Introduction to Time-Frequency Analysis and Wavelet Transforms
Prof. Arun K. Tangirala
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture - 6.2
Properties of WVD

Hello friends. Welcome to lecture 6.2, where we will discuss properties of Wigner-Ville distribution. In the previous lecture, we learnt the definition of Wigner-Ville distribution. We also saw a few illustrations on some standard signals and of course, also came across the concept of interferences.

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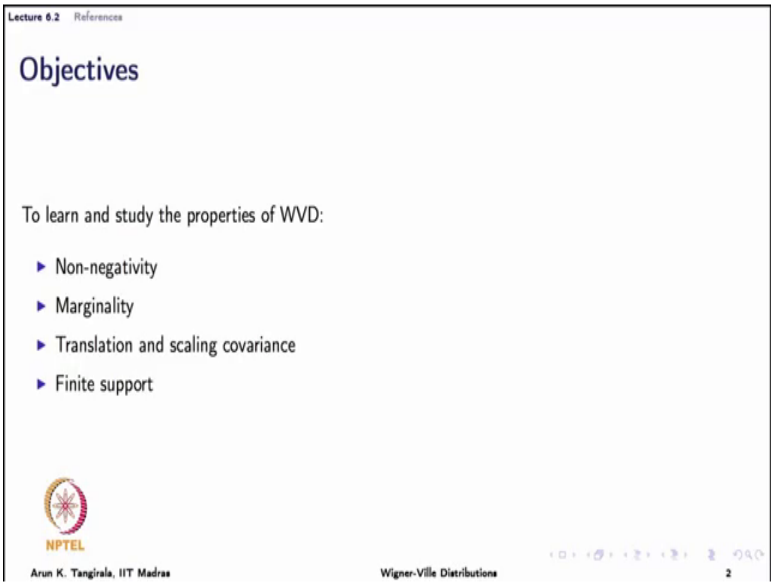

What we are going to do in this lecture is examine a few useful and interesting properties of Wigner-Ville, namely the non-negativity, the marginality, translation and scaling covariance property and the finite support property. We will in addition also look at a concept, the property of total energy recovery and the property known as a unitary property of the Wigner-Ville. All these properties if you recall are the requirements that joint energy density should have, and we had stipulated a list of requirements that joint energy density should ideally satisfy, but of course as we had mentioned in that lecture, that there is no joint energy density which will satisfy all the properties. One should expect the Wigner-Ville distribution will also not be able to satisfy all these properties. There are a few more other properties that we shall study in the next lecture.

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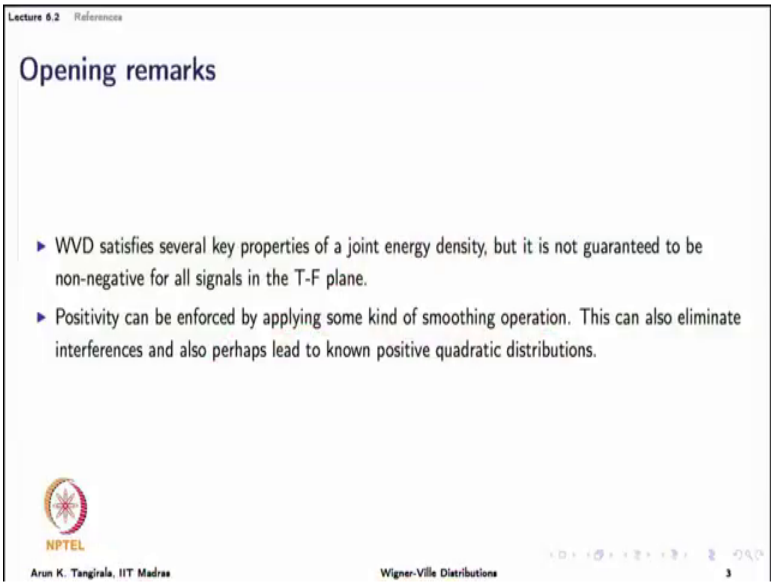

Now, we have mentioned this before that the Wigner-Ville distribution has some very nice properties, of course which we are going to discuss in detail today. But one of the key drawbacks of this WVD is that it is not guaranteed to be non-negative in the entire time frequency plane for all signals. The only exception being the modulated chirp, but then if you look at Cohen's book, there is a nice explanation of why the Wigner-Ville distribution gives you non-negative values for the modulated chirps primarily because you can say that it is no longer a quadratic energy distribution. If it is a quadratic energy distribution, Wigner himself proves that there exist no positive quadratic energy distribution that will satisfy both the marginality and the positivity requirements.

Wigner-Ville satisfies the marginality, but does not satisfy the positivity or non-negativity. We shall use these terms positivity and non-negativity interchangeably. Of course, we can enforce positivity by applying some kind of smoothing which means local averaging. When you average bunch of negative and positive numbers, there is a possibility depending on how you average that you end up with the positive number. We shall study this idea more in detail in lecture 6.4.

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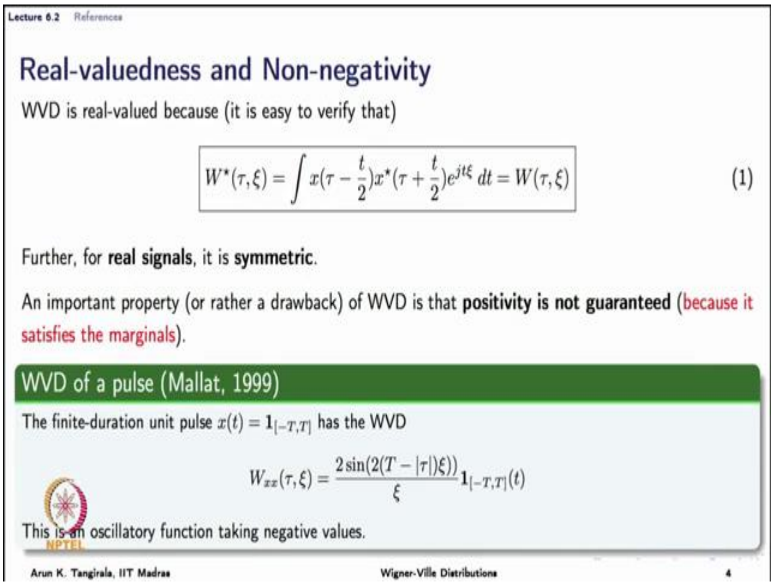

As of now we will look at the properties of the Wigner-Ville. In passing I should also mention them such as smoothing will help us remove the interferences. Again, we will study that in lecture 6.4. So, let us study the most elementary property of the Wigner-Ville which is, that it is real value regardless of whether the underline signal is real value or complex. A proof of that can be easily seen in equation 1, where I have written expression for the complex conjugate of the Wigner-Ville. Therefore, the star on x of tau minus t by 2 disappears and appears only x of tau plus t by 2 if you compare with the definition of Wigner-Ville. And e to the minus j tau psi is replaced to the e to the j tau psi by change of variables you can show that this is nothing, but the original definition itself and therefore, the complex conjugate and itself they are identical. Consequently the Wigner-Ville distribution is the real value quantity. If this signal is real further you can show there it is symmetric. And in any case, most of the signals at we deal with that real signals, but of course, we may use analytic representations in which case the symmetricity property may not hold any more.

Now, one of the most important properties that I had mentioned is the lack of positivity properties which is the major drawback of Wigner-Ville and to understand this, let us look at a simple example that I have taken from Mallet's book. This signal he is a finite duration pulse. It is also sometimes called the unit pulse indicator function. This pulse is of width 2 t and a magnitude unity. When you work through the math, that is when you plug in this expression in the formula or expression for Wigner-Ville, then you can arrive at this expression for the WVD for this unit pulse. Now, rather than really worrying about the exact nature, exact expression itself, let us look at the nature of the expression. It tells me two things. One which is quite relevant what we are discussing that the Wigner-Ville can take on negative values. In fact, we will see that in the next lecture as well that the Wigner-Ville can take on negative values should have an illustration, but this sin c function certainly takes on negative values at some point in time.

Therefore, this is not a proof, but an ample evidence of the fact that Wigner-Ville can produce negative values. The other observation that we can make is that we have this sin c function multiplied by this unit pulse itself which exists only for a short period of time, a final duration of time telling us that the Wigner-Ville satisfy the weak finite support property. What is the weak finite support property? If the signal is non-zero for a finite interval and zero outside this interval, then the Wigner-Ville also that is the joint energy density should also have the same feature, and Wigner-Ville does have this property. We will talk about this finite support property towards end of this lecture.

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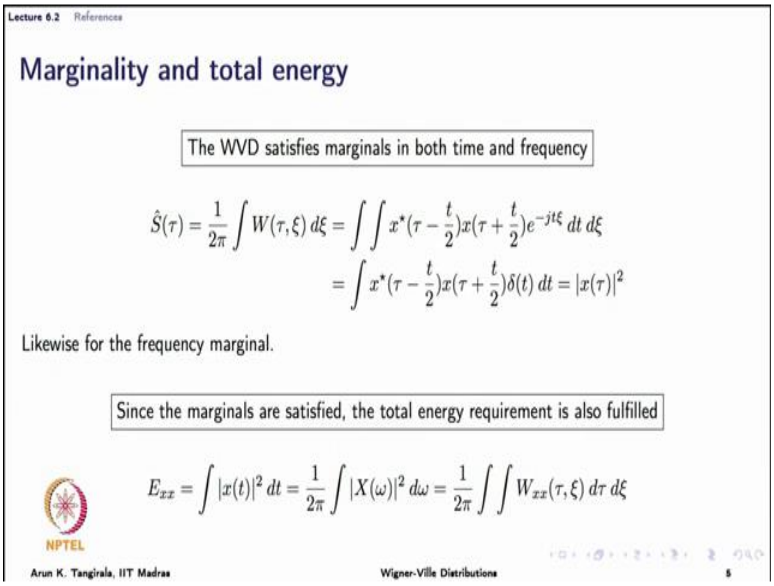

So, the key point is the WVD does not necessarily produce positive values in the entire time frequency plane. However, the WVD satisfies marginal which is a very useful and interesting property in many respects. One of course is a practicality that I can just walk along the time or the frequency access and recover the energy densities, and the other consequence is on the calculation of instantaneous frequency and so on which you will study in the next lecture.

So, let us look at a quick proof of this marginality property. I am only proving for one-dimension that is time you can prove. Similarly, for the other dimension as well are that is a frequency dimension. So, we start with the Wigner-Ville, an integrated along the frequency access. This 1 over 2 pi once again because working with the angular frequency and let us call this as s hat of tau, some standing a tau and I am walking across all the frequencies summing up, I substitute for the definition of w here and then, use the property that integral of minus j tau psi d psi is delta of t. Therefore, 1 integral vanishes and again using the sampling property of the direct delta function, we can simplify this integral to module x of tau square which is nothing, but the energy density. That point in time, tau proves marginality. Likewise you can prove the frequency one by using the definition of Wigner-Ville in terms of the Fourier transform of x. Recall the definitions, two different definitions we are given for Wigner-Ville.

Now, since the marginal are satisfied, the total energy requirement is automatically satisfied. You do not have to really verify, but just as a matter of information I have given you, this expression for energy preservation or conservation. Recall now contrast is with spectrogram, where the total energy requirement is fulfilled, but the marginality requirements are not satisfied. On the other hand, the short time Fourier transforms rather than the spectrogram. This spectrogram is actually positive value, right. Again you should recall the Wigner-Ville distribution is not a transform of the signal. It is a transform of the local or the instantaneous auto-correlation. Therefore, it directly gives you the energy distribution. Unlike the case in short time Fourier transform, where you have to first compute short time Fourier transform and then, compute the spectrogram.

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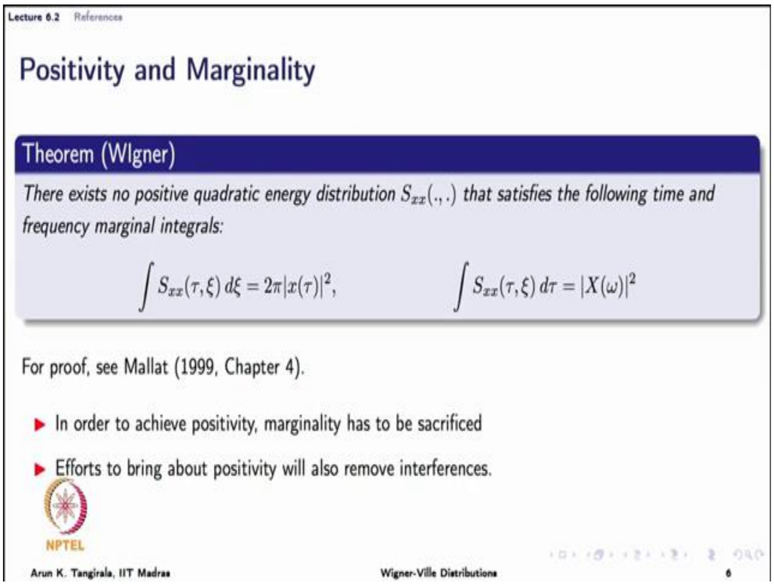

So, earlier we mentioned the positivity and marginality requirements are satisfied are actually related. This is by virtue of Wigner's own result or theorem that it proved in 1930s, where I said that there exist no positive quadratic energy distribution. What we mean by quadratic or billionaires that you have product of signal terms. You do not have any cubic terms and so on. There exist no such positive quadratic energy distribution that is both positive and satisfying the marginality requirements. So, for a proof you can look up Mallet's book, especially the chapter 4 of the Mallet's book. The proof is fairly short and not complicated. So, I suggest that we go through the proof.

So, what is the consequence of this result? The consequence is if I have to achieve positivity of the energy density which is very important to make any meaningful interpretation, I have to sacrifice marginality. That means, in practice whatever energy densities that I work with will never ever satisfy time or the marginality requirement as long as I working with quadratic once. So, please remember that condition you may have other kinds of energy distribution functions that you can compute, may be 2 the power of 4 and so on that can satisfy the marginality requirements, but not definitely the quadratic one. As I mentioned earlier, when we put in efforts to bring about positivity in the Wigner-Ville’s, so what we will do is, we will modify the Wigner-Ville distribution and impose the modified one satisfies the positivity requirement. This will also help us remove interferences, but of course we should expect the modified one do not satisfy the marginality requirement.

In fact, we will prove this formally that one type of modification will lead us to the spectrogram, and another type of modification will lead us to this scalogram. There is also another result due to Moyal which we will talk about apartly now and later on which shows the connections between the spectrogram and scalogram.

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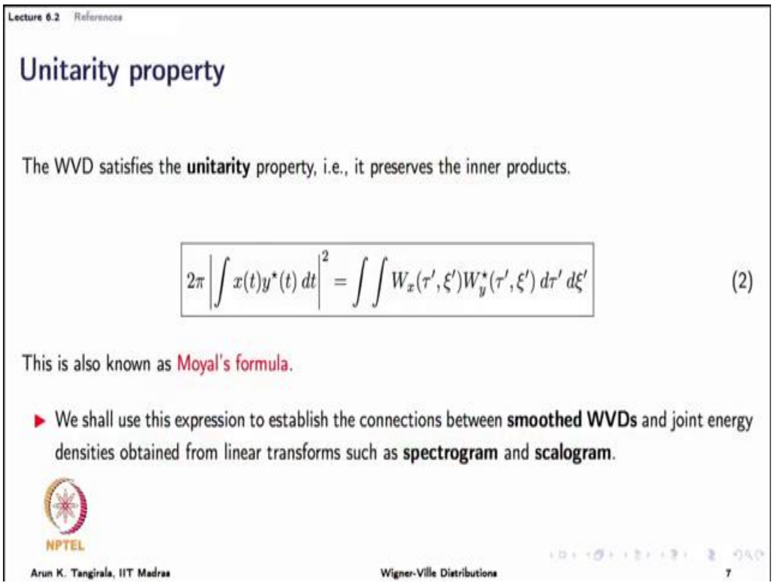

This moved off the modified Wigner-Ville and this is exactly that property due to Moyal which is called Unitarity property. Essentially what did he say is, the inner products are preserved in the signal domain and the Wigner-Ville domain that is in the time frequency plane. Once again we have 2 pi here because we have we are working with angular frequencies. You should be careful. The integral x of t y star of t, essentially the inner product and what Moyal proved is essentially and not exactly the preservation of the inner product, but the preservation of this squared magnitude of the inner product. It says that whether I compute the squared inner product in time or the double integral of the product of the Wigner-Ville of x and complex conjugate of the Wigner-Ville of y, it is one and the same, right. We will use this result to establish the connections between smooth Wigner-Ville distributions, that is modified ones and the quadratic energy densities obtained from linear transform such as spectrogram and scalogram. This will play a critical role in the proof.

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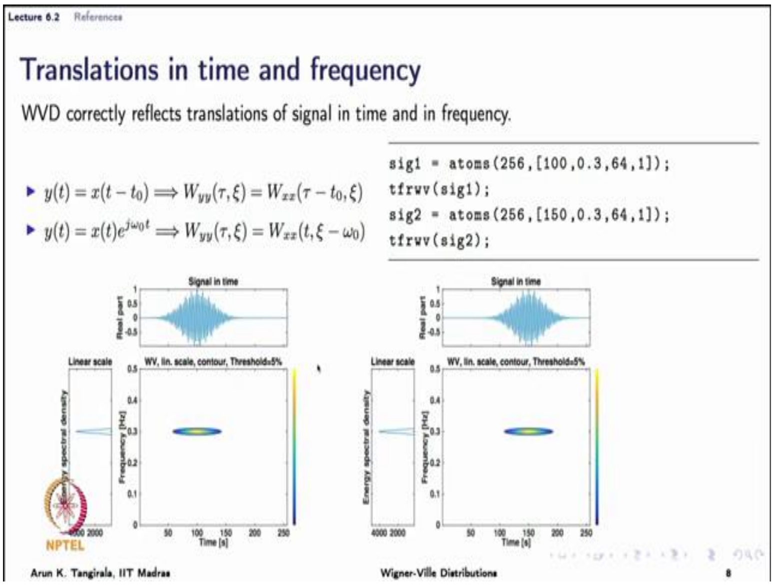

So, let us move on and look at this property of translation. Covariance translation invariance is different from translation covariance. Sometimes in the literature you see this term invariance such incorrect that is in the context of joint energy densities. When we say invariance, we say that it is insensitive to shift and when we say covariance, it is ((13:39)). We want covariance, not invariance. What we mean by covariance is if this signal shifts in time, the joint energy density should also reflect contrast this with linear time invariance system. For example, we say a system is time invariant if a shift input does not produce the different output. It produces a same output regardless of when I give the input. That is why call them as time invariance system.

In fact, when we studied d w d, we will recognize that the d w d is not sensitive to shifts in time, unlike the c w d or the Wigner-Ville and so on. So, coming back to the discussion, the Wigner-Ville distribution correctly reflexes the translation of signal in time and in frequency. The theoretical proof is fairly straight forward. You just plug in a new signal y which is shifted x shifted by amount t naught. Then, you can show that the Wigner-Ville of y is simply the Wigner-Ville of x shifted in time by the same amount t naught, and likewise for frequency. Remember frequency shifts or archived in time by multiplication of the signal with e to the j omega naught t which we call as a frequency modulation. For the frequency, we can say it is frequency modulation. Now, the Wigner-Ville of such a signal is also shifted version of x, but shifted in frequency where same amount omega.

Now, I am just illustrating the first property here which is the translation covariance in time. I have generated and amplitude modulated a sin wave here. The amplitude modulation is a Gaussian modulation. You can see that I have shifted the signal by 50 units here on the right. On the left, you have the original x. You can think of the right one is y. Y is shifted version of x by 50 samples and the Wigner-Ville also shows exactly the same shift, all right which is and again and illustration of what we have seen as a theoretical property. So, the Wigner-Ville is also sensitive to the scaling of the signal which is again a very important requirement. When I scale the signal or when I compress or expand a certain feature of the signal, the joint energy density should reflect that.

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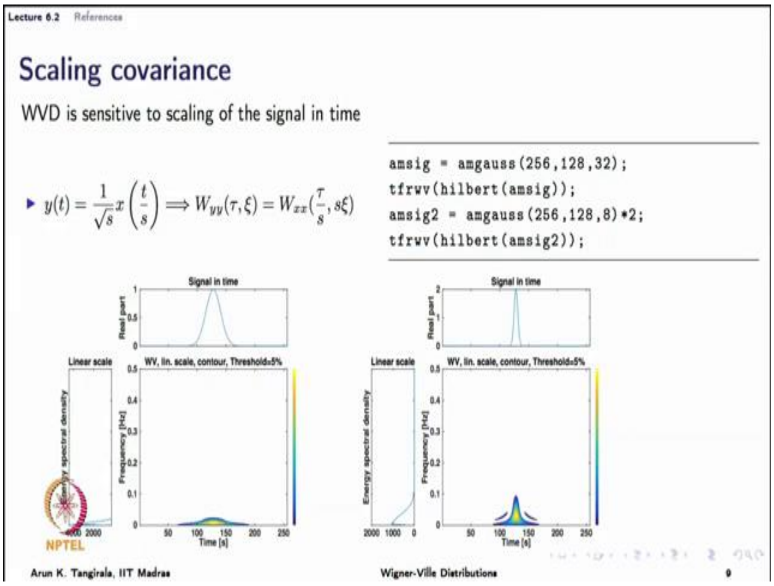

Otherwise, I would have difficulties in feature extraction and fall diagnosis and so on. So, here I have a scale version of a signal x of t. We have already noted down this convention x of t by s is actually this scale version that s is a scaling factor. S values greater than 1 will result in dilation. Values less than 1 will result in compression. Many a times in literature you may see x of s times t. There the situation is different, that is you have to be careful how you write the scale version in this form of expression. The values of s greater than 1 will result in dilation and 1 over root of s here is to simply introduce normalization, so that y has a same unit energy as x. You can once again show that the Wigner-Ville of y reflexes scaling which means now the Wigner-Ville of y is a scale version of x. In the sense, that you have tau over s and s psi.

What this means is here if look at this example, if I have a signal x of t here, a Gaussian wave and its compressed version, then the Wigner-Ville also is compressed and stretched in frequency. So, this s psi becoming the psi in y for y being the x psi for x is what this means is that the frequency spread of the Wigner-Ville of y is going to be a spread version or the stretched version of the frequency spread of x. What this also means is that the center frequency is going to be shifted to omega psi by s. If omega psi is the center frequency of x, then the center frequency of y is going to be omega psi by s. We have discussed this earlier when we talked about the scaling covariance requirement of a joint energy density. I suggest that you work out each of this example by yourself by typing these commands that I have given in Mat lab. Again we are using the time frequency tool box for this.

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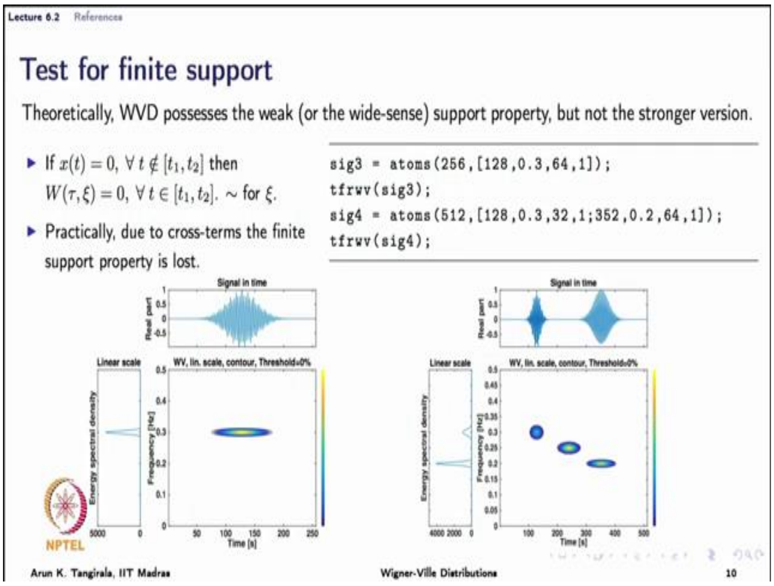

So, the last property that we would like to study in this lecture is the finite support property. Theoretically, the Wigner-Ville distribution satisfies the weak support property or the so-called white sense support which essentially requires the joint energy density to be non-zero when I have a signal which is non-zero over a finite interval and zero outside an interval. So, you can prove that fairly easily starting from the definition of Wigner-Ville. So, I have an example here on the left, where the signal, again the same amplitude modulated sin is non-zero only over some finite interval and zero outside this interval. So, the Wigner-Ville is also exactly reflecting that it does not really go outside the boundary of the existence of the amplitude modulated sin wave.

On the other hand, if I look at this strong finite support property, what is the strong finite support property? This signal starts. It exists for a while and then, stops and then starts again. Such an example here is shown where I have two amplitude modulated sin waves of different frequencies. The first one is of a higher frequency. So, the signal is modulated sin initially for a period of time and then, remains inactive. Then, you have another amplitude modulated sin wave of a lower frequency. What happens here is the Wigner-Ville has yes definitely detected the presence of this frequency and durations, but then you have this interference terms appearing which essentially spoils the strong finite support property. We know why this interference term occurs? It is because of the cross terms. The other way of looking at it is remember the Wigner-Ville folds the signal at any point in time. It looks at the left and the right (( )) both segments and examines if there is an overlap. That is what happens here when you are standing at any time in which a signal is inactive. It is likely that there will be some overlap, some correlation in the finite duration segments that produces this interference, right.

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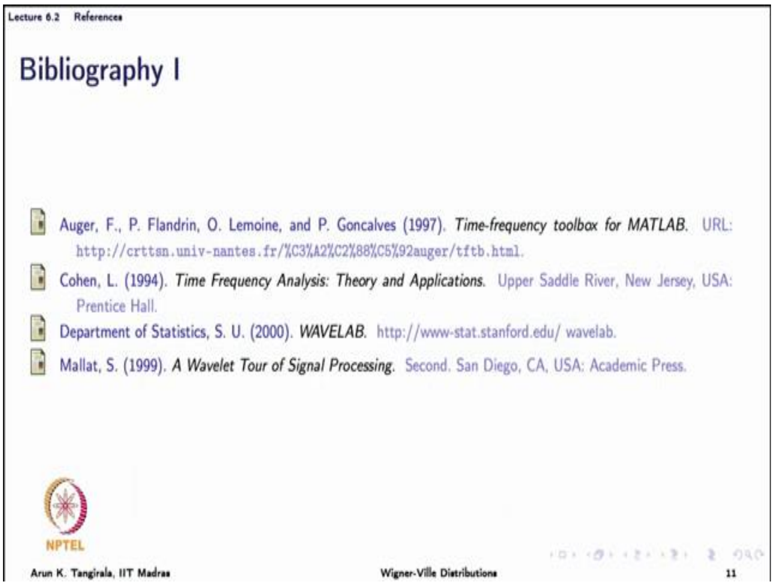

So, the message is that WVD satisfies the weak finite support property, but practically you are not going to have signals which only exist for a short period. Then, do not exist before and after you are the more practical situation is example that you see on the right. Therefore, it does not really satisfy the finite support property in practice. So, with this we come to close of this lecture. Again there are few references that you have seen earlier. There are of course other books up. I would go, I would suggest that you go and search. There are few books again time frequency analysis by few authors like Auger and so on. So, those books are also very nicely written. They are more up to date. I suggest that you read those books as well. In the next lecture, we are going to look at a few other properties of the Wigner-Ville and in particular, we will talk about the global and local averages and the effects of noise, and talk about the need for analytic representations versus real representations. Thank you.