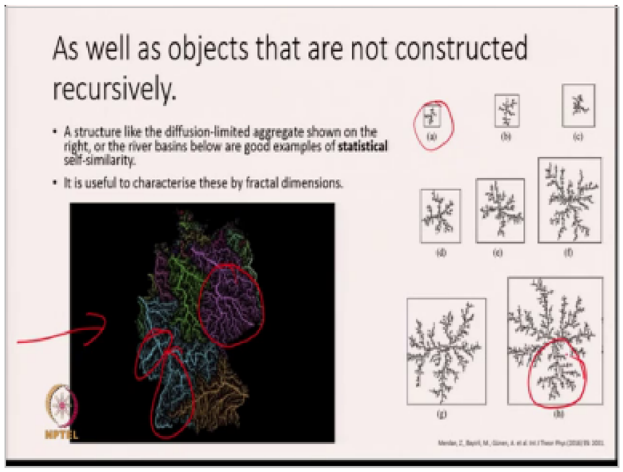
Now, given this data over here that when r changes as 1, 3, 9, 27 etcetera all the way down to 3 to the n, the number of boxes goes as 1, 4, 16, 64 all the way up till 4 to the n. Putting all this data in this expression over here, you get that 4 to the n is 3 to the $n^D$ from which you can deduce that the dimension, this box counting dimension of the Koch curve is log 4 by log 3. (Refer Slide Time: 15:36)



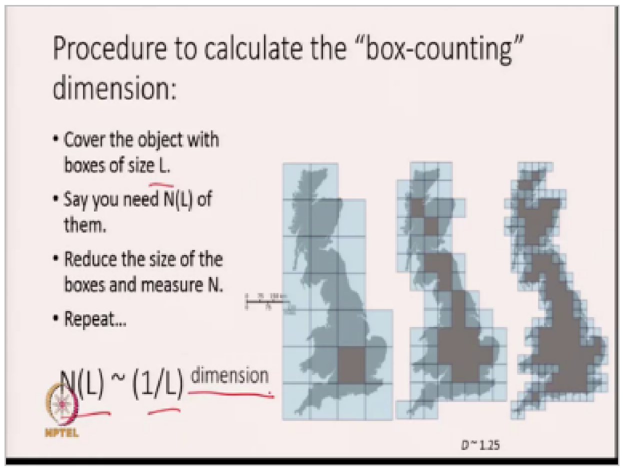

Now, log 4 by log 3 is not an integer, it is a fraction of some kind; and hence the name fractal was given to it to such objects by Mandelbrot. In this particular case by evaluating log 4 by log 3, you can deduce that the value is something like 1.261. So, the curve is not as thin as the line, which would have dimension 1 and yet it is not space filling was not filling the plane in which case it would have had dimension 2, but it is somewhere between 1 and 2. So, this curve has got features on all scales and this is very typical and characteristic of a fractal object. (Refer Slide Time: 16:24)



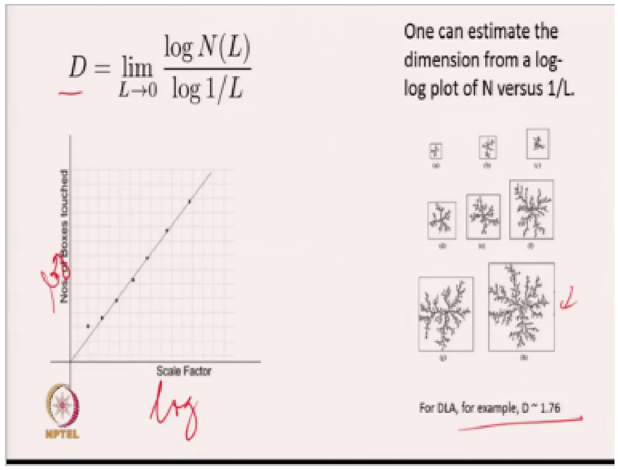

This kind of construction can be applied to other objects can take a triangle, remove the middle third of it, and keep on doing this recursively over and over again; and a little calculation will show you that the fractal dimension, following exactly the same procedures that we have done over here is something like 1.58 You can do this in more than 2 dimensions, you can do it in 3 dimensions, you can do it in 1 dimension, we can do it in any number of dimensions; you can have this recursive manner of construction. And here as the Sierpinski gaskets in 3 dimensions and this object will have dimension 2. So, fractal need not necessarily have a fractional dimension, it is just that it should obey this kind of scaling in a non-trivial manner ok. So, many interesting objects can be constructed and they will have their own particular value of the fractal dimension. (Refer Slide Time: 17:42)



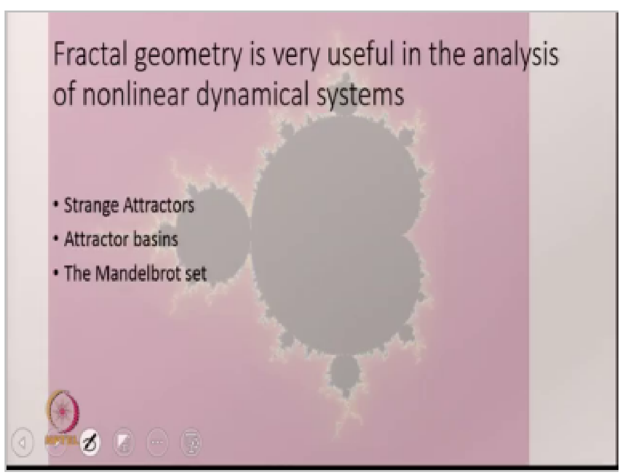

But you can go beyond this, you do not have to look at constructed objects, many natural objects also seem to have the same feature. Of course, these are not exactly replicas of themselves at every stage; but statistically speaking they are very similar to the construction at another stage. So, here are two examples that I have taken from the internet, various different sources, here is the rivers of Germany for example; and what has been done over here is to plot the river basins showing the main rivers and their various tributaries and you can see that visually at least. The moment you see a particular river basin, if you just isolate a small part of it that looks pretty much like a scaled down version of the entire thing. And you can scale this one down and this one down and so on, so this has structure on all scales. Another common example is that of diffusion limited aggregation, it is a model out of statistical physics and this is a model which this is an example of a growth process. And this object grows in this following way, it starts out as a small little nucleus and then it accumulates particles on the edges and keeps growing and you can see pretty much that one part of it looks very much like itself. (Refer Slide Time: 19:21)



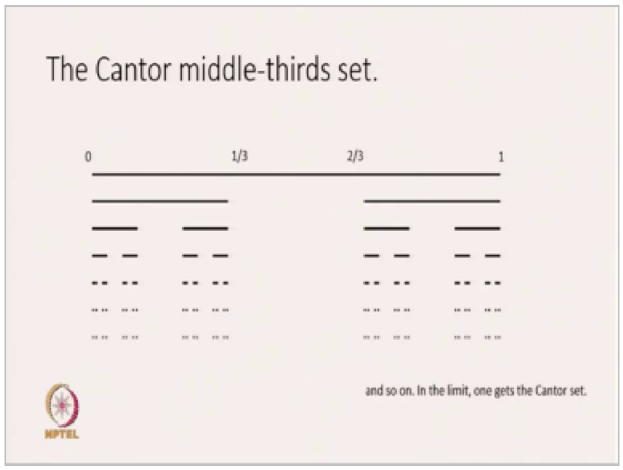

So, to for the coastline of Britain, it is not a smooth self-similar curve; but it is only statistically self similar. So, there is a procedure for calculating this dimension by the process of box counting. And how this is adapted to objects that are not exact copies of themselves is as follows: You cover the object with a box of size L let us say over here. Once you have done that, you figure out how many boxes you need, reduce the size of the box say by a factor of 2. And you ask how many boxes I am going to need now and reduce it again and ask how many boxes we need now and so on and so on and so forth. And if you keep scaling the box by size L at each iterate, the number of boxes that you require goes as $N(L) \sim (1/L)^{dimension}$. (Refer Slide Time: 20:31)



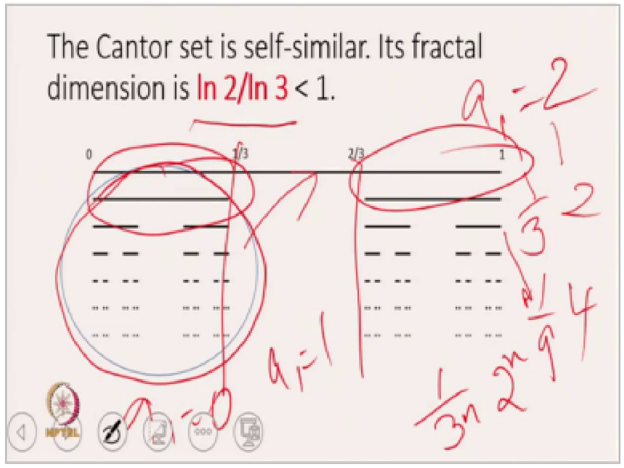

So, on a log log plot, so on this side you plot the log of the number of boxes which have been touched by the object let us say. And on this side the scale factor which is 1 over L; so the dimension D is just the slope of this line. And this is a way in which the fractal dimension is operationally calculated for a number of different objects; when you apply this to DLA for example, you get a number like 1.76. Again pointing to the fact that, this object is not quite aligned; but it is also not quite filling the plane and somewhere between 1 and 2. (Refer Slide Time: 21:26)



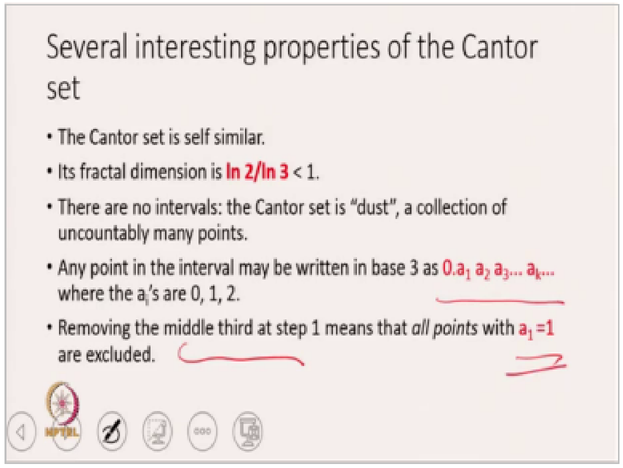

It turns out that fractal geometry is very useful in the analysis of non-linear dynamical systems. Strange attractors tend to be on fractals; what I have at the background over there is the Mandelbrot set which you would have seen in countless places or screensavers and so on. The boundary of this object is a fractal and it is well characterized, there are lots of fractals everywhere in the Mandelbrot set. When you have an attractor, the basin of an attractor is frequently a fractal, a lot of natural objects are fractals, a lot of constructed objects are fractals and so, fractal geometry is a very useful way of describing things. (Refer Slide Time: 22:07)



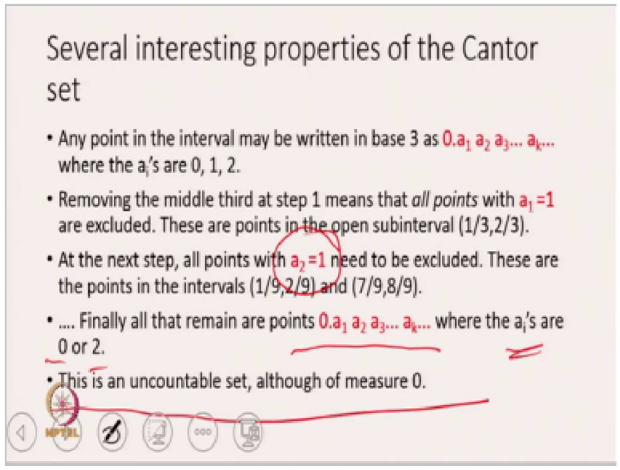

A famous fractal was described by the Mathematician Georg Cantor over a century ago. And this is the famous cantor middle third set, which is constructed in the following way; one starts with a line of length 1 as is marked over here and from that you remove the middle third portion. So, the portion from one thirds to two thirds is removed at the first stage; at the second stage you remove a third of each of the remaining intervals. So, now, we have 4 intervals of length 1 by 9 each, at the next stage you remove one third of that; and then you remove third of the next one and so on and so on and so forth. Now, in the limit one gets the so, called Cantor set and this Cantor set is a fractal, because as one can see there is a element of self-similarity in it. (Refer Slide Time: 23:22)



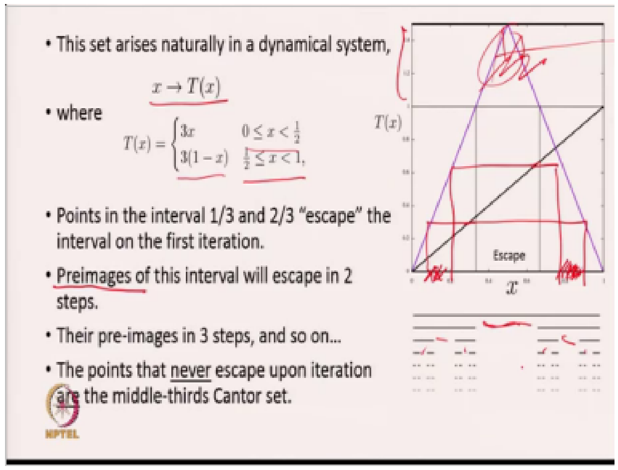

As you can see this portion of the set, if expanded by a factor of 3 would look like the entire picture itself. So, this construction displays the self similarity of this set, the fractal dimension of this set is log 2 by log 3 and this is a number which is less than 1; and one can see why this you know why the fractal dimension is log 2 by log 3 by doing the usual box counting. At the initial stage I need 1 box of length 1 to cover the entire set. If I reduce the length of the box to one third, I will now need 2 such boxes. If I reduce it to 1 by 9 that is at this level, I will need 4 boxes. And you can easily see that if I reduce the length to 1 by 3 to the power n, I am going to require 2 to the power n boxes. In the limit as n goes to infinity, you will find that the fractal dimension computed in exactly the same way as we have done in the earlier examples is log 2 by log 3 ok; and this is a number which is bigger than 0, but less than 1. (Refer Slide Time: 25:00)




There are several interesting properties of this Cantor set; the first of course, is that it is self similar and as we have just computed the fractal dimension is $ln(2)/ln(3)$. But more than that, there are no intervals because as you have noticed over here; I have removed the middle thirds and then the middle thirds and the middle thirds and so on and so on recursively. So, there are absolutely no intervals in this set and the Cantor set just consists of a set of points. In higher dimensions this construction is termed a Cantor dust and it is a collection of uncountably many points. Now, how do we see that, there are uncountably many points in this set, this is done by looking at the ternary representation of any number in the interval [0,1]. Ternary representation or base 3, any point between 0 and 1 can be written as $0.a_1a_2a_3...$ all the way you know up like $a_n soontendingtoinfinity; butbecausethisisinbase3the a_i$'s of is are either 0, 1 or 2. Removing the middle third at step 1 means that all points with $a_1 = 1$ are excluded. Because points on the left hand side, points on this side have got a 1 equals 0, points on this side I have got a 1 equals 2; and what has been removed are all the points with a 1 equals 1. So, at the first step at the first stage removing points in the middle one third removes means; that you remove all those points which have got a1 equals 1. (Refer Slide Time: 27:17)




At the next stage you want to remove all the points in the sub intervals 1 by 9 to 2 by 9, 7 by 9 to 8 by 9 the middle thirds of these are the 2 sets and; that means, that all the points with a 2 equals 1 need to be excluded. At the next stage we will remove all the points with a 3 equals 1; the next stage is all the points with a 4 equals 1. And in the limit all the points that remain are points 0, a 1, a 2, a 3 all the way you know going up to infinity; where the a is are either 0 or 2, 1 cannot appear anywhere. And it is also easy to see, that any sequence of zeros and twos can be mapped identically to a sequence of zeros and ones because treating 0 and 2 as just symbols, 0 and 1 would be equivalent symbols. So, in binary notation this is entirely all the points in the interval; and therefore, the remaining points that we have in this construction is an uncountable set, but because there are no intervals at all this is a set of measure 0. This set applies very interestingly to a dynamical system to which we will now turn. (Refer Slide Time: 29:00)



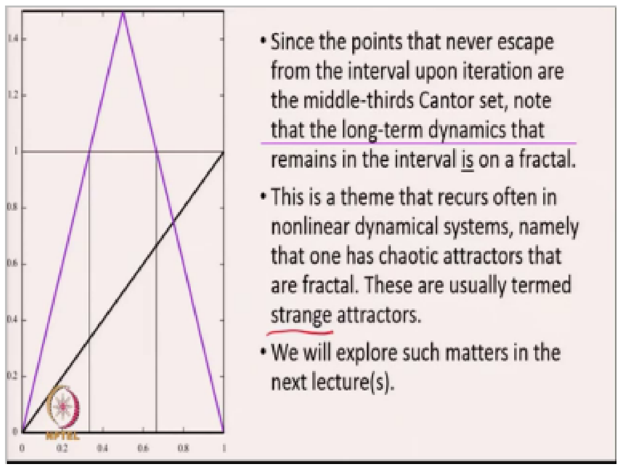

But it also seems to arise naturally in many dynamical systems. As for example, the tent map; recall the tent map that we introduced several lectures ago $x \to T(x)$, where

$$T(x) = 3(x), 0 \le < 1/2$$

or

$$T(x) = 3(1 - x), 1/2 \le < 1$$

, but now I want to make this tent map with the slope 3. So, the tent map is 3x on the interval 0 to half and is 3 into 1 minus x into from the interval half to 1. Since this map goes outside the interval, you see this is outside the interval, points that map here will eventually go to minus infinity; namely all these points will escape. Everything that maps above the line over here, once you increase the value it will just keep escape. Now, by construction you can see that this is precisely the interval one third to two thirds and all these points have escaped. What about the points that will come to the interval on the first iteration? This is the pre-image of this interval and as we know the pre-image of this interval can be easily constructed by looking at that or going out here and finding those points ok. So, these two intervals will be exactly one ninth to two ninths and seven ninths to eight ninths over here. So, these points will escape after two iterations; namely they will come to one third in the first iteration and then the next iteration they will go out. Pre-images of these two subintervals there will be 4 of them and they will escape in 3 steps and so on and so on. So, that the points that will never escape on iterating in the system, they actually are the middle one third Cantor set. As you can see, we started with this line 0 to 1; all these points left in one iterate, these points are leaving in 2 iterations, these points will leave in 3 iterations, these will leave in 4 and so on and so on and so forth. So, that the only points that will always remain in the interval are going to be the middle thirds Cantor set. (Refer Slide Time: 31:40)




The points that never escaped from this interval upon iteration will form a fractal and the dynamics will be on this particular fractal, but this will be contained inside the interval 0 to 1. This is a theme that recurs often in non-linear dynamical systems; namely that you have chaotic motions on attractors and these attractors happen to have fractal geometries. Attractors that are fractal, and on which the motion is chaotic are usually termed strange attractors, the terminology that goes back to the seventies and is both standard and it has a mathematical definition. And it is important therefore, to look at dynamics not just in terms of instabilities and the geometry of the dynamics. But also look at the structure of the object on which the dynamics takes place, and these are best described by the fractal geometry. In the next lecture or maybe more, we will explore such matters.