Radioactive equilibria

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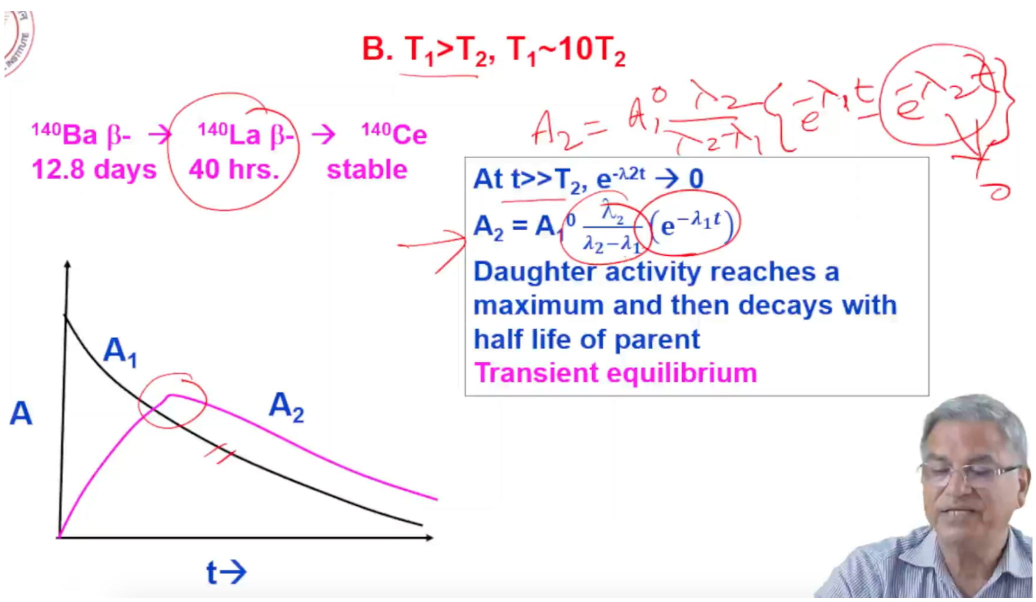
Lecture-2, Module-2

Hello everyone. In the previous part of this lecture, we discussed the radioactive decay chain and also derived the expression for the activity of a daughter isotope which is also radioactive. And based on that equation, in fact, we generalize that equation for multiple decays to find out the activity of granddaughters. We also discussed that if the parent is short-lived compared to the daughter isotope, then this is a case of no equilibrium. After the parent activity has decayed down, the daughter isotope decays with its own half-life. Now I will discuss a different situation where the parent is longer lived than the daughter isotope and that is where we will call it as a radioactive equilibrium.



So this is the case of a parent being longer lived than daughter isotope and there are two cases of this type here. In the first case, the parent half life is roughly 10 times that of daughter, and there is another case where the parent is much longer-lived, maybe 100 times or even more than that. So we will discuss these two cases separately and they have slightly different implications, that's all. So let us take the case of this 10 times more parent half-life than daughter and I have given an example here.

^{140}Ba having a half-life of 12.8 days, decays to ^{140}La having half-life of let us say 1.6 days, so roughly 10 times the parent. It is not very hard and fast that it has to be 10 times. It could be 9 times, 8 times or 12 times, but just order of that and why it is so, I will explain very soon and, it is decaying to ^{140}Ce which is stable.

So let us see mathematically what happens and then see graphically what happens. So when we have a large time elapsed, so again here I will put the equation for activity of daughter

$$A_2 = A_1^{\circ} \frac{\lambda_2}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t})$$

This is the general equation for the activity of the daughter product here ^{140}La as a function of time. So what we are discussing here that when the time, so here λ_1 is smaller than $\boldsymbol{\lambda_2}$. So when the time elapsed is much more compared to the half-life of the daughter.

The daughter is short-lived. So after several half-lives of the daughter, , because λ_2 is very high, this term $e^{-\lambda_2 t}$ tends to 0. When this term tends to 0, you can see here, activity of daughter A_2

$$A_2 = A_1^{\circ} \frac{\lambda_2}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t})$$

You see here what is happening now? This is the decay of parent whereas we are seeing the change in the activity of daughter.

So daughter isotope is decaying with the half-life of parent after a sufficiently long time. That means several half-lives of daughter. So after several half-lives of daughter, the daughter activity starts decaying with the half-life of parent. So even after several half-lives because the daughter activity is being fed from the parent, the daughter activity is there but it is decaying with the half-life of parent. So as a result of that, the daughter activity reaches a maximum and then decays with the half-life of parent.

So that means this is like a transient. That is why it is called a transient equilibrium. For a moment, the daughter activity reaches a maximum and then starts decaying with the half-life of parent. So I have tried to explain this in this graph which is again in the linear scale. We are plotting log of activity as a function of time.

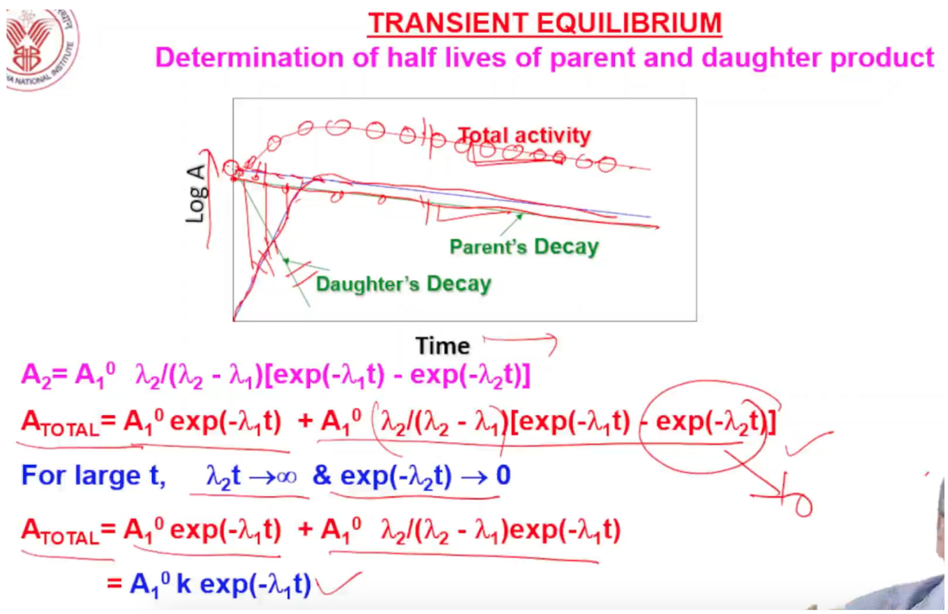
The parent is decaying in this way and the daughter activity starts growing from 0 because initially there is no daughter activity, reaches more than that of parent. Now you see the implication of this term in this equation. This term $\frac{\lambda_2}{\lambda_2 - \lambda_1}$ will be more than 1. So at that case you will see A_2 will become, so you can write this as

$$A_2 = A_1 \frac{\lambda_2}{\lambda_2 - \lambda_1}$$

$$\text{where } A_1 = A_1^{\circ} e^{-\lambda_1 t}$$

so A_2 will be more than A_1, and this is the condition which is meaning by this equation.

So the activity of daughter becomes more than that of parent and again after some time starts decaying with the half-life of parent. So that is what we call as the transient equilibrium.



Okay, so now let us do an exercise of resolving the total activity into that of the parent and daughter in the case of a transient equilibrium. So when we have a freshly purified parent isotope then the total activity of this sample, which is freshly purified, the total activity will increase and then subsequently decrease in this fashion. Here I am plotting total activity in the logarithmic scale as a function of time and from this data let us try to find out the half-life of parent and that of the daughter.

And also their initial activity we can find out. So you know the total activity can be given as

$$A_{TOTAl} = A_1^{\circ} e^{-\lambda_1 t} + A_1^{\circ} \frac{\lambda_2}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t})$$

So now what happens that when that λ_2 has become quite large because λ_2 means the decay constant of daughter that is higher than λ_1 then this exponential term, this term tends to 0. And therefore the total activity becomes parent activity and both of them following the half-life of parent. So you can see here the parent activity at a later time is in logarithmic scale in straight lines that means it is exponentially decaying and it should be the decay constant of parent.

So this is the data which represents the half-life of parent. So you can extrapolate this to 0 time and draw a line parallel to this line. This is the decay of the parent in the total activity. So from here you can find out the half-life of parent. Now the excess activity from here to here is due to the growth of the daughter.

So if you subtract point by point from here then you will get the growth of the daughter and which will grow become more than the parent activity and then again decay with the half-life of parent. So this is the growth of daughter activity and if we had not done the separation of daughter from parent then if you can extrapolate this to 0 time that gives you activity of daughter which was present in the sample before the separation was done. So this was the activity of daughter which was in the sample and this is the activity which is growing in the sample. For the subtraction of this to this, this data will give you the daughter decay. The daughter which had been separated from the parent as a result of the radiochemical separation which is now kept separately will be decaying with its own half-life and so from the decay data we can find out the **λ** and hence the half-life of daughter.

So this is how you can resolve the total activity into that of the parent and the daughter. You can find out their half-lives and you can find out their initial activities. This was the exercise that I thought that it better to do it for the transient equilibrium. Similarly you can do for the secular equilibrium. Now we discussed the situation that the daughter activity grows, reaches a maximum and then starts decaying with the half-life of parent.

So what is the time to reach the maximum daughter activity?

Time to reach maximum daughter activity

$$A_2 = A_1^0\left(\frac{\lambda_2}{\lambda_2-\lambda_1}\right)\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$

$$dA_2/dt = K(-\lambda_1 e^{-\lambda 1tm} + \lambda_2 e^{-\lambda 2tm}) = 0$$

$$\lambda_2 e^{-\lambda 2tm} = \lambda_1 e^{-\lambda 1tm}$$

$$e^{(\lambda 2\ tm\ -\ \lambda 1\ tm)} = \lambda_2/\lambda_1$$

$$t_m(\lambda_2 - \lambda_1) = \ln(\lambda_2/\lambda_1)$$

$$t_m = \ln(\lambda_2/\lambda_1)/(\lambda_2 - \lambda_1)$$

$$t_m = \frac{2.303}{(\lambda_2-\lambda_1)}\log\left(\frac{\lambda_2}{\lambda_1}\right)$$

$$t_m = \frac{\ln\frac{\lambda_2}{\lambda_1}}{\lambda_2-\lambda_1}$$

Let us discuss this in this slide. The activity of daughter changes with time using this expression $A_1^{\circ}\frac{\lambda_2}{\lambda_2-\lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t})$. So the time at which the activity of daughter becomes maximum then the derivative of A_2 will be zero. So let us differentiate this A_2 as a function of time and equate this equal to zero to get the time. So when this expression is equal to zero t becomes t_m, the time of maximum activity.

So this you can say if you put the constant terms as K like A_1° lambda factors will be constant. So we have differential of this

$$\frac{-dA2}{dt} = K\left(-\lambda_1 e^{-\lambda_1 t_m} + \lambda_2 e^{-\lambda_2 t_m}\right) = 0$$

So this equation boils down to

$$\lambda_2 e^{-\lambda_2 t_m} = \lambda_1 e^{-\lambda_1 t_m}$$

So you can now try to separate the time factor.

So you bring the λ_1, λ_2 this side, it becomes

$$e^{(\lambda_2 t_m - \lambda_1 t_m)} = \frac{\lambda_2}{\lambda_1}$$

And now you can arrange them

$$ln\frac{\lambda_2}{\lambda_1} = t_m(\lambda_2 - \lambda_1)$$

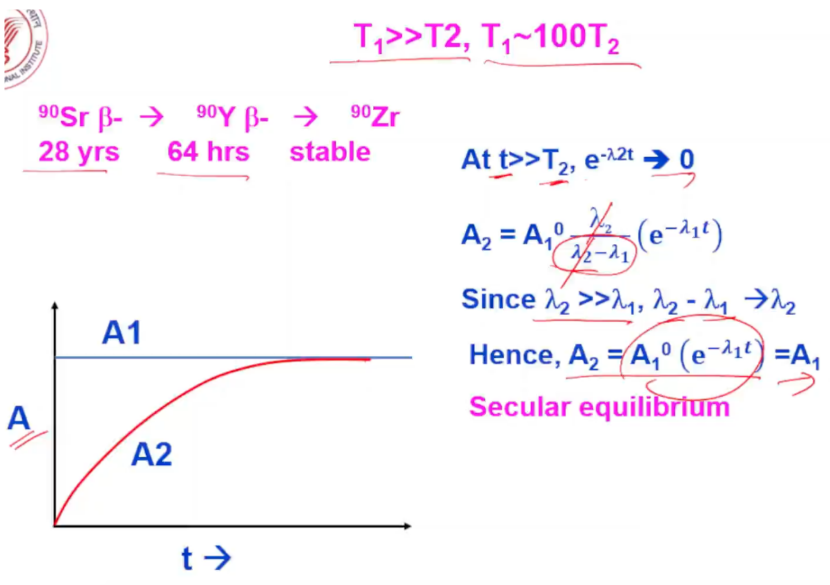
So t_m the time to reach maximum daughter activity is equal to

$$t_m = ln\frac{\lambda_2}{\lambda_1}/(\lambda_2 - \lambda_1)$$

Using this expression, you can find out t_m and if you want to put this ln in terms of the logarithm you can put

$$t_m = \frac{2.303}{(\lambda_2-\lambda_1)}log(\frac{\lambda_2}{\lambda_1})$$

So using this formula you can calculate the time when the daughter activity will reach maximum in a case of a transient equilibrium. For that matter even in the case of a no equilibrium case the daughter activity will reach a maximum and you can calculate at what point of time it will happen.



Okay now I will come to the second case of equilibrium and that is called as the secular equilibrium where the parent is much much longer lived than daughter roughly of the order of 100. So it is not a very hard and fast it has to be exactly 100 times but of the order of it could be even few hundreds or even thousands also. but it is more than let us say typically more than 100 or more than that. And an example I have given here is ^{90}Sr undergoes β- decay to ^{90}Y which also undergoes β-decay to ^{90}Zr. The half-lives are 28 years and 64 hours (2.64 days) and Zr^{90} is stable. So again use the assumptions that at very large time compared to the half-life of the daughter $e^{-\lambda_2 t}$ term tends to 0 because λ_2 into t will become very large positive number.

So exponential of negative of a large number will become 0. So

$$A_2 = A_1^{\circ} \frac{\lambda_2}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t})$$

Now compare this with transient equilibrium case. Here λ_2 is much much larger than λ_1 because t_1 is much larger than t_2. And so in such a case you can neglect λ_1 with respect to λ_2.

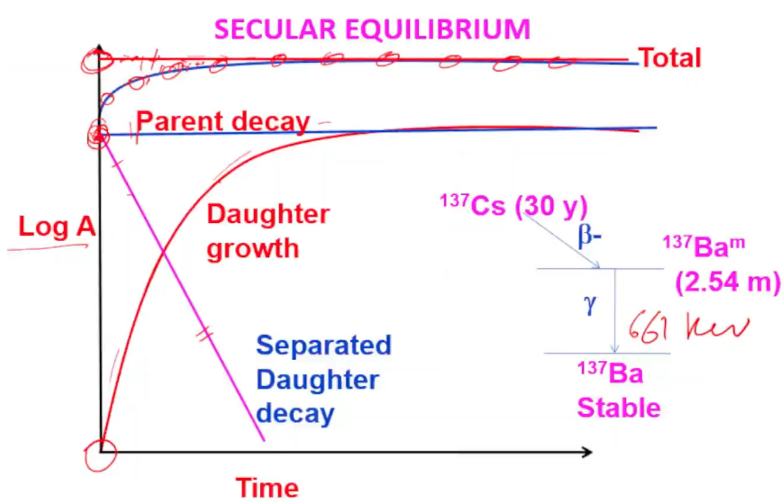
So in the denominator this term $\lambda_2 - \lambda_1$ will become λ_2 and so $\boldsymbol{\lambda}_1$ will get cancelled out. So because of this approximation the **λ** term vanishes it becomes 1 and so what happens to this term,

$$A_2 = A_1^{\circ} (e^{-\lambda_1 t})$$

and this term is nothing but activity of parent. So you can see here the very term meaning of secular means they are same. In the case of transient equilibrium the daughter activity is decaying with the half-life of parent but it is not same. In fact, it is more than the parent activity because of the lambda factors.

In the case of secular equilibrium once the equilibrium is established the activity of daughter and parent become same, A_2 equal to A_1. This I have tried to illustrate using this graph here. The activity now, though plotted on linear scale I have put the graph as linear the activity that means during the period of our observation the activity is not changed, the parent activity is not changed and so A_1 is becoming flat but the daughter activity will start growing and finally will become equal to that of the parent activity. So this is the difference between the case of the secular and transient equilibrium. In transient equilibrium the parent is not that long-lived that it will not decay with time.

Parent is also decaying but the daughter grows and becomes more than that of parent activity and then starts decaying with the half-life of parent.



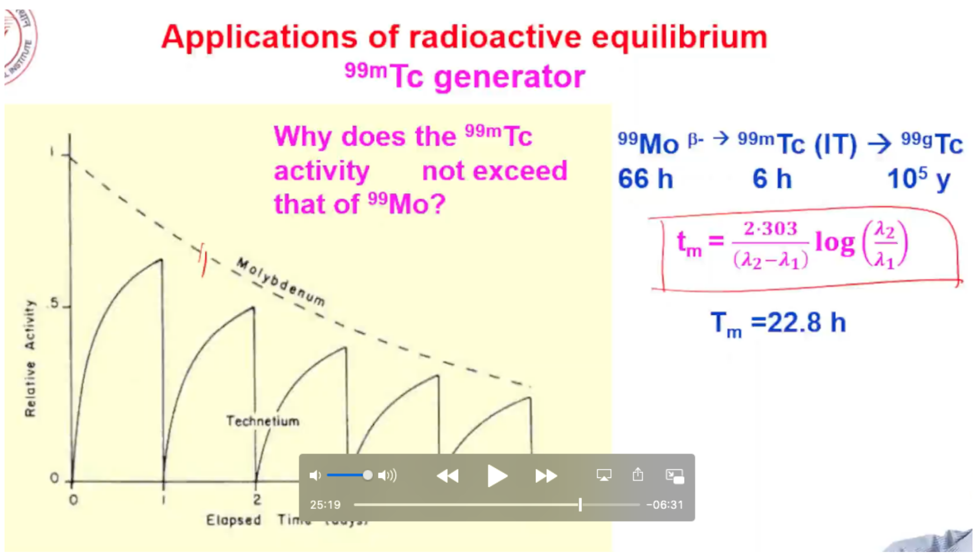
Now I will give you an example of this secular equilibrium here. Here now I have taken this as a log scale because the growth of the daughter is like linear that means the activity is plotted in logarithmic scale. So again this is an experiment one can do in the laboratory to resolve the total activity into parent and daughter activity and then find out their half-lives. So suppose you take a parent isotope and you do a chemistry to purify it.

You have pure freshly purified parent activity then the total activity will grow like this. And then you extrapolate to zero time. So extrapolate this to the zero time. Now in secular equilibrium the parent is very long-lived so you can draw a parallel line to this, which will represent the parent decay. So these two these two lines you know this is the decay of a freshly purified parent isotope and this is the decay of activity total activity if you did not purify the parent.

So when you did not purify the parent activity the total activity is sum of parent plus daughter and they are in equilibrium. Once we separate then there is no daughter activity as it becomes zero and it starts growing in the parent. So you can resolve the parent and daughter activities. These points you subtract this data from this data this is the total activity this is without separation and with separation if you subtract then you will get this data that is representing the daughter activity. It is like you know if you do a separation of daughter activity put it in a separate flask then there it is not growing from parent it will decay in its own half-life and that is what is represented by this graph.

So from this graph we can find out the half-life of daughter and the activity of daughter is nothing but the same as that of parent because they are in the secular equilibrium. So that is why the total activity will become double after equilibrium is established. Now you can see here this is the total activity and this is total activity at equilibrium and this is the parent activity. So because of the logarithmic scale this is equal to this that is what I wanted to convey that in logarithmic scale it is such that like 1 to 2 and 2 to 3 will look differently on log scale and the typical example I have shown here the experiment in fact we've done in the laboratories you take ^{137}Cs having half-life of 30 years and it undergoes beta minus decay to the isomeric state of ^{137}Ba, which has a half-life of 2.54 minutes and which emits a gamma ray of 661 kV to ^{137}Ba ground state which is stable.

So in the laboratory experiments the ^{137}Cs is held in a column which is selectively taking up cesium and then when you elute the barium you take some sodium chloride solution ^{137}Ba will be eluted and cesium will not be eluted and so whatever barium is eluted will decay into the half-life of 2.54 minutes because it is not growing now it is in a separate test tube and so you can get this data only from the separated ^{137}Ba and again you know it will again grow like the time to reach the maximum activity will be typically four times that daughter half-life so you can do this experiment multiple times from a column which is containing the ^{137}Cs and for years together you can use the same column so this is a experiment in the laboratory for radio chemical separations. Now I will just discuss why I took so much time to discuss this equilibrium cases because they have lot of applications. So the applications of radioactive equilibrium particularly in nuclear medicine or even if you want to use a radioisotope frequently then if you have a parent which is long lived you can make a generator. So radio isotope generators are based on the radioactive equilibrium.

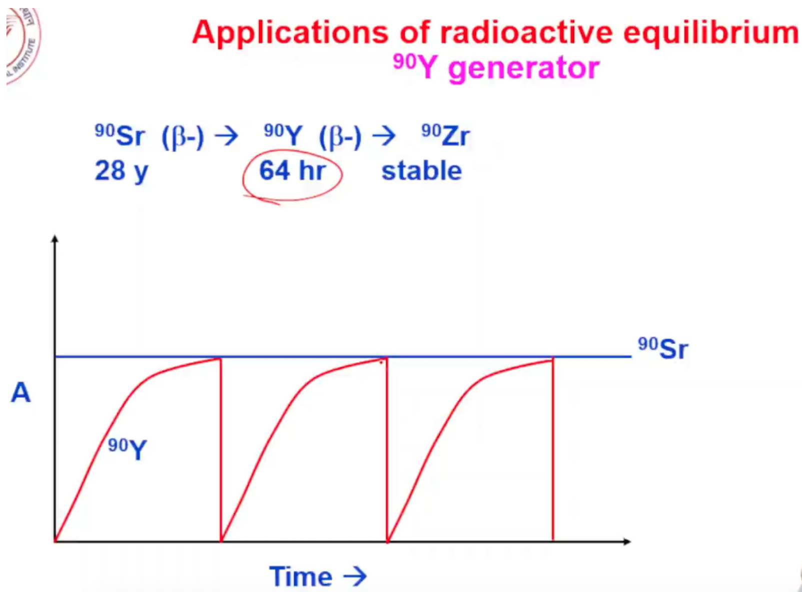


I will give you two examples one of transient equilibrium and one of secular equilibrium. This is the case of ^{99m}Tc generator. ^{99m}Tc is called as the workhorse of nuclear medicine. It is used in diagnosis of the diseases in the body. it has a half-life of six hours.

So you cannot produce this isotope take to the hospitals and then you have to keep on supplying every day the ^{99m}Tc activity. But if you have a generator system you have a parent which is long lived like ^{99}Mo having sixty six hours half-life and it is decaying by beta minus to ^{99m}Tc which undergoes internal transition by gamma decay to ^{99g}Tc having half-life of about 10^5 years. So now let us see you calculate the time when the activity of ^{99m}Tc will reach a maximum. Using this formula just now we derived, the time taken for ^{99m}Tc to reach its maximum is 22.8 hours. So the ^{99m}Tc activity will grow in the sample reach a maximum and you can then separate ^{99m}Tc use it for the investigation diagnosis like SPECT analysis and then after one day 22.8 hours again you have same amount of technetium in the column. So this is what is the concept of a radio isotope generator. So here on the left hand side I try to plot the activity of ^{99}Mo. So this is the activity of ^{99}Mo which is decaying exponentially it is a linear scale and now the technetium activity grows till let us say typically one day it reaches this value and then you elute this ^{99m}Tc from that column molybdenum remains in the column and this activity now you can use in the hospitals for diagnostic purposes.

Next day again you elute^{99m}Tc now the ^{99}Mo is decaying so this activity also will be less you again elute ^{99m}Tc next day morning carry out the tests that day, third day you again do the separation fourth day and fifth day and so on. So normally you know Monday the hospitals are supplied with molybdenum-99 and for that week because this half life is 66 hours 2.6 days or so. So you can carry on these tests for the whole week and next week again you have fresh lot of molybdenum-99. Now I have put a question mark here why does the technetium 99m activity do not exceed that of ^{99}Mo.

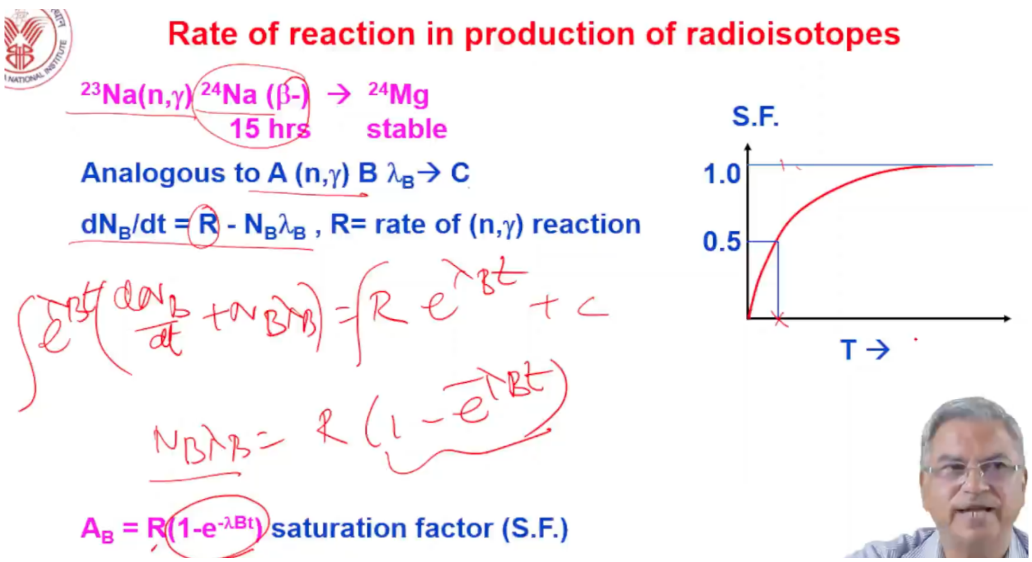
I explained that in the transient equilibrium the daughter activity will be more than that of the parent so it should have actually gone up and more than molybdenum but in this particular case what happens you are measuring the gamma ray from the technetium 99m sample and the gamma ray have certain abundance for 100 decays this gamma is emitted only 87 times so this is called abundance in percentage 87 percent so this is not the complete decay of the 99 molybdenum. Only the 87 percent of the time technetium 99m decay this gamma ray is emitted and that is why the gamma ray activity becomes less than the parent activity instead of going more than per molybdenum it is becoming less and so this is a typical case of a transient equilibrium where you can use generators based on this concept for the applications in healthcare.



Now another application of this generator is the ^{90}Sr and this is the case of a secular equilibrium where the parent is much longer than daughter. So you see here that strontium-90 is having half life of 28 years undergoing beta minus decay to Yttrium-90 having half life 64 hours and this is undergoing beta minus decay to ^{90}Zr which is stable. So now this is a case of secular equilibrium because the parent is much much longer than that of the daughter.

The ratio of half life is more than 100 it is quite high. So what I have plotted here is the activity of a case where activity of the strontium-90 will not change with time so half life is 28 years you are measuring for a day or two or few days it will there will be no change but the activity of Yttrium-90 will grow for about 10 days, that is, three four half lives the activity of Yttrium 90 will grow and then it will become flat it will become equal to that of the parent. So at this point of time Yttrium-90 can be separated and again it will start growing and you can again you reach the maximum value. So here both strontium-90 and Yttrium-90 are pure beta emitters that means the activity of the daughter will be equal to parent. There is nothing like a gamma ray abundance factor coming into picture here and also every time over a period of even say two weeks the activity of daughter will again become equal to that of parent.

So you can use this generator to separate ^{90}Y which is also another important isotope useful in therapeutic applications. Yttrium-90 is used in many therapeutic applications in nuclear medicine. So this is the case of secular equilibrium where you have Yttrium-90 generator.



Now lastly I come to the case of a rate of reaction in production of radioisotopes. We have just seen molybdenum-99 or strontium-90. So they are also produced in some process like fission or in a nuclear reaction. So the decay growth system of activity produced in a nuclear reaction also follow similar pattern as we discussed in the radioactive decay chain previously. So here I will give you an example of a nuclear reaction sodium-23 captures a neutron becomes sodium-24 and which emits a gamma ray. So prompt gamma is emitted and we have the ^{24}Na in ground state undergoing beta minus decay to magnesium-24 which is stable. So what we are essentially seeing the how the activity of sodium-24 will change with time when it is produced by n,γ reaction on sodium-23.

So this is analogous to A going to B by nuclear reaction and B going to C that C is magnesium-24. So instead of A going to B by beta minus decay or alpha decay here we have A going to B by (n,γ) reaction. So the profile will be similar. So here we can set up the equation for the formation of B

$$\frac{dN_B}{dt} = R - N_B\lambda_B$$

and you can again solve the same way that is

$$\frac{dN_B}{dt} + N_B\lambda_B = R$$

and if you recall the integration factor $e^{-\lambda_B t}$ and if you integrate what you will be getting is

$$N_B\lambda_B = R(1 - e^{-\lambda_B t})$$

So the activity of daughter, those steps I have not shown because they are similar to what we showed earlier for the case of A going to B going to C and so this factor is called as saturation factor that means the activity of B will grow by this factor $(1 - e^{-\lambda_B t})$ and what I have shown here is the same thing as a function of time the saturation factor this is the rate of the reaction so saturation factor will grow in this fashion and become close to 1 after some 4-5 half-lives.

So this in fact this graph is used to decide for how much time we need to irradiate this particular target in the reactor or accelerator. So suppose you have one half-life then 50 percent is produced and to get 100 percent you get about four half-lives and more than that there is no gain. So this gives an idea to fix the time of irradiation and the same profile you get for the activity of daughter when you are producing it in the irradiation. So that's all I have to say and in the next lecture now I will talk about nuclear structure and stability. Thank you very much.