Continuum Mechanics And Transport Phenomena
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Lecture - 20
Reynolds Transport Theorem: General form - Part 2

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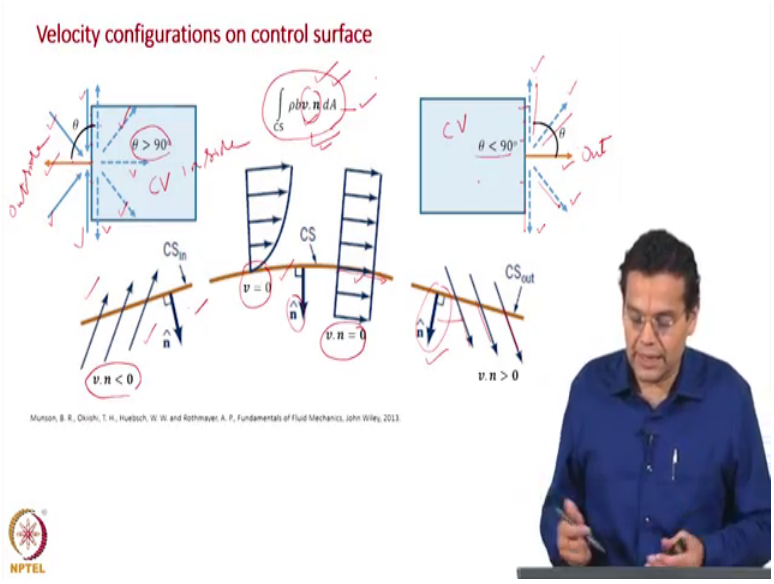

$$\int_{CS} \rho b\, v.n\, dA$$

So, we will see how this expression automatically takes care of inflow and outflow. We had two different terms for inflow and outflow; now we have only one term, but that automatically takes care of outflow and inflow. As earlier we will start with outflow which is easier to explain first; this is the control volume (top right hand side image) and the outside and there is outflow through this control surface and that is why we are drawn an outward normal. So, inside of control volume and outward normal for this control surface.

Now, what I have shown here are few typical velocity vectors (dotted arrows in the right top image), because it is an outflow surface so fluid can flow along any direction and what is shown is also the angle theta between the normal vector and the velocity vector. Now based on this configuration, we can easily conclude that angle theta is going to be less than 90 degrees for this configuration which means for the outflow control surface and that is what is also shown here (bottom right hand side image) in another way from this book, the outward normal to the outflow control surface, the direction of outflow vectors and the angle is less than 90 degrees; which means that $v.n$ is going to be positive. So, this integral expression is positive for outflow.

Let us see what happens at the inflow surface, this is the control volume (left hand side top image) and there is inside, and the outside and remember, we always draw outward normal. So, the normal points away from the control volume and the left side surface represents the inflow control surface.

Now here again I have shown, different possibilities of the direction of inflow velocity vectors and to represent the angle, the velocity vectors have been extended in this direction and the angle between the normal and these extended velocity vectors is shown. Now, what is the value θ can take? Based on this configuration we can easily conclude that θ will always be greater than 90 degrees; which means that $v.n$ will be negative. So, this expression will be negative for the inflow surface.

So, $v.n$ automatically takes care of inflow and then outflow boundary based on the angle and another representation of inflow boundary is shown here (bottom left hand side image) and once again the angle between the normal vector and the velocity vector will be more than 90 degrees, resulting in $v.n < 0$. So, the integral expression for the inflow boundary is negative, and for the outflow boundary is positive. Let us see what happens at the walls and that is what is shown here (middle image) at walls the fluid clings to the surface of the wall; which means that the velocity is 0 ($v = 0$) at the surface of the wall, which means that there would not be any contribution from this term.

If the fluid flows or slides along the surface of the wall, then the angle between the normal vector and the velocity vector is 90 degrees once again $v.n = 0$. So, whenever you have a wall either it clings to the surface of the wall or it slides along the surface of the wall, in both cases the integral term will not contribute. And that is why we represented in the previous case, summed up just control surface out plus control surface in as control surface; because you do not have a contribution from the walls where there is no either inflow or outflow.

Now, because this expression is positive for outflow and negative for inflow, this integral represents the net rate at which property leaves the control volume through the control surface. That word leaves is important because if this is positive which means there is outflow through the control surface; if it is negative which means there is inflow.

So, there could be some surfaces where there is inflow, some surfaces where there is an outflow and for all the inflow this will be negative, for outflow this will be positive. And if this turns out to be net value after taking care of all the outflows and inflows is going to be positive; which means that there is a net outflow of the property from the control volume through the entire control surface, no longer we say in or out.

So, $\int_{CS} \rho b\, v.n\, dA =$ net rate at which the property leaves the control volume through the control surface; we do not say inflow or outflow, it includes the entire control surface.

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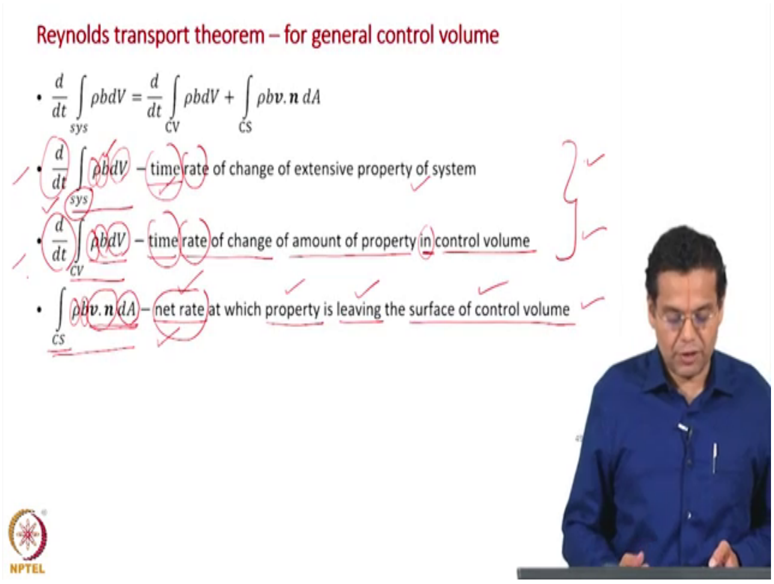

So, having discussed in detail the new term, other terms are known to us; let us look at the physical significance we had discussed that few times just to summarize here.

$$\frac{d}{dt}\int_{sys} \rho\, b\, dV = \frac{d}{dt}\int_{CV} \rho\, b\, dV + \int_{CS} \rho b\, v.n\, dA$$

So, this is the general form of the Reynolds transport theorem; it is applicable for a general control volume means that now we allowed for any part of the surface to be inflow, any part of the outflow, which means any number of inlets and outlets, allowed for variation of properties across inlet surface outlet surface, we also allowed for any angle between the velocity vector and the normal to the face.

So, whatever assumptions we had, they all taken care of that now. Now the left hand side represents

$$\frac{d}{dt}\int_{sys} \rho\, b\, dV = \textit{the time rate of change of the extensive property of the system}$$

I specifically added the word time, because the moment you say the rate of change usually it means with respect to time to be more explicit included the word time rate of change of extensive property of the system and how do you quickly interpret?

You are taking a small volume dV multiplying by density get a mass of the small volume, multiply by the property per unit mass, you will get the property for the small volume, integrate over the entire volume you get the property for the system and take $\frac{d}{dt}$ then that gives you a time rate of change of extensive property for the system. Analogously that is for the system, rate of change of extensive property of the system.

$$\frac{d}{dt}\int_{CV} \rho\, b\, dV = \textit{time rate of change of the amount of property in control volume}$$

Now, similarly, take a small volume in the control volume, multiply by density, multiply by b that gives the amount of property in the control volume. When you integrate, gives the amount of property in the entire control volume. Take time derivative gives you the rate of change, more specifically time rate of change of the amount of property in the control volume. A sometime back discussed, we usually say extensive property of the system; we do not say property of control volume not so very correct, we say a property in control volume or property of contents of the control volume.

$$\int_{CS} \rho b\, v.n\, dA = \textit{net rate at which the property is leaving the surface of control volume}$$

We have discussed in detail the significance of this term, which represents, the net rate at which the property is leaving the surface of control volume. We have seen why is it leaving, because it is $v.n$ is positive for outflow and $v.n$ is negative for inflow that is why it is leaving. Why is it net rate? Because part of the surface can be outflow, part of the surface there can be inflow and of course, that is why it is the net rate at which the property is leaving the surface of control volume.

How do we interpret this? We take a small area and then let us say velocity taking into account the projection of velocity along the direction of area. So, this gives the volumetric flow rate we have area multiplied by velocity when I say velocity along the normal direction to the area, so this gives you volumetric flow rate, you multiply by density get the mass flow rate; and then when you multiply by b which is property per unit mass, you get the rate at which the property leaves enters etcetera and when you integrate to entire surface you get the net rate at which property is leaving.

So, to distinguish the net rate and the time rate, specifically I used time rate for first two terms; the first two represent rate of change with respect to time because of with respect to time, there is variation in the property of system, a property in control volume. The net rate is because of fluid flow, so this rate is because we have a fluid flow; a flowing fluid has with it associated mass, momentum, energy, and then species mass, so whatever carried by the fluid flow is represented by the third term. The first two represents whatever change with respect to time, one for system, one for control volume.

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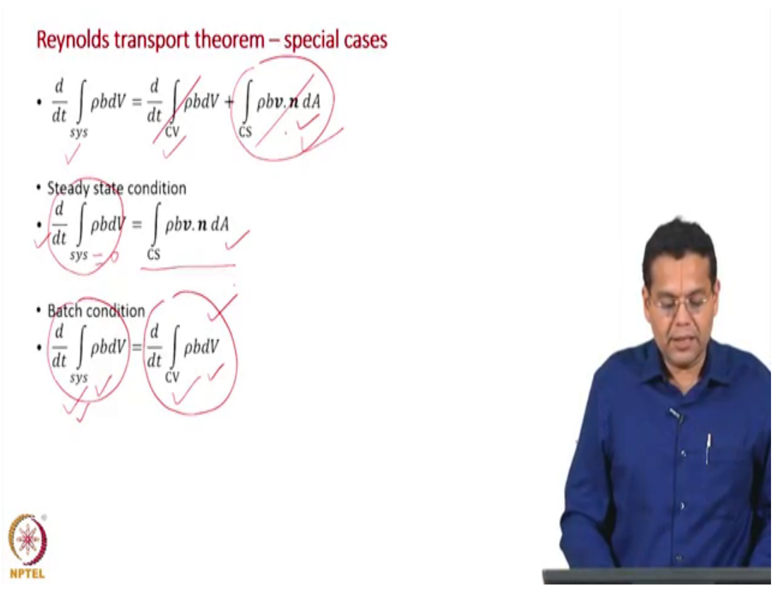

Let us look at two special cases of the Reynolds transport theorem. So, let us write down the Reynolds transport theorem first;

$$\frac{d}{dt}\left(\int_{sys} \rho\, b\, dV\right) = \frac{d}{dt}\left(\int_{CV} \rho\, b\, dV\right) + \int_{CS} \rho b\, v.n\, dA$$

The rate of change of property for the system is equal to rate of change of property for the control volume plus net rate at which the property leaves the control volume through the control surface. There are some special case of Reynolds transport theorem

1. Suppose let us say, we are going to apply this Reynolds transport theorem for a tank, and let us say the tank is operating under steady state condition and since the tank is operating under steady state condition when I say tank it is a control volume. So,

$$\frac{d}{dt}\left(\int_{CV} \rho\, b\, dV\right) = 0$$

So, we are left out with the left hand side, rate of change of property for the system, and right hand side we have only the term corresponding to the net rate at which the property leaves the control surface.

$$\frac{d}{dt}\left(\int_{sys} \rho\, b\, dV\right) = \int_{CS} \rho b\, v.n\, dA$$

Now, these two become equal. So, under steady state conditions, the rate of change of property for the system and net rate at which the property leaves the control surface both are equal. Now depending on the property, the left hand side may be 0 or may not be 0. Suppose we are applying for mass conservation, then we know that rate of change of mass for a system is 0, so

$$\frac{d}{dt}\left(\int_{sys} \rho\, b\, dV\right) = 0$$

So, the net rate at which mass leaves the control surface is 0. But suppose if we are applying this for a momentum balance then the left hand side, we have seen from the law of physics; that the rate of change of property in this case momentum is equal to the sum of forces which means there is nonzero. So, in that case, the net rate at which momentum leaves the control surface will be equal to the sum of forces.

$$\frac{d}{dt}\left(\int_{sys} \rho\, b\, dV\right) \neq 0$$

So, this net rate at which property leaves maybe 0 or may not be 0, under steady state condition.

2. Now let us look at the second special case; a batch condition means we have a tank there is no inflow and outflow that is what we mean by batch condition, which means that

$$\int_{CS} \rho b\, v.n\, dA = 0$$

There is no inflow there is no outflow. So, this term represents the net rate at which property leaves the control surface that term is 0. So, now, what are we left with,

$$\frac{d}{dt}\left(\int_{sys} \rho\, b\, dV\right) = \frac{d}{dt}\left(\int_{CV} \rho\, b\, dV\right)$$

On the left hand side we have rate of change of property for the system and on right hand side we have rate of change of property for the control volume and both of them are equal under batch conditions.

There can be another situation where both of them are equal; where there is inflow and outflow, but it so, happens that the net rate of flow is 0. Once again under that condition, the rate of change of property for the system and the rate of change of property for the control volume both are equal. Now if you look at the Reynolds transport theorem, the rate of change of property for the system and control volume are same, but for this term which represents net rate at which property leaves the control surface, otherwise, both of them are same.

So, this term is what makes a difference between the two time derivatives. So, since in this condition batch condition or where the net rate term is 0, then the rate of change of property for the system and control volume both become equal. Now once again, whether the left hand side is 0 or not depends on the property for which we are applying. If you are applying for mass balance the left hand side is 0, rate of change of property for the system is 0. And so the rate of change of property in this case mass, so the rate of change of mass for the system is 0 and the rate of change of mass for the control volume is also 0.

Suppose we are applying for energy balance; what will happen? The left hand side is rate of change of energy for the system which is nonzero. Because that is equal to the net rate at which heat transfer takes place, work transfer takes place, so which is nonzero. So in that case, the rate of change of energy for the control volume will be equal to net rate of heat transfer and work transfer. So, the left hand side could be zero or nonzero depending on the property or the balance for which we are going to apply this equation.

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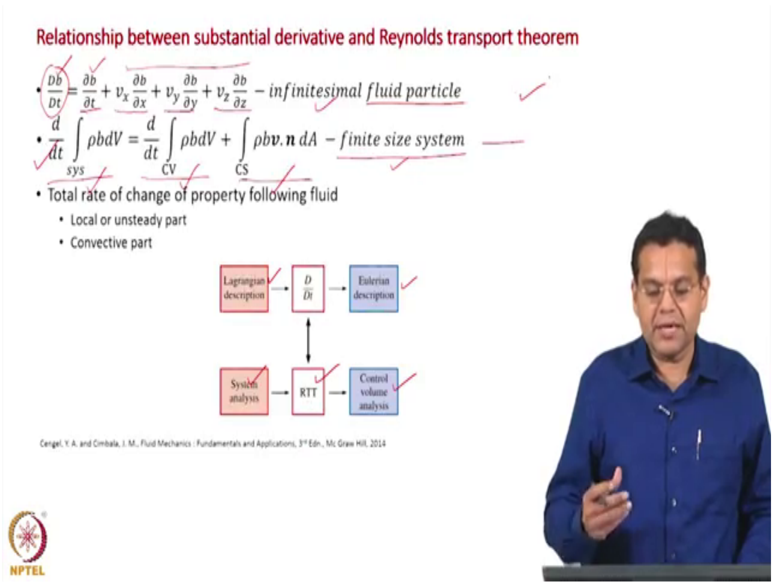

Now, what we are going to discuss here is the relationship between the substantial derivative and Reynolds transport theorem. We have discussed a substantial derivative a few classes earlier and just now we have discussed the Reynolds transport theorem.

There is a relationship between these two and that is what we are going to discuss now. So, let us write down the substantial derivative for a property b,

$$\frac{Db}{Dt} = \frac{\partial b}{\partial t} + v_x \frac{\partial b}{\partial x} + v_y \frac{\partial b}{\partial y} + v_z \frac{\partial b}{\partial z}$$

Now let us write down the Reynolds transport theorem also and then let us discuss the relationship.

$$\frac{d}{dt}\int_{sys} \rho\, b\, dV = \frac{d}{dt}\int_{CV} \rho\, b\, dV + \int_{CS} \rho b\, v.n\, dA$$

Rate of change of property for the system on the left hand side, rate of change of property for the control volume, and net rate at which property leaves the control surface.

Now, what is the significance for the substantial derivative on left hand side $\left(\frac{Db}{Dt}\right)$ represents, the rate at which property changes as you follow a fluid particle, the fluid particle is important terminology here. So, we applied the substantial derivative or we derived it for expressing the rate of change of property as we follow a fluid particle. So, the left hand side is a Lagrangian representation, which tells about how property changes as you follow a fluid particle. Right hand side we said it has two components, two contributions; the first term $\left(\frac{\partial b}{\partial t}\right)$ represents the local contribution that is because of the unsteadiness of the field. When I say field, it could be temperature field, velocity field, whatever field we are applying for, and the next set of three terms $\left(v_x \frac{\partial b}{\partial x} + v_y \frac{\partial b}{\partial y} + v_z \frac{\partial b}{\partial z}\right)$ are because of convection, because of fluid flow.

So, the first component is because of the unsteadiness which is a local component; the second contribution is from the convective component and the two contributions on the right hand side are the Eulerian representations. So, as we discussed substantial derivative expresses a Lagrangian derivative in terms of Eulerian representation. Left hand side we have Lagrangian description, right hand side we have Eulerian description.

Now, let us look at the significance of the Reynolds transport theorem; the left hand side $\left(\frac{d}{dt}\int_{sys} \rho\, b\, dV\right)$ is rate of change of property for a system which is once again a Lagrangian description or Lagrangian concept. And right hand side, the two terms correspond to the Eulerian description because they are for a control volume and then corresponding control surface.

What is the significance of the left hand side, rate of change of property for the system; right hand side we have rate of change of property for the control volume $\left(\frac{d}{dt}\int_{CV} \rho\, b\, dV\right)$ that once again represents the unsteady component, because that component arises because of the transience; and the second component $\left(\int_{CS} \rho b\, v.n\, dA\right)$ arises because of flow through the control surface or because of convection. So, left hand side is the Lagrangian representation, right hand side is the Eulerian representation and we have written the Reynolds transport theorem for a system.

Now, let us compare both and that is what is shown here Reynolds transport theorem relates a system analysis to a control volume analysis. Now let us look at the analogy between the two; the substantial derivative also left hand side was Lagrangian representation, in the Reynolds transport theorem also the left hand side was Lagrangian representation. Coming to the right hand side in both the substantial derivative and Reynolds transport theorem they represent the Eulerian representation. Even in terms of contribution, we have two contributions in both the cases; the first contribution is the local contribution or unsteady state contribution, the second contribution is from convection or because of fluid flow and that is what is shown here, the total rate of change of property following fluid has two contributions local or unsteady part and then the convective part.

So, both in terms of description, that is a left hand side for Lagrangian and right hand side is for Eulerian; in both the cases, substantial derivative and Reynolds transport theorem. Not alone that the right hand side even the two contributions are analogous; the unsteady part, the local part, and convective part. Now, what is the difference between these two?

Now if you look at the substantial derivative as I told you the keyword is the fluid particle, the substantial derivative is for a fluid particle. The Reynolds transport theorem, we have written for a finite size system, fluid particle are infinitesimal so more precisely infinitesimal fluid particles; the Reynolds transport theorem is for a finite size system.

Now, if you recall back we said a system can be considered to be consisting of several fluid particles and that is what is brought out here as well. So, while $\frac{D}{Dt}$ is for an infinitesimal fluid particle; the corresponding integral representation in the Reynolds transport theorem for a finite size system. So, $\frac{D}{Dt}$ is for infinitesimal fluid particle, to a formal word the integral representation when I say whenever if say integral it is for a finite size.

So, the integral representation of $\frac{D}{Dt}$ is the Reynolds transport theorem. Now looking at the other way, suppose if you take the Reynolds transport theorem and make the volume shrink to 0, you will get the substantial derivative. In fact, that can be mathematically proved. So, if you want to go from the Reynolds transport theorem to the substantial derivative, you shrink the volume to 0. So, you have a finite size system there are several fluid particles, you make it to 0 which means; you will have only one fluid particle and that is what the substantial derivative represents.

So, in terms of physical significance, they are analogous, in terms of size they are different. One is for infinitesimal fluid particles, the other one is for finite size system, and that is the relationship between the substantial derivative and the Reynolds transport theorem. A nice representation is shown in the figure (above referred slide) from this book cengel and cmbola fluid mechanics fundamentals and applications. And that is the reason, we discuss because we are discussed substantial derivatives sometime earlier. Now we are discussing the Reynolds transport theorem. So, want to emphasize the analogy between these two and what is the difference in terms of scale or length scale; one is for very small infinitesimal particle scale, other is for finite system scale. That is why whenever we represent fluid particles, we represent by a dot, but the finite system we will represent by a region.

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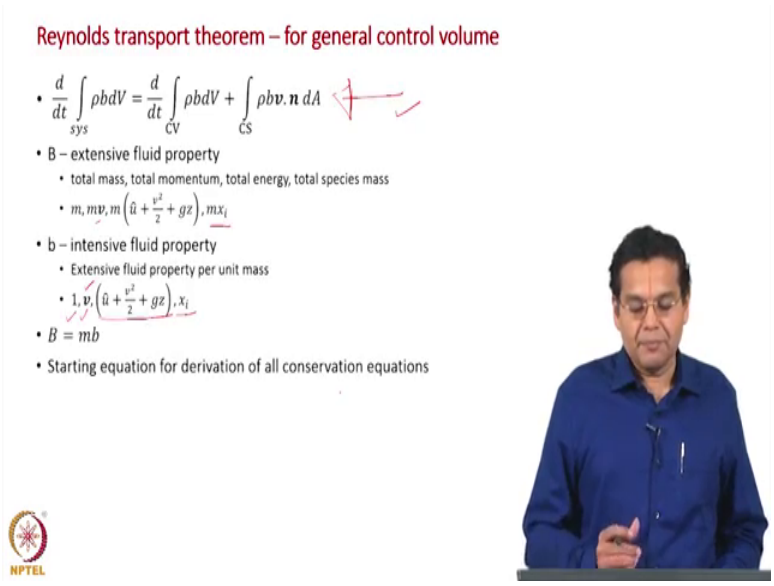

We have discussed the significance of B, we said B is extensive fluid property, which could represent total mass, total momentum, total energy, total species mass. In terms of variables, they are mass multiplied by one, velocity, the internal, kinetic, and potential energy per unit mass, and then the mass fraction of species.

The b represent the intensive fluid property, the extensive fluid property per unit mass

$$\frac{d}{dt}\int_{sys} \rho\, b\, dV = \frac{d}{dt}\int_{CV} \rho\, b\, dV + \int_{CS} \rho b\, v.n\, dA$$

This same equation represents or we going to use it for different values of b; namely, for mass total mass it is 1, for the momentum it is v and for energy, it has sum of all this internal kinetic and potential energy and for species, it has this mass fraction (x_i). Of course, both are related by $B = mb$.

This equation which represents a Reynolds transport theorem is a starting equation for the derivation of all the conservation equations, which we will see as we go along; of course, it looks little scary you have an integral expression and that integral expression is within a differential expression and that likewise, you have two expressions one on the left hand side right hand side have another integral expression.

So, on the look of it looks a little scary, but you will see that we will simplify them in when it comes to the applications. We will apply this for deriving the conservation equation for mass, momentum, energy, and species mass and when we apply we will simplify that, and becomes very easy for an application.

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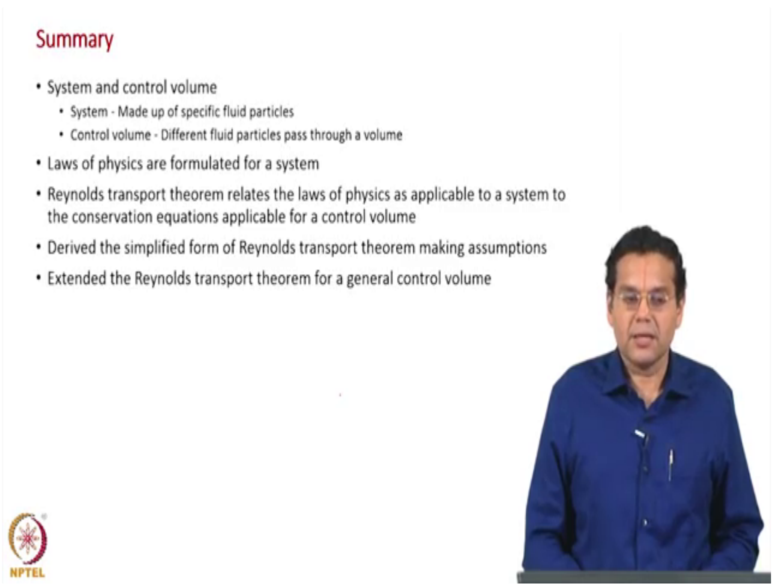

So, that brings us to the close of this part of the fundamentals. We defined and discussed system and control volume, the system made of specific fluid particles, control volume different fluid particles pass through; and we discussed the need for Reynolds transport theorem, laws of physics are formulated for a system and Reynolds transport theorem relates the laws of physics as applicable to as system to the conservation equations applicable for a control volume. Now this sentence makes more sense, the derivative of the left hand side was for system the right hand side all related with a control volume.

How exactly we are going to apply that, we will see as we go along. We derived the simplified form of a Reynolds transport theorem making assumptions and then extended the Reynolds transport theorem for a general control volume.

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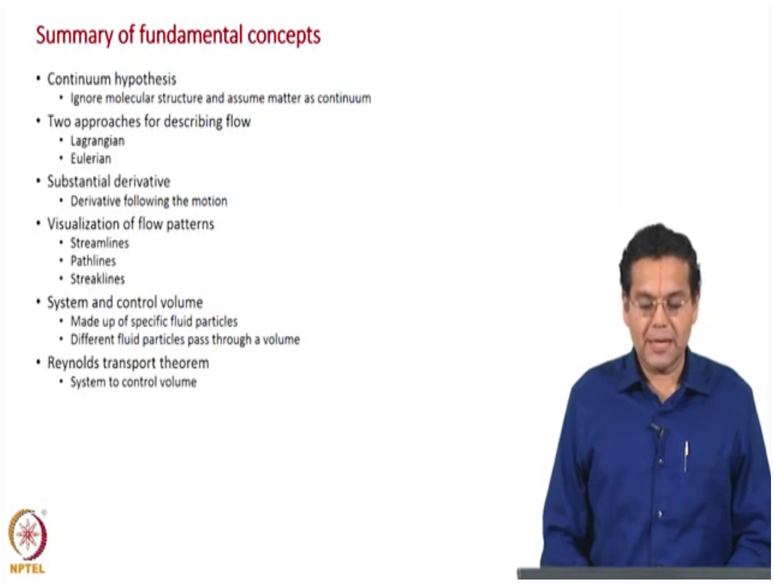

So, this brings us to the close of the first part of the course and fundamental concepts. Let us quickly summarize them,

- Started with the continuum hypothesis which ignores molecular structure and assumes matter as continuum.
- Two approaches for describing fluid flow namely Lagrangian and Eulerian;
- Looked at the substantial derivative which is derivative following the fluid motion; looked at different ways of visualizing flow patterns namely streamlines, pathlines, and streaklines.
- We distinguished system and control volume, the system made of specific fluid particles, different fluid particles pass through a control volume, and finally
- The Reynolds transport theorem which links the laws of physics for the system through the conservation equations for a control volume.

I wish to emphasize that almost all the topics which are discussed under fundamental concepts are not restricted to fluid mechanics alone. Usually, we discussed all these topics under fluid mechanics but should be kept in mind that almost all the topics except; possibly visualization of flow patterns, all the topics are not restricted to fluid mechanics alone. They are applicable to the flow of fluids, flow of energy, and flow of species mass as well.

For example when we discussed Lagrangian, Eulerian we took examples of temperature measurement, concentration measurement. Substantial derivative; we discussed about derivative following the motion for the case of temperature, the concentration, of course, velocity as well. Similarly, the concept of system and control volume and Reynolds transport theorem are generic; and in fact, Reynolds transport theorem is going to be applied for the derivation of conservation equations, all the conservation equations for mass, momentum, energy, and species mass. So, and that is why all these topics are been put under fundamental concepts. So, these concepts can be applied and for the derivation of conservation equations of total mass, momentum, energy, and then a species mass.