

CH5230: System Identification

z-Domain Descriptions 1

It's always the case, the reason for moving to a transform domain is because some things become convenient. What we cannot do easily in the raw domain. The raw domain for us is a time domain, becomes easier in the transform domain. For example, why do I actually soak cloth in water, I'm

actually transforming, I'm going to a new domain, because it becomes easier to separate the dirt from a cloth. Right?

In the raw domain that means in air, it becomes difficult to separate the dirt from the cloth but the moment I soak it in water, which is equivalent to a transform, the dirt can be easily, the separability is improved. And in signal processing that is one of the prime reasons why signals are transformed for separating the signal from noise, because separability is enhanced by leaps and bounds. Why do I actually heat a rod-- metal rod informing, if you look at foundry, why do I heat the metal rod because when it is very hot, I can actually turn it into any shape that I want.

But when you bring it back to room temperature it becomes very difficult. Heating the rod and taking it to a certain temperature it's also the analogy of a transform. In signal analysis and in system analysis, the reasons are almost the same. We go into a new domain, a transform domain because the analysis becomes easier. And always remember this, map is all about, I mean, largely about representing. It's all about representing numbers, if you take languages, there are many, many languages that we have on this planet, and certain things are best said in certain languages, right? For it to be effective. Likewise, certain types of analysis are best done in a certain representation of numbers, representation of signals and so on. So here we are talking of representation of signals and that why I say z-transforms and representations.

So let's now understand how one goes from time domain to z-domain, to this new domain. Why do we study that because, the natural domain in which we work is a time domain. All right. So I've already given you the motivating reasons, for in general working with transform domains, specifically when it comes to looking at z-domains for the analysis of LTI systems there are some advantages. And the first advantage is that the difference equation form that we have seen or even the convolution form that we have seen, transforms into an algebraic form.

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z-Domain Descriptions

z-Domain Representations

The main advantages of analyzing an LTI system in the z -domain are:

- ▶ **The difference equation description takes an algebraic form in the new domain**
- ▶ **The characteristics of the system and the response are captured in the singularities and zeros of the function in the z -domain.**
- ▶ Computing the response using DE forms is made easy using z -domain representations.
 - ▶ All computations are first performed in z -domain. In the final step, the response is obtained by performing an **inverse transform**.

It is not just the signals that undergo a transform, the moment the signals undergo a transform, their relations also undergo a transfer, which means-- and also it means that certain operations that you're performing in the time domain also undergo a transform. So, for example, one of the standard operations, you would perform is shifting a signal in time, that means delaying a signal. That's an

operation, in time domain. That may not be the same in the transform to that, so, there is a thread that connects these two.

When you do something here this thread actually cause something else in the transform domain. Therefore, it is important to understand these properties and in the end we will realize that the difference equation or even the convolution equation form takes on an algebraic form. Why is it an advantage because it's easier to work with algebra, algebraic forms are much more easier to work with than difference equation forms. This is the same in continuous time case as well. I'm sure all of you are familiar with Laplace transforms. The reason is the same. The reason for working in Laplace domain when it comes to continuous time LTI systems is the differential equation forms.

Assume an algebraic form in the Laplace domain. And we know very well, solving a differential equation is much more painful than solving an algebraic equation. Therefore, you do whatever you want to do in the Laplace domain, the solution and the final step, come back. So in other words, I want to clean the cloth, I can't do it easily in the raw domain. I soak it in water, I can actually go to a specific part of the cloth, apply maybe some detergent, some soap whatever operation I want to perform, I operate it in the water domain, where the cloth is being soaked. I'll perform all kinds of operations there spinning, rinsing, whatever I want to do. Finally, when the cloth is ready to be used, I'll dry it and get it back to the domain.

Same idea here, I will perform whatever operations I want to. Suppose I want to compute the response of the system, we have talked about impulse response, right? But that's only one response, tomorrow we may be interested in knowing the response or system to some arbitrary input. Solving a difference equation form or a differential equation becomes very difficult. It can be time consuming, whereas in the new domain, maybe it's much more easy and that is the case in fact, you perform all your computations there. Finally, when the response is ready to be mapped back, you take an inverse transform and you compute it. So that is one of the prime reasons.

The other reason of course, is that the characteristics of the system, what we mean by characteristics is stability, dynamics, that means certain features of dynamics whether I will see-- get to see an oscillatory response, whether I will get to see an inverse response or a particular kind of damped oscillatory response, what is it? What are the different characteristics that, I mean, normally get to see can be easily discovered by analysing the relationship in the transformed domain.

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So, we'll see that shortly. First let us quickly understand what is a z -transform? The z -transform as you see on the screen is defined in a similar vein; you see the similarity between the shift operator and the z but the z now here is a complex variable.

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z-Transform

Definition

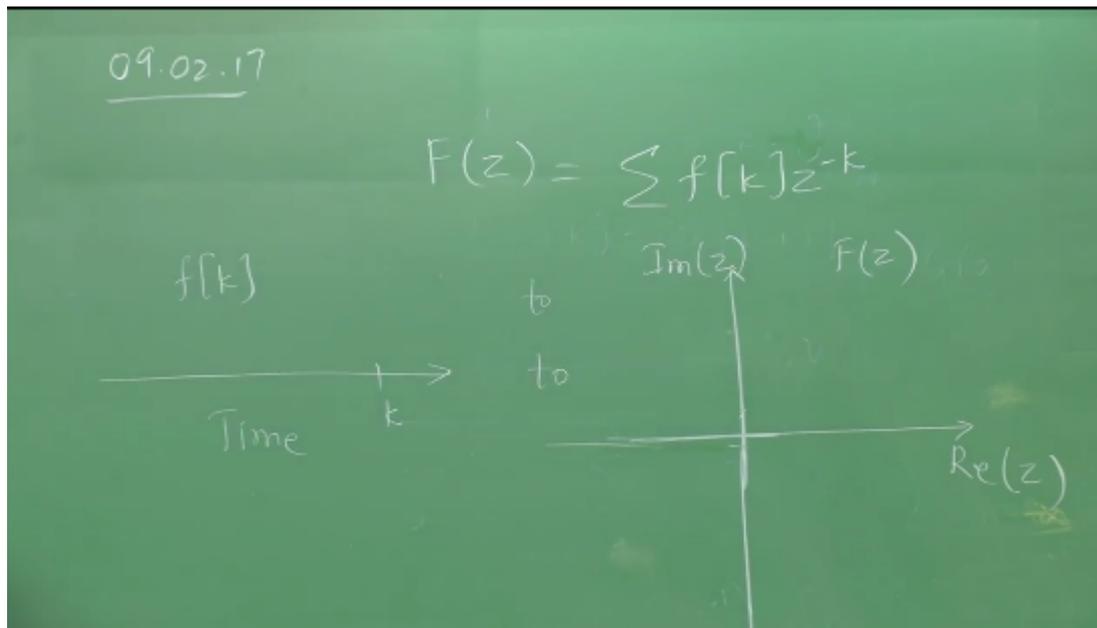
The **one-sided** z -transform of a signal $f[k]$, denoted by $F(z)$, is defined as

$$F(z) = \sum_{k=0}^{\infty} f[k]z^{-k} \quad (1)$$

- ▶ The signal is now characterized by a complex variable $z = \sigma + j\omega$
- ▶ It is assumed that signal $f[k]$ is zero for negative times, i.e., $f[k] = 0, k < 0$

So, what you're doing is, you're moving from time domain you can say this is time to a complex plane. So, this is real part of z and here is imaginary part of also. So f of k some function here is being mapped to some function here. z is in general a complex variable.

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Okay? Now by the way, this is called a unilateral z-transform. There is something called a bilateral z-transform, in which a summation runs from minus infinity to infinity. But the unilateral z-transform is the one that is particularly useful in the analysis of LTI systems. You can still use bilateral but there are more advantages to work with unilateral transforms, of course, in working with this unilateral that transform we will not keep saying unilateral and just telling you right now, but later it's understood we're going to work with this one-sided z-transform. The assumption-- the implicit assumption is that the function, or the sequence or the signal is zero at negative times. If it isn't, then you will have a problem, right? You would have ignored a portion of the function. Because the summation start from k equals zero. So this unilateral or one-sided, z-transform is useful in the analysis of either causal signals and we say causal signals, signals that begin from time zero, that are zero valued at negative times or causal systems. If you know that the system was not zero or the signal was not zero at negative times, then you should keep that in mind and exercise a lot of caution, okay?

Alright, so let's look at quickly some z-transforms of standard signals once again, because we will keep coming back to this. So if I have-- now we're studying the mapping, you're trying to understand. It's like taking from-- if your mother tongue is Tamil and if you're going into English or Sanskrit, and so on, so you need a Tamil to Sanskrit dictionary. So what words in Tamil? How do they map two words in Sanskrit? So here I have an impulse signal in time domain and I want to know how it looks like in z-domain. How does it look like?

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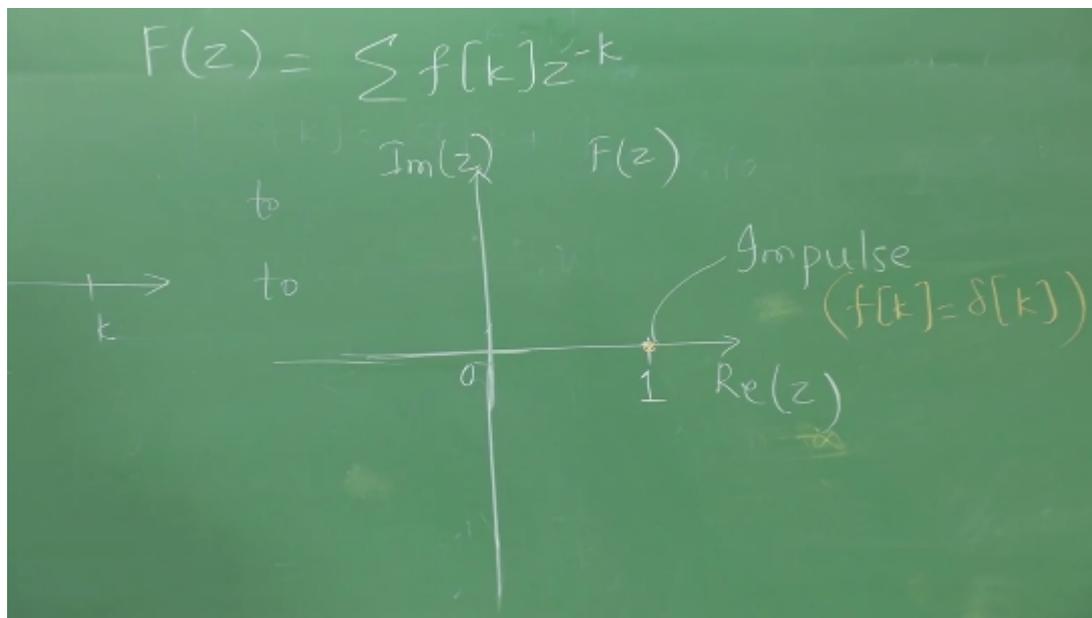
z-transforms of standard signals

1. Unit impulse:

$$f[k] = \delta[k] = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} \implies F(z) = 1$$

Well, we know that the impulse description is as follows, it's one at lag zero. Unit impulse is what we are looking at. It turns out that in z-domain, the function-- this impulse maps to one, just a number one, which means if I consider an impulse let say, here is my zero and here is one, this is the representation of the impulse. Right? That is af of k is the delta k. So this point in the z-plane here corresponds to an impulse signal.

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Look what has happened? A signal, which typically exists from zero to infinity, has now been map to a point. Even that's the case in Fourier transform, right? You can see the similarity of this with the DTFT. This is a more-- this is considered a generalized version of the Fourier transform, in fact, that is another route for arriving at z-transforms. If you look at signal books written on signal processing, they would say, well, look Fourier transforms exist for signals that are only absolutely convergent, but there may be signals that are not absolutely convergent, that may blow up. Can I still transform them?

Yes, you can, but not with Fourier transform, but it's with z-transform. Okay? Here z is a complex variable; it has a real and an imaginary portion. An imaginary portion is indeed the Omega, the frequency that we talk off in Fourier transforms.

So the point to remember is this infinitely long sequence in general, impulse you may not think of is infinitely long sequence but if you look at a step, a step exist forever, starting from zero it exist forever. Where does it get mapped to? Well, it gets mapped to one over one minus z- inverse, but in a region only in a z-plan that region is called the Region of Convergence. So if you look at the math carefully here, I plug in the value of f(k) and I get a summation z to the minus k and I have one over one minus z-inverse. Right? I've written is at one over one minus z-inverse, assuming that this summation converges to one or one minus z-inverse only in this region, otherwise, this f of z doesn't exist. In other words, the step does not map to anywhere in a z-plan.

The impulse also does not map anywhere the z-pane. The impulse also does not map anywhere. The impulse sequence maps to a point called unity. The step now maps to some region in the z-plan, you have to guarantee that there exists a region that there exists a--see as soon as a bus arrives, what do we do in India? Volvo buses do not allow us to do that, but if you take the village bus, right? The bus that goes from point A to-- village A to village B, and City A to City B, what do we do? Whatever we have-- if I've babies, I'll put the baby in the seat, if I have kerchief, I'll put a kerchief in the seat, and claim my position, correct?

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--:Domain Descriptions

z-transforms of standard signals

1. Unit impulse:

$$f[k] = \delta[k] = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} \implies F(z) = 1$$

2. Unit step:

$$f[k] = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases} \implies F(z) = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1 - z^{-1}}; \quad \text{ROC: } |z^{-1}| < 1$$

So then they'll be war of kerchiefs, when you get in. Here also-- that is to make sure that I'm guaranteed a seat when I board, doesn't matter even if a board after an hour I have put my kerchieffirst. So that's my position. Here also, when I'm looking at signals there is no kerchiefbusiness here, but you have to guarantee that their exist a region, in which the mapping exists. And that region is called Region of Convergence. If you cannot find even a point at which f of z exist for a given signal, then you say, this signal does not have a z-transform. Okay? So here the step signal-- unit step signal does have a region that region is this z-inverse greater than one, which means you can write it as z greater than one. Sorry, z-inverse less than one in magnitude, which means z greater than one.

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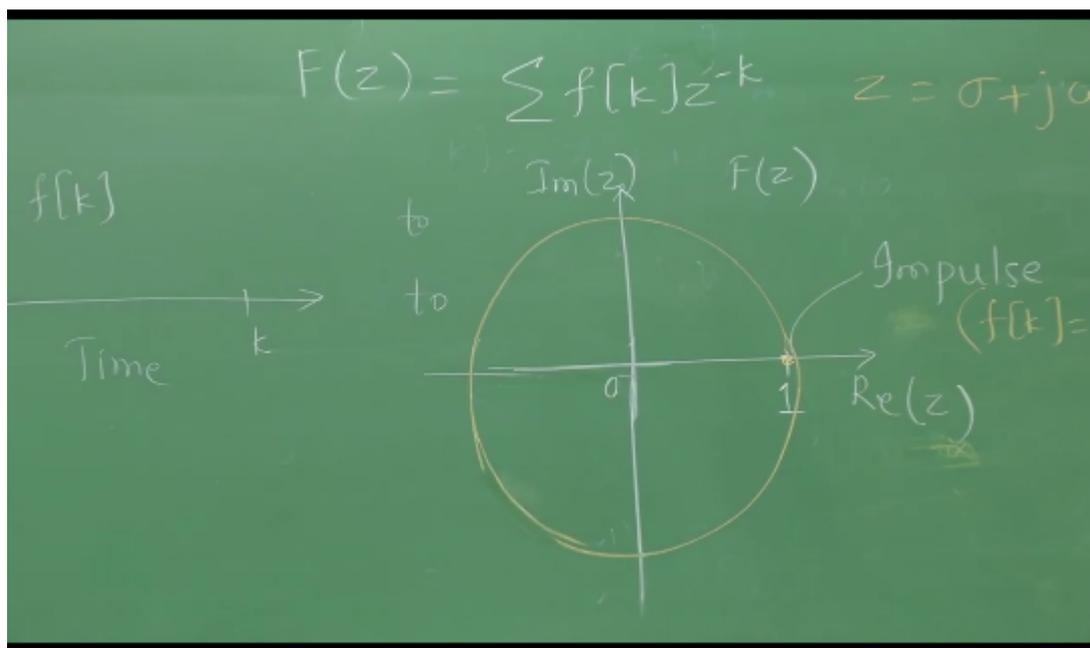
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System Identification

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What is that mod z greater than one region? How does it look like? Unit circle, right? So the mapping is all the points on the-- any point exterior of the unit circle, unlike the impulse which map to a point, now if you look at a unit circle here, i don't, i don't know if i have drawn a circle better than this in my life before, but, so, this is the unit circle you can say. Here is a minus i and this is i or j , whatever. Electrical engineers prefer the use of j , not ask them why? Maybe because they say i stand for current. But I always thought Electric Engineers didn't like the word i because it means egoistic, so j is preferred, so that we use j and we'll stick to j . f of z , when it comes to step exist anywhere outside this but not within this. [speaking in Hindi].

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Okay, that is what it is. When it comes to exponential signal-- exponential, when we say exponential, it's a to the k , don't think of e to the k . Then the region again-- f of z exists in a certain region of convergence. Slowly we will drop specifying this ROC assuming that there exists an ROC, you should verify it. Right? Again now also the Region of Convergence is a circle, exterior of a circle of radius a , correct? And you can straight away see that the step is a special case of this. Henceforth, we will only remember f of z , by the way of this elementary signals you're supposed to remember it and are fairly easy to remember, if you can't you can derive them also on the spot. And finally for the sinusoid you have this f of z and not specified ROC, don't worry, there exists a ROC. You look at the sinusoid; the z -transform looks a bit more complicated. But in all of this almost except for the impulse, the z -transform has a numerator and the denominator, correct? Both could be functions of this variable z or one of them could be.

The fact is that by looking at the roots of the denominator, particularly, I can comment on the nature of the signal. What I mean by nature is whether it is exponentially decaying, whether it is blowing up, whether it remains constant or whether it is oscillatory. The same applies to the Laplace transforms as well. So a very quick example, if I take the exponential signal here, I know very well that if a is greater than one, then it's a growing signal, the signal that's waiting to blow up. What can I say about the root of the denominator here, in terms of z ?

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z-Domain Descriptions

z -transforms of standard signals

3. Exponential signal:

$$f[k] = a^k, \quad |a| < 1, \quad k \geq 0 \implies F(z) = \sum_{k=0}^{\infty} a^k z^{-k} = \frac{1}{1 - az^{-1}}; \quad \text{ROC } |az^{-1}| < 1$$

4. Sinusoid:

$$f[k] = \sin(\omega_0 k) \implies F(z) = \sum_{k=0}^{\infty} \sin(\omega_0 k) z^{-k} = \frac{z \sin(\omega_0)}{z^2 - 2z \cos(\omega_0) + 1}$$

Note: ROC stands for region of convergence in the z -plane, in which $F(z)$ exists.

Navigation icons

So, if the root lies outside, the unit circle, then we say that it corresponds to all signals that are waiting to blow up. If the root of x of z , that is a we call as the pole, you can, but typically poles are used for systems transfer functions and so on, but we can use that here also.

If the root lies within the unit circle, then I can say that corresponds to, so, all z -transforms that have roots within the unit circle correspond to signals that have exponentially decaying. So, now we are developing a dictionary, here the mapping from time domain to z -domain. Whatever inferences we are drawing here pretty much apply to systems as well, these are for signals we can apply to systems as well. Remember, now we have learned how to go from time to z -domain but we also need to learn how to come back, otherwise it will be like Abhimanyu's case. So we will have-- that is why you see

the chakra there. There is a chakravayuham there, the integral, the circular integral. So, it's a snake with a circle.

This is how theoretically, you map from z to k back. But practically you don't use this expression, fortunately, right? What we need to do is actually, we do is, we do a partial fraction expansion.

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z-Domain Descriptions

Getting back in time

The inverse transform is obtained using

$$f[k] = \frac{1}{2\pi j} \oint F(z) z^{k-1} dz \quad (2)$$

In practice, expression (2) is seldom used. Rather a partial fraction expansion of $F(z)$ with a table of standard z -transforms is used.

I'll just show you an example and then we'll adjourn. So here is an X of z that is given to me, of some signal, don't worry about it. The goal is to actually discover that. So what I do is I do a simple partial fraction expansion. The idea is to break up this X of z into simpler terms for which I can easily recognize the inverse z -transform. That's the idea, that's all. So you can do your partial fraction expansion any way you want. So, here I have broken this up into two terms here using simple partial fraction concept, and now z -transform is a linear one. That means sum of signals, z -transform of sum of signals, is a sum of the respective z -transforms, the inverse also satisfies this property.

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z-Domain Descriptions

Examples: Computing Inverse z -Transforms

Example 1:

$$X(z) = \frac{1}{(1 - 0.5z^{-1})(1 - 0.3z^{-1})} = \frac{2.5}{1 - 0.5z^{-1}} - \frac{1.5}{1 - 0.3z^{-1}}$$
$$\Rightarrow x[k] = 2.5(0.5)^k - 1.5(0.3)^k, \quad k \geq 0$$

So, the inverse of X of z is the sum of the inverse of these two terms. We can straight away recognize that the first term is the z -transform of 2.5 times, point 5 raise to k and likewise for the second term. So that's why I write the solution straight away as 2.5 times point five raise two k , minus 1.5 times point three raised k , I also have to specify the domain of the solution. Otherwise, your solution is incomplete. Remember that, whether it's an assignment or exam and so on, you have to specify for what times your solution is valid. So, here I'm clearly saying k greater than or equal to zero, sometimes or many times, the solution may be valid only after a certain time. How do I figure that out? By looking at the z -transform again, and we'll learn that tomorrow. So, tomorrow will complete the discussion on the use of z -transforms. Okay?

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