

**Rheology of Complex Materials**  
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**Lecture – 35**  
**General linear viscoelasticity**

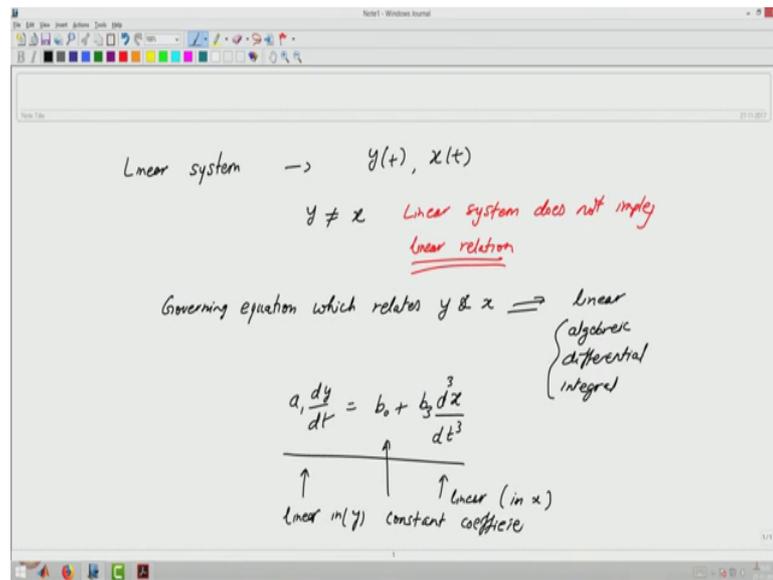
Having learnt about viscoelastic response through many of material functions such a storage modulus, loss modulus, dynamic viscosity, stress relaxation through relaxation modulus, creep compliance and many such material functions. Now what we will do is to look at the overall governing equations that are useful in linear viscoelasticity.

Most of our discussion earlier was based on looking at specific type of response. We looked at let say for a solid like material; we looked at response for a material which had more Maxwellian response. So therefore, we looked at a very specific example of different types of responses. In these set of lectures, we will look at linear viscoelasticity from a general point of view and see that the governing what are the governing equations that are used in linear viscoelasticity.

There is a close correspondence between the governing equations of linear viscoelasticity and other types of linear response theories. In communication engineering, for signal processing, we can use linear response theory. In controls where we have input and output and relations between input and output are what have to be understood and used in order for us to decide control actions so called transfer functions.

So, in control also when we have linear systems, the overall analysis is similar. So, all of what we discussed under linear viscoelasticity is in general related to what is called linear response theory where the overall system is linear. And as we have said earlier linear does not imply linear relation between a variable and another.

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So, for example, when we say a linear system and let us say for this particular system,  $y$  and  $x$  are the 2 variables. Of course, for rheology, we have been discussing let us say stress and strain as the 2 variables; so, any 2 variable; linear system does not imply this. So, therefore, linear system does not imply linear relation; however, what it does imply is governing equation which relates  $y$  and  $x$  will be linear.

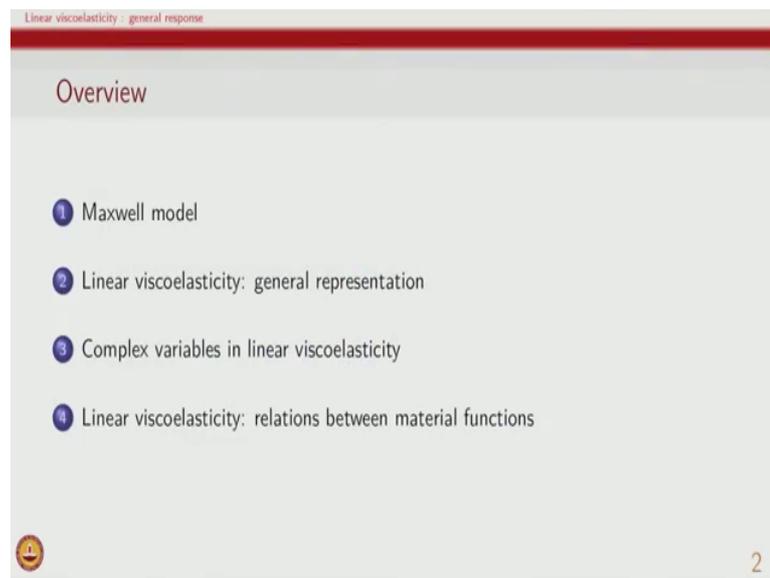
It can be a linear algebraic equation, it can be linear differential equation, it can be a linear integral equation, it can be a combination also. So therefore, for example, this a 1 times  $d y$  by  $d t$  is equal to some  $b_0$  plus  $b_3$  times  $\frac{d^3 x}{d t^3}$  is an example of; we will not use partial derivatives because both  $x$  and  $y$  are functions of time. So, you can see this is an example of. So, this term is linear in  $x$ , this term is linear in  $y$  and this term is a constant coefficient.

So, therefore, whenever we say linear system, we imply that the variables of interest are related to each other through linear equations and these linear equations can be algebraic, differential or integral equations.

So, as we have seen already; for example, we had Maxwell model which is a differential equation which relates stress and strain, we could have hooks law of elasticity which is stress and strain relation, but a pure algebraic relation. And we could also have the integral version of Maxwell model which is an integral statement. So, therefore, all of those are examples of linear systems.

So, what we will do in these set of lectures is try to quickly review our learning which was related to the Maxwell model itself and some general discussion which is related to viscoelasticity. Then we will look at governing equations or how the equations are written for a general linear viscoelastic material. Given that, we have in phase, out of phase, response. We also use the complex variables. So, therefore, we will use real and imaginary parts of a response.

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And so, we will see how complex variables are used in linear viscoelasticity. And given that, we are using several different modes to understand the rheological response are these different response functions related to each other. So for example, can if I measure creep compliance using that can I get storage modulus? So, of course, since it is the same material which is being investigated using different ways, theoretically it is one material function should be related to another material function and so, we will just look at one example of such material functions how they are related to each other.

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Linear viscoelasticity : general response  
Maxwell model

### Response, material functions, constitutive models

**Material response**

- Class of response, qualitative description
- Viscous, viscoelastic, thixotropic, yield stress material

**Material functions**

- Quantification of material response
- Measurement under controlled conditions
- Viscosity, relaxation modulus, storage modulus, creep compliance, extensional viscosity, stress growth viscosity, ...

**Constitutive models**

- Phenomenological models
- Carreau Yasuda model, Maxwell model, Structural model, Herschel Bulkeley model, ...

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So, we have been using this framework to look at and where we define the material response qualitatively first, then we look at specific material functions and then we also look at some simplistic models.

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Linear viscoelasticity : general response  
Maxwell model

### Viscous response and elastic response

**Viscous response**

- Current state of stress and current state of strain rate

**Elastic response**

- Current state of stress and current state of strain

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And so, we also discussed that in general for viscoelastic response, viscous response is where current state of stress and current state of strain rate or relevant and they are related to each other. For elastic response it is the current state of stress which is related to current state of strain.

But when we look at it from an energy perspective, viscous response is dissipative responsive where energy gets dissipated, while elastic response is storage response. In case of viscous response the dissipation rate is related to the stress and strain rate being.

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The image shows a digital whiteboard with handwritten mathematical notes. The notes are organized into three sections:

- Dissipation rate:**  $\propto \frac{\text{Stress} \times \text{strain rate}}{\text{Newtonian fluid}}$   
 $\frac{\text{viscosity} \times (\text{strain rate})^2}{\text{viscosity} \times (\text{strain rate})^2}$
- Storage:**  $\propto \frac{\text{Stress} \times \text{strain}}{\text{Hookean elastic solid}}$   
 $\frac{\text{modulus} \times (\text{strain})^2}{\text{modulus} \times (\text{strain})^2}$
- Maxwell model:**  
 $\underline{G(t)} = G e^{-t/\lambda}$   
 $\underline{G(t-t')} = G e^{-\frac{-(t-t')}{\lambda}} \Rightarrow \underline{G(s)} = G e^{-s/\lambda}$   
 (Note:  $-(t-t') = -s$ )

So, dissipation rate is related to stress into strain rate. Or in case of a Newtonian fluid, it is related to viscosity into strain rate squared. So, this is the dissipation, the rate of dissipation in fluid or rate of dissipation in a Newtonian fluid.

Similarly, the storage energy storage is proportional to stress into strain. And for a Hookean elastic material, it is related to modulus into strain square. Of course, we know this because stress for Newtonian fluid will be related to viscosity into strain rate. Similarly for a Hookean elastic solid, the stress will be related to modulus into strain.

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Linear viscoelasticity : general response  
Maxwell model

### Viscous response and elastic response

- Viscous response**
  - Dissipative response
- Elastic response**
  - Storage response

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So, therefore, in these 2 viscous and elastic responses, we have energy dissipation in one case and energy storage in another case.

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Linear viscoelasticity : general response  
Maxwell model

### Viscoelastic response

- Contributions**
  - Viscous response and elastic response
  - Relative contributions vary depending on time-scale of interest
- Formulation**
  - Current states of stress, strain, strain rate, rate of strain rate, rate of stress, ...
  - Current state of stress depends upon past history of deformation
- Terminal response: time, frequency, ...**
  - Viscous response
  - Elastic response

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So therefore, we can think of viscoelastic response as contributions from both of these. So therefore, there will be energy dissipation as well as energy storage and then there will be relative contributions of these 2, depending on what is the time scale of interest. And so, we defined relaxation time and we know that based on the how we examine the

material, how fast or slow we examine, the material the overall material response may change and we may have more of viscous or more of elastic contributions.

So, in terms of the mathematical formulation for describing the behaviour, we can relate the overall response in terms of current state of stress, strain, strain rate, rate of strain, rate of stress, rate of stress rate and so on. So basically, this is a differential equation which can relate all the different quantities. And we have already seen examples of standard linear solid model or Maxwell model in the differential form.

We could also look at the overall formulation in terms of current state of stress depending on the overall history of deformation in the material. And this is similar to an integral formulation or this is the same as an integral formulation and we have also seen how Maxwell model can be represented in an integral form where the current state of stress depends on all the past deformation that the material was subjected to.

And generally when we look at the viscoelastic response, given that we are probing the material at different times and frequencies, we would generally tend to also observe that under some terminal condition, which implies that either very high frequency or very low frequency or very long time or very short time, we would most often see either completely viscous response or completely elastic response.

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Linear viscoelasticity : general response  
Maxwell model

Maxwell model

$$\tau_{yx} + \lambda \frac{\partial \tau_{yx}}{\partial t} = \eta \dot{\gamma}_{yx} \quad (1)$$
$$\tau_{yx}(t) = G \gamma_{yx}^0 \exp\left(-\frac{t}{\lambda}\right) \quad (2)$$
$$G(t) = \frac{\tau_{yx}(t)}{\gamma_{yx}^0} = G \exp\left(-\frac{t}{\lambda}\right) \quad (3)$$

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And so, quickly looking at Maxwell model which we have used quite a lot in the course where the stress and stress rate are related to strain rate. And we saw that in a stress relaxation experiment, the stress is an exponentially decaying function. We saw that for a viscous fluid the decay will be instantaneous and for a Hookean elastic material, it will never decay. But for a general viscoelastic material with relaxation time lambda, it is an exponential function and lambda determines how fast or slow the overall decay will be. And based on this, we had defined a relaxation modulus which is related to the stress as a function of time divided by strain. And so, that is also an exponential function.

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Linear viscoelasticity : general response  
Maxwell model

Maxwell model

Using integrating factor approach

$$\tau_{yx} \frac{1}{\lambda} \exp\left(\frac{t}{\lambda}\right) + \frac{\partial \tau_{yx}}{\partial t} \exp\left(\frac{t}{\lambda}\right) = \frac{\eta}{\lambda} \dot{\gamma}_{yx} \exp\left(\frac{t}{\lambda}\right) \quad (4)$$

$$\frac{\partial}{\partial t} \left[ \tau_{yx} \exp\left(\frac{t}{\lambda}\right) \right] = G \dot{\gamma}_{yx} \exp\left(\frac{t}{\lambda}\right)$$

$$\tau_{yx}(t) = \int_{-\infty}^t \left[ G \exp\left(-\frac{t-t'}{\lambda}\right) \right] \dot{\gamma}_{yx} dt'$$

stress at time t =  $\sum_{past} [G(t-t')] \times [\dot{\gamma}_{yx} \Delta t']$

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We saw that given that the governing equation is a differential equation, we can use some methods of solving ordinary differential equation; for example, an integrating factor approach to simplify the overall equation of Maxwell model. So, if we multiply everything with this factor integrating factor which is exponential t by lambda then we can manipulate and basically show that the overall stress at current time can be shown as an exponential function integrated multiplied with strain rate.

And when we discuss this earlier, we had also mentioned that, stress at time t seems to be sum of all past effects and each and every effect is (Refer Time: 12.52) modulus multiplied by strain, modulus multiplied by strain at different time and this t minus t prime is the variable which keeps track of how much time in the past do we go. So, when

t prime is minus infinity that basically implies that very long time ago. When t prime is t, that is the present time and when this overall factor is actually just unity.

So, this in case of Maxwell model, the specific functional form of a relaxation modulus is being used in this equation. So, if you look at the relaxation modulus, it is just G times minus t by lambda. And in this equation which is the integral representation of Maxwell model, we can again see it is G exponential some time unit divided by lambda.

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Linear viscoelasticity : general response  
Maxwell model

### Several Maxwell models in parallel

$$\tau_{yx}^i + \lambda_i \frac{\partial \tau_{yx}^i}{\partial t} = \eta_i \dot{\gamma}_{yx} \quad (5)$$

Linear equations:

$$\sum_i \tau_{yx}^i + \sum_i \lambda_i \frac{\partial \tau_{yx}^i}{\partial t} = \sum_i \eta_i \dot{\gamma}_{yx} \rightarrow \tau_{yx} + \sum_i \lambda_i \frac{\partial \tau_{yx}^i}{\partial t} = \eta \dot{\gamma}_{yx}; \eta = \sum_i \eta_i \quad (6)$$

$$G_i(t) = G_i \exp\left(-\frac{t}{\lambda_i}\right) \quad (7)$$

$$G(t) = \sum_i G_i \exp\left(-\frac{t}{\lambda_i}\right) \quad (8)$$

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So, by comparing these two, we can say that this is overall G of t minus t prime. So, given that for a Maxwell model, we define G of t as G time's e to the power minus t by lambda. We can define G of t minus t prime as G of minus t minus t prime divided by lambda. Or if we write let us say t minus t prime as minus s, then we could also write this as G of s G e to the power minus s by lambda. So, depending on whichever variable we use, there are different ways of writing the overall relation.

So, all of these are saying that this G is a function and that is why we call it material function; it is a function of one variable time. We could write it in terms of t minus t prime which indicates how much is the difference between current time and anytime in the past. We could also indicate in terms of a new variable s which is basically this difference in time.

Similarly, we also saw that many Maxwell models can be combined. And so, when you combine many Maxwell models basically now you have a differential equation which has several different contributions.

The  $i$  here indicates that it is a combination in parallel of several such models. So,  $i$  is equal to 1, 2, 3 will imply that there are 3 such Maxwell models in parallel. And so, the overall relaxation modulus then is summation of individual contributions. So, individual relaxation modulus is just  $G$  exponential minus  $t$  by  $\lambda_i$  and then when we sum all of it we get the overall relaxation modulus.

So, given this is the overall background that we have already discussed.

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Linear viscoelasticity - general response  
Maxwell model

### General model for linear viscoelasticity

$$a_0 + a_1 \tau_{yx} + a_2 \frac{\partial \tau_{yx}}{\partial t} + a_3 \frac{\partial^2 \tau_{yx}}{\partial t^2} + \dots = b_0 + b_1 \gamma_{yx} + b_2 \frac{\partial \gamma_{yx}}{\partial t} + b_3 \frac{\partial^2 \gamma_{yx}}{\partial t^2} + \dots \quad (9)$$

- Maxwell
- Kelvin-Voigt
- Standard linear solid model
- Burgers

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Now, what we can do is to look at the overall most general model which can be there for linear viscoelasticity. So, given that, we are saying that stress and strain are related to each other for a linear response it implies that there has to be a linear equation which will govern it. And the most general linear equation which can be described will have stress and it is various derivatives related to strain and it is various derivatives. So therefore, this equation which includes all the stress and stress rates and strain and strain rates, it is the most general statement of viscoelasticity.

So, for any linear viscoelastic material, this governing equation with different number of terms will be possible choice. For example when we say this dash dash here, what we are

implying is there are additional higher order derivatives which are possible. Of course, we have seen some specific examples of these in case of our earlier discussion; for example, if let us say we have only the these terms nonzero. So,  $a_1$  is nonzero and  $b_0$  is nonzero, if that is the situation then the overall governing equation as we can see. So, if  $a_1$  is nonzero and  $b_0$  is nonzero.

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Handwritten notes on a digital whiteboard showing constitutive equations for different material models based on parameters  $a_1$ ,  $b_0$ ,  $b_1$ , and  $b_2$ .

$a_1, b_0$

$$a_1 \tau_{yx} = b_0$$

$$\tau_{yx} = \frac{b_0}{a_1} \Rightarrow \text{constant stress perfectly plastic}$$

$$\tau_{yx} < \frac{b_0}{a_1} \Rightarrow \text{no deformation}$$

$$\tau_{yx} \geq \frac{b_0}{a_1} \Rightarrow \text{continuous deformation}$$

$a_1, b_1$

$$a_1 \tau_{yx} = b_1 \gamma_{yx} \Rightarrow \tau_{yx} = \left( \frac{b_1}{a_1} \right) \gamma_{yx} \rightarrow \text{Hookean elastic model}$$

$a_1, b_2$

$$a_1 \tau_{yx} = b_2 \frac{\partial \gamma_{yx}}{\partial t} \Rightarrow \tau_{yx} = \left( \frac{b_2}{a_1} \right) \frac{\partial \gamma_{yx}}{\partial t} \rightarrow \text{Newtonian fluid}$$

Labels in the image: modulus (pointing to  $\frac{b_1}{a_1}$ ), viscosity (pointing to  $\frac{b_2}{a_1}$ ).

So, we will have  $a_1 \tau_{yx}$  is equal to  $b_0$  or in other word  $\tau_{yx}$  is equal to  $b_0$  by  $a_1$ .

So in this case, it is a constant stress material. And this we will see is a perfectly plastic material. What this material implies is up to this value of stress, when stress is less than  $b_0$  by  $a_1$ , no deformation and when  $\tau_{yx}$  is greater or equal to  $b_0$  by  $a_1$ , there is continuous deformation. So, what we have seen is, are the cases where only  $a_1$  and  $b_0$  are nonzero.

Now, we can look at several such other examples. If  $a_1$  and  $b_1$  are nonzero; so, if  $a_1$  and  $b_1$  are nonzero, then we have  $a_1 \tau_{yx}$  is equal to  $b_1 \gamma_{yx}$  and therefore, this implies that  $\tau_{yx}$  is equal to  $b_1$  by  $a_1 \gamma_{yx}$  and this is nothing but Hookean elastic model.

Now, similarly if  $a_1$  and  $b_2$  are not 0, then I am sure you can guess that the overall material is again a known material. So,  $a_1$  and  $b_2$ ,  $a_1 \tau_{yx}$  is equal to  $b_2 \frac{\partial \gamma_{yx}}{\partial t}$

$y$  x by del t. And again; so, therefore, you can see that this is nothing but a Newtonian fluid.

And so, of course, this coefficient is what we recognise as modulus. Similarly this coefficient we recognise as viscosity. So, therefore, from this general equation depending on various coefficients that are chosen, we have all the different possible models. And so, most of the models which are used in linear viscoelasticity will usually be some terms related to stress and stress rates and some terms related to strain and strain rates though in theory a infinite number of terms can be chosen to represent the overall material behaviour.

See you can do this exercise to try to find out what would be the Maxwell or the void or standard linear solid model or the Burgers model, what coefficients will be 0 or nonzero. When we discuss any such term; for example, for Maxwell model we will have a 1 nonzero, a 2 nonzero and b 2 nonzero. So, it will be useful for us to just become comfortable with this overall governing equation by seeing that many other models that I have already been discussed in class in the course of our are already subset of this overall equation. For example, Burgers model which is used also in describing many material systems is again a subset where a 1, a 2, a 3 and b 1, b 2 are nonzero.

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The image shows a Notepad window with handwritten mathematical notes. The text is as follows:

$$a_1, a_2, a_3, b_1, b_2 \Rightarrow \text{Burgers model}$$

$$\text{parameters} \Rightarrow \frac{a_2}{a_1}, \frac{a_3}{a_1}, \frac{b_1}{a_1}, \frac{b_2}{a_1}$$

$$y(t) \rightarrow \text{Laplace transform} \Rightarrow Y(s)$$

$$\frac{dy}{dt} \rightarrow \text{Laplace transform} \Rightarrow sY(s) - \frac{y(0)}{\downarrow}$$

$$y + k \frac{dy}{dt} + m \frac{d^2y}{dt^2} = 0$$

$$Y(s) + k [sY(s)] + \dots$$

So, it is a 5 parameter nonzero and Burgers model. And of course, the parameters of the model because the 5 parameters are there in 5 terms, we can divide by one of them and

then it becomes 4 independent parameters. So, the 4 independent parameters of this would be the following.

So, these are the parameters of burger model burgers model. And so, similarly Maxwell and standard linear solid model and all the other models are representations of these. Now we can; so, this is a statement which is related to what we earlier said is current state of stress and strain and strain rate are all related to each other. So, therefore, this statement that we made is the overall differential form that the current state of stress, strain, strain rate, rate of strain rate, rate of stress, rate of stress rate and all these variables are related to each other.

Now, let us try to see if we can get an overall governing equation for linear viscoelasticity where we can again get the current state of stress based on all the past deformations.

So, to do that, we will just take a step back and look at mathematical tools which you must have encountered in engineering mathematics which are called Laplace transforms or Fourier transforms. So, using these 2 tools, we can actually attempt solutions of differential equations also. So in that context, we learn about these transforms. This transforms are also useful some time to do the analysis in either the Laplace domain or the frequency domain. In linear viscoelasticity and especially when we use oscillatory deformation, we saw that we use frequency and how fast or slow. So, if we do oscillation very fast, then we are probing the material using one time scale. If we are doing oscillation very slowly, then we are using the different time scale to probe the material.

So, therefore, frequency is a very important variable and there it may be useful to look at the overall response and the frequency domain by doing Fourier transforms. So, therefore, we use Laplace transform and Fourier transforms to analyse response of linear systems and not surprisingly we also see in linear viscoelasticity extensive use of Laplace transforms and Fourier transforms.

So, we could do the Laplace transform of such an equation and you can consult the engineering mathematics book to say that if you have a function  $y$ , which is a function of time it is Laplace transform will be denoted as let us say capital  $L$  of  $s$ , the question that we have is if I now have  $\frac{dy}{dt}$ , what will be its Laplace transform. And if you look

at the resource material, you will see that the overall let me just retrace this is, we will write it as capital Y and. So, this will be capital Y s minus the function at 0 time.

So, we will see that quite often before we start the deformation, the stress and strain and all of them will be 0. So, many of these terms will be not present in our case. So, if I have now an equation which is y plus let us say some k times d y by d t plus m times d squared y by d t squared is equal to 0, I can take Laplace transform of this and so, I will get Y of s plus k times s times Y of s and so on.

So, therefore, I can write the overall equation in terms of Laplace transform itself. And so, now in our case we have a relation between stress and strain.

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The image shows a whiteboard with handwritten mathematical notes. The board is divided into two columns: "derivatives of stress" on the left and "derivatives of strains" on the right. The first row shows the Laplace transform of stress  $\sigma_{yx}(t) \rightarrow \mathcal{J}(s)$ . The second row shows the Laplace transform of strain  $\gamma_{yx}(t) \rightarrow \Gamma(s)$ . The third row shows the Laplace transform of the first derivative of stress  $\frac{\partial \sigma_{yx}}{\partial t} \rightarrow s \mathcal{J}(s)$ . The fourth row shows the Laplace transform of the first derivative of strain  $\frac{\partial \gamma_{yx}}{\partial t} \rightarrow s \Gamma(s)$ . Below these, a note reads "Laplace transform of the differential equation for general viscoelastic material" followed by the equation  $\frac{a_0}{s} + a_1 \mathcal{J}(s) + a_2 s \mathcal{J}(s) + \dots$ .

So, we have derivatives of stress on one side and then we have derivatives of strain on the other side. And each of them can be using the rules of Laplace transform of a derivative or a Fourier transform of derivative can be simplified. And so, what we can see is, if upon simplification we will be able to write the overall governing equation for linear viscoelasticity in the Laplace domain and in the Fourier domain.

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Linear viscoelasticity : general response  
Linear viscoelasticity : general representation

### General model for linear viscoelasticity

Laplace domain

$$\mathcal{T}(s) = \mathcal{G}(s) \dot{\Gamma}(s) \quad (10)$$
$$\mathcal{T}(s) = \mathcal{M}(s) \Gamma(s) .$$

Fourier domain

$$\mathcal{T}(\omega) = \mathcal{G}(\omega) \dot{\Gamma}(\omega) \quad (11)$$
$$\mathcal{T}(\omega) = \mathcal{M}(\omega) \Gamma(\omega) .$$

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So, for example, let us just denote the Laplace transform. So, we have the stress component,  $\tau_{yx}$  in the time domain it is Laplace transform is  $T$ . So, this symbol is basically script  $T$   $s$ . So, this is the Laplace transform of stress and similarly we have Laplace transform of strain. So, if  $\gamma_{yx}$  which is a time function, it is Laplace transform is  $s$ .

So now, what we will see is, if there is  $\delta \gamma_{yx}$  by  $\delta t$  it is Laplace transform will be  $s$  times  $s$ , similarly  $\delta \tau_{yx}$  by  $\delta t$ , it is Laplace transform will be  $s$  gamma times  $s$ .

So, what we will have is several terms. So, we will basically when we do the, now let us look at the overall differential equation. And so, we have a 0, a 1 and all of these terms. So, if I take Laplace transform of the differential equation for general viscoelastic material, then what I have is basically a 0 plus a 1. So, this will be a 0 by  $s$  plus a 2  $s$  times and so on.

So, we can club all the terms together and in the end get a relationship where we have all the terms which are related to the Laplace transform of stress and all the terms which are Laplace transform of strain. And so, we can club together all the terms and collect them together.

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memory function

$$\mathcal{T}(s) = M(s) \Gamma(s)$$

$G' / G'' \rightarrow$  viscous / out-of phase  
 $\hookrightarrow$  elastic  $\rightarrow$  in-phase

$$\sqrt{G'^2 + G''^2} = G^* \text{ complex modulus}$$
$$G^* = G' + i G''$$

$\hookrightarrow$  real  $\hookrightarrow$  imaginary

$$\epsilon^* = \epsilon' - i \epsilon'' \rightarrow \text{dielectric loss}$$

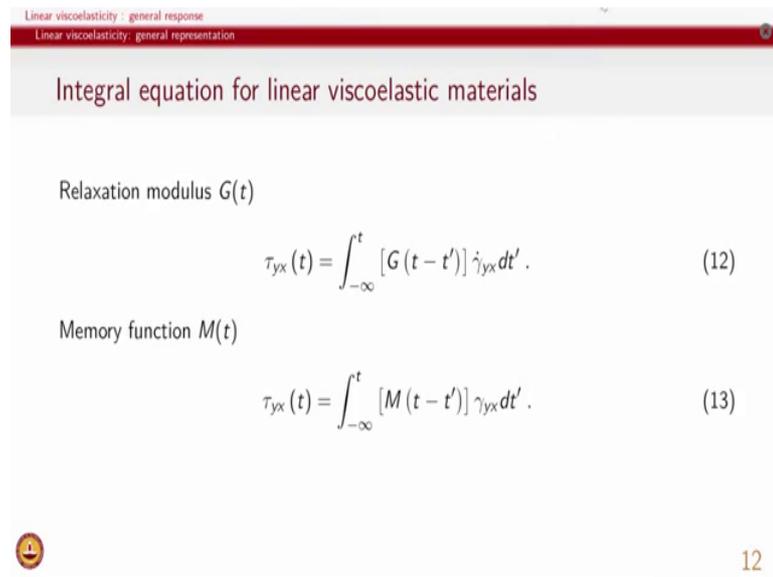
$\hookrightarrow$  permittivity

And in the end, we can write all the terms which are mentioned here it will be a 0 plus a 1 plus a 2 s plus so on and similarly on the other side b 0 plus b 1 s plus b 2 a square and so on. All of these terms can be clubbed together and that will be called the Memory Function. And so, the general representation of this particular model in Laplace domain is given by such equation where stress is related to strain rate, the Laplace transform of stress is related to Laplace transform of strain and similarly Fourier transform of stress is related to Fourier transform of strain rate and Fourier transform of stress is related to Fourier transform of strain.

And so, now, having seen these overall governing equations; next we will look at what happens to this representation in the time domain and then we will go on to look at the use of complex notation to represent some of the general material functions in linear viscoelasticity.

So, the equations that we saw in Laplace domain or Fourier domain can be transformed to the time domain by using convolution.

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Linear viscoelasticity : general response  
Linear viscoelasticity : general representation

### Integral equation for linear viscoelastic materials

Relaxation modulus  $G(t)$

$$\tau_{yx}(t) = \int_{-\infty}^t [G(t-t')] \dot{\gamma}_{yx} dt' . \quad (12)$$

Memory function  $M(t)$

$$\tau_{yx}(t) = \int_{-\infty}^t [M(t-t')] \dot{\gamma}_{yx} dt' . \quad (13)$$

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And the equations are given as either stress in terms of strain rate or stress in terms of strain and  $G$  is called the Relaxation modulus and  $M$  is called the Memory function. So, the equations which are listed here on this slide are the general representation for linear viscoelastic material. If we choose a specific form of  $G$ , for example, for Maxwell it will be exponential automatically they will be another specific form for  $M$  and so, then this model this equation can be reduced to Maxwell model.

Similarly, therefore, there are infinite possibilities of  $G$  and  $M$  that can be chosen and various viscoelastic models can be represented using the integral form.

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Linear viscoelasticity : general response  
Complex variables in linear viscoelasticity

### Complex notation for material functions in oscillatory deformation

- Complex representation
$$G^*(\omega) = G' + iG'' . \quad (14)$$
$$\eta^*(\omega) = \eta' - i\eta'' . \quad (15)$$
- Real and imaginary parts, in-phase and out-of-phase parts, storage and dissipative parts
- Various representations - Cole-Cole plot, Bode plot

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Having seen the general description of this viscoelastic modulus, we will end with looking at them in the complex notation also.

The idea of a complex notation arises because we have  $G'$  and  $G''$  and this we know is elastic contribution and also in phase. And similarly,  $G''$  is a viscous contribution and out of phase.

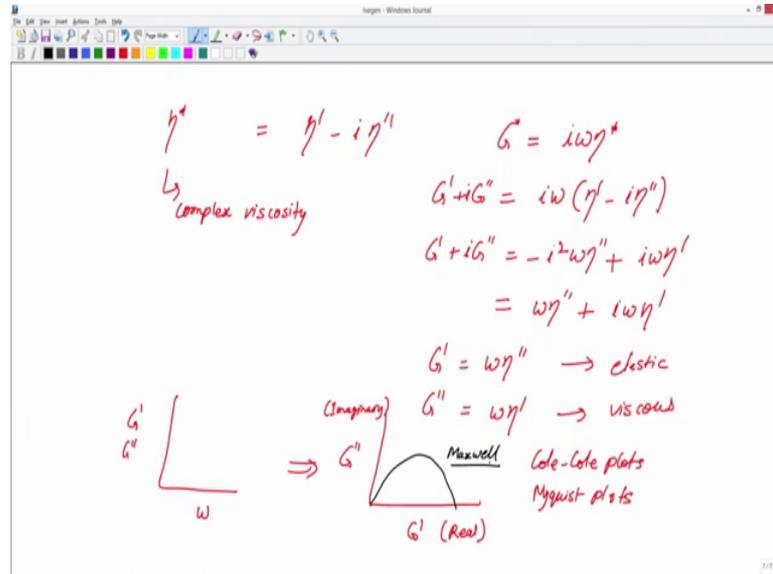
So, we could think of a combined model modulus; for example, if I look at this quantity, so this is the what is called the Complex modulus. In other words we are using the complex number notation to say that we have a real and an imaginary part of the complex modulus. So, this is the real part and this is the imaginary part.

And those of you who are familiar with dielectric materials may also know that the overall complex permittivity is again related to real and imaginary and this is usually called the permittivity and this is called the dielectric loss.

So, in each of these cases, the overall complex quantity is signifies the material response and the overall complex quantity has 2 components; one which is in phase with the input. So, for example,  $G'$  is in phase with the strain input and  $G''$  is out of phase with the strain input. In the case of dielectrics,  $\epsilon'$  is where the voltage derivative and current is in phase while voltage derivative and current are out of phase for dielectric loss. In other words the voltage and current are in phase for dielectric loss.

So, similarly looking at the same type of behaviour using viscosity, we could also write the complex viscosity.

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So, this is called the complex viscosity as eta prime minus i eta double prime and. So, all of these quantities can be related to each other. For example, you can try doing this exercise where G star can be related to eta star again using complex numbers. So, if we substitute from the previous work that we have already defined these quantities, what we have is i eta double prime. And so, G prime plus i G double prime is equal to i squared omega eta double prime minus and plus i omega eta prime.

And so, I should be careful here, this is actually minus that should be plus. And so, what you have is this also becomes minus i squared minus 1. So, you have omega eta double prime plus i omega eta prime.

So, you can see the G prime is equal to omega eta double prime and the G double prime is equal to omega eta prime. And of course, these are elastic contributions and these are viscous contributions. So, all the oscillatory shear material functions that we have defined can be related to each other. And there are also various representations to try to look at how does the material response vary. We have for example, looked at G prime or G double prime as a function of frequency the same plot, you could do as G double prime as a function of G prime. So, this is usually done in controls and variety of other subjects whenever there is linear systems analysis where we have the imaginary on the y

axis and the real part on the x axis and the plot is. So, these are called Cole-Cole plots or also Myquist plot. So, these can also be used to represent the behaviour of materials in viscoelasticity.

So, for example, for a Maxwell model, one can see that the response will be semicircle.

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Linear viscoelasticity : general response  
Complex variables in linear viscoelasticity

### Complex notation for material functions in oscillatory deformation

- Complex representation
$$G^*(\omega) = G' + iG'' . \quad (14)$$
$$\eta^*(\omega) = \eta' - i\eta'' . \quad (15)$$
- Real and imaginary parts, in-phase and out-of-phase parts, storage and dissipative parts
- Various representations - Cole-Cole plot, Bode plot



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And so, to summarise what we have is a complex notation can also be used for the oscillatory functions that we define and these are of course functions of frequency and we can define complex modulus or complex viscosity. We could also define complex compliance which will be inverse of complex modulus and so, all of these are useful in representing the overall response.

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Linear viscoelasticity : general response  
Complex variables in linear viscoelasticity

### Complex notation for material functions in oscillatory deformation

Stress and strain

$$\sigma_{zz}(t) = \sigma_{zz}^0 \exp(-i\omega t) . \quad (16)$$

$$\epsilon_{zz}(t) = \epsilon_{zz}^0 \exp(i\phi) \exp(-i\omega t) . \quad (17)$$

Tensile complex modulus

$$E^*(\omega) = E' + iE'' . \quad (18)$$

Extensional complex compliance or tensile complex compliance

$$D^*(\omega) = \frac{\epsilon_{zz}}{\sigma_{zz}} = D' - iD'' . \quad (19)$$

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We can also do the same for tension or Tensile or Extensional deformations in which case we use  $e$  star instead of  $G$  star and we use  $D$  star instead of  $j$  star. So, complex compliance in shear is  $j$  complex compliance in extension is  $D$  and all of these quantities are related to each other.

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Linear viscoelasticity : general response  
Linear viscoelasticity: relations between material functions

### Material functions : inter-relations

- Material functions represent behaviour of a given material in different modes of characterization
- One material function can be used to estimate other material functions

Relaxation modulus to steady viscosity

$$\sigma_{yx}(t) = \int_{-\infty}^t G(t-t') \dot{\gamma}_{yx} dt' \rightarrow \frac{\sigma_{yx}}{\dot{\gamma}_{xz}} = \int_0^{\infty} G(s) ds = \eta . \quad (20)$$

Where  $s = t - t'$

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Now, the other important thing in linear viscoelasticity is, since it is the same material that is being examined with multiple different types of deformations; for example, creep

or stress relaxation or steady shear or oscillatory shear. So, given that it is the same material can material function of one be related to another.

So, here there is a small example how we can relate the oscillatory shear response or stress relaxation response to another function which is steady shear. So, for example, Relaxation modulus we know is  $\sigma_y x$  and this function and if we do a steady shear then the strain rate is constant and therefore it is  $\dot{\gamma}_y x$  not which implies it is a constant quantity, this can be taken out of the integral and then basically what you have is this integral. So,  $G_s D_s$  and we know that this ratio  $\sigma$  the shear stress to shear rate is nothing but viscosity.

So, if we know the Relaxation modulus  $G(t - t')$  or  $G(\omega)$  where we do the transformation of variables, then we can find the viscosity. So, relaxation modulus is related to steady viscosity based on this.

So, therefore, the general framework of linear viscoelasticity is very powerful in terms of any infinite number of combinations of models are available and also there are general relationships which can be used to relate one material function to the other. With this, we have finished the overall discussion on linear viscoelastic materials and going on further we will discuss other classes of material systems.