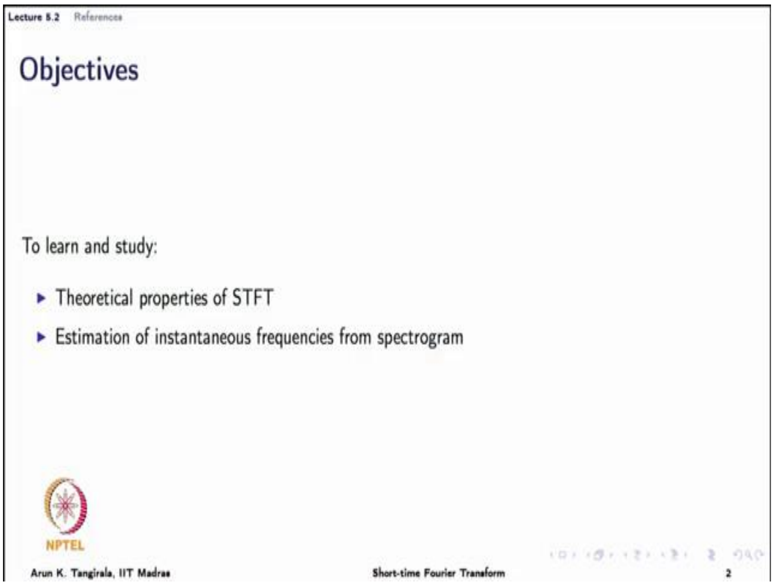
Introduction to Time-Frequency Analysis and Wavelet Transforms
Prof. Arun K. Tangirala
Department of Chemical Engineering
Indian Institute of Technology, Madras

Lecture - 5.2
Properties of STFT

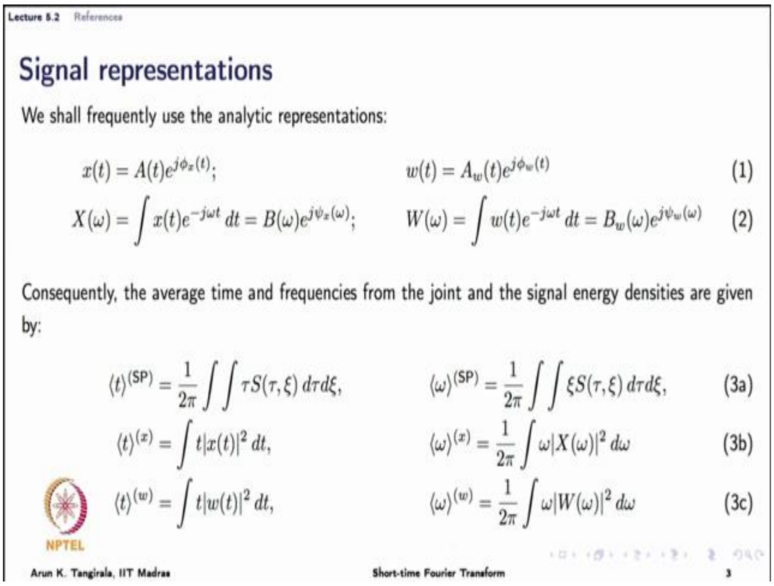
Hello friends. Welcome to lecture 5.2 of the course on Introduction to Time Frequency Analysis and Wavelet Transforms. In this lecture, we are going to discuss mainly the theoretical properties of short-time Fourier transform. In the previous lecture that is 5.1, we introduced the definition of short-time Fourier transform. And we briefly discuss the filtering perspective which tells me that the short-time Fourier transform essentially acts like a band pass filter of constant bandwidth. Then also briefly discussed about the smearing that occurs due to the use of the window functions, smearing of the energy density due to the use of the window function.

(Refer Slide Time: 01:05)



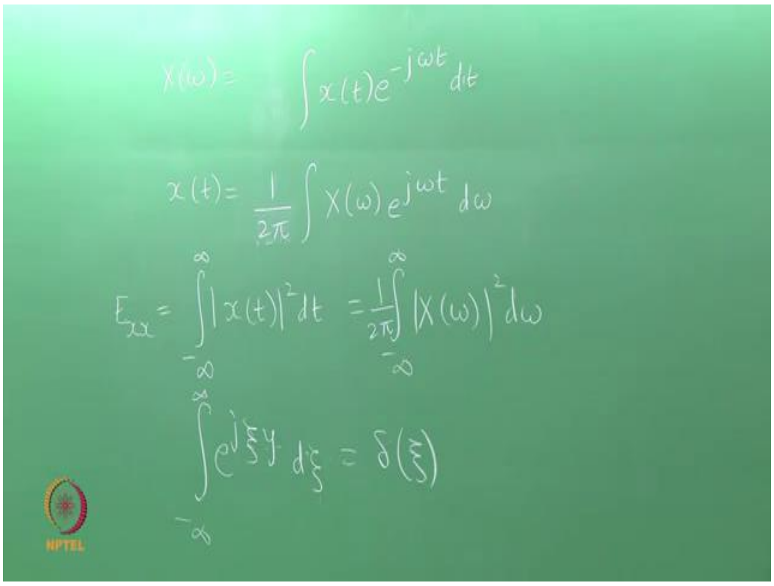
Today or in this lecture, we are actually going to talk about the theoretical properties. When we say theoretical properties, primarily we are referring to this spectrogram as a joint energy density, where we have already seen yesterday that perfect recovery of the signal is possible from short-time Fourier transform. Therefore, I can use it for filtering applications. Primarily we are going to talk about how the joint energy density in other words, a spectrogram behaves. In this respect, you should recall the properties or the requirements that we had spelt-out for the joint energy density in any time frequency analysis, namely the translation invariance or non-negativity, marginality and so on. We are also going to talk about estimation of instantaneous frequencies from spectrogram. Again the estimation part is going to be kept less theoretical, and I will show you how to estimate this in Matlab. We are also going to use today Wavelab toolbox. So, if you are going to follow this lecture, you may want to keep your Matlab window ready and have Wavelab installed and set on your path.

(Refer Slide Time: 02:21)



Before we begin to discuss the theoretical properties, let me make certain notations and mathematical representations clear to you. On the left hand side here of the equation 1, I have the signal representation, and on the right hand side I have the time domain representation for the window. Given beneath each of these are the Fourier transforms of the signal and window. At this point, I want to caution you or alert you on an important thing. In discussing the basics of time frequency analysis, that is in unit 4 on time frequency analysis, we had 1 over root 2 pi in front of this integral here. So, let me make it clear once again.

(Refer Slide Time: 03:23)



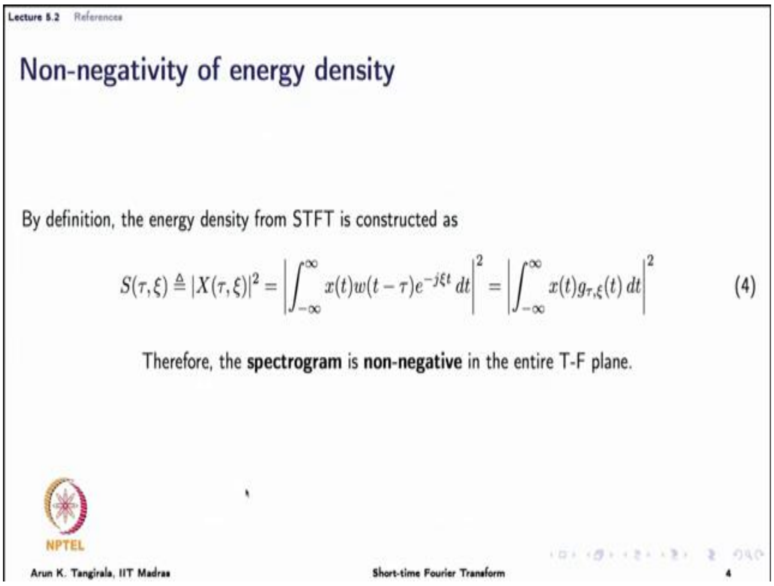
In deriving the or in discussing the time frequency basics or foundations, we had used this definition for the Fourier transform of the continuous time signal, and likewise x of t being 1 over root 2 pi, both running from minus infinity to infinity. Once again here by omega, we are referring to the continuous time frequency. So, we had these definitions, so that I could write the energy density. Sorry, the energy itself has the area under mod x of t square, and also as area under mod x of omega square. Remember we call this as a unitary transform. This convention we followed primarily keeping in mind the source of all the time frequency analysis foundations, which is the book by Cohen. In order to avoid any confusion, I had followed this convention, but now we switch over to the regular definition that I have given for the continuous time Fourier transform, where 1 over root 2 pi disappears here in front of the x of integral for x of omega, and this 1 over root 2 pi here is replaced with 1 over 2 pi, so that the energy density is now 1 over 2, the energy itself is 1 over 2 pi integral of this.

What is a consequence of this? Well, when you are computing averages, and the spreads, the time, particularly the frequency is spread and the frequency averages. You will now see A 1 over 2 pi appearing as you see on the slide. In the previous unit, you would not see this 1 over 2 pi. So, just this notation change occurs in this unit, and also this makes it easy for you to follow the rest of the literature, particularly in the theory develop in Mallat's books and so on. So, please note this simple change. We are moving now from unitary transforms to the regular transforms.

So, with that note of caution, let us proceed. Now I have here the expression for the time averages computed from spectrogram, that is joint density and a signal energy density is themselves. These expressions will become useful to us later on. What I am doing here is, I am giving you expressions for the global average that you would compute from spectrogram. So, if I give you a spectrogram and I ask you how would you compute the average location or the average center in time for the signal and the average frequency for the signal, you would use these expressions here.

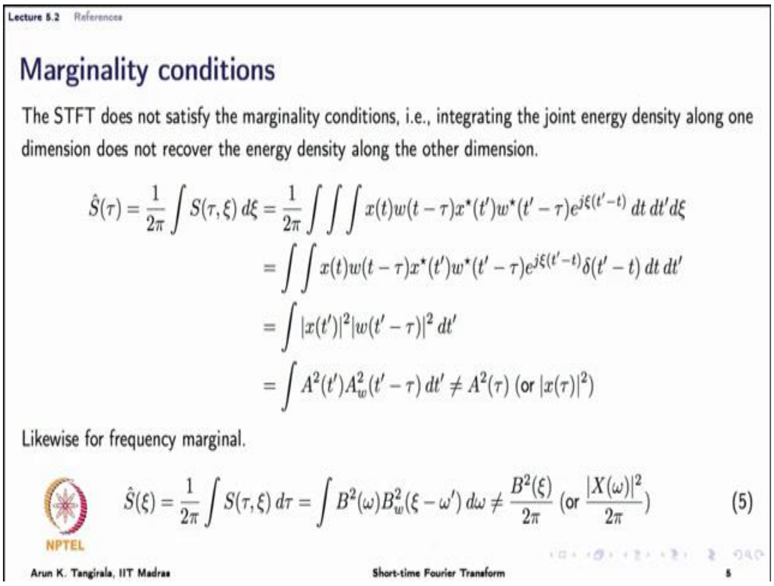
Notice that I am using, in fact this should be tau strictly speaking on the left hand side, but as long as I remember that it is time, you should be ok. So, remember that we are evaluating a double integral here, because the spectrogram is a two-dimensional function. Likewise here, the center frequency computed from the spectrogram involves a double integral once again. The only difference is here I am computing with respect to the first moment with respect to frequency, and here the first moment with respect to time, these expressions in 3 b and 3 c are familiar to you. As I just said in the unit 4, when we wrote the expressions for the center frequency of a signal, we did not have 1 over 2 pi, but now 1 over 2 pi appears, because now we are switched over from unitary transforms to the regular ordinary transform. Therefore, I have 1 over 2 pi here, but the rest of the expressions are remains the same.

(Refer Slide Time: 07:51)



So, let us proceed. First property of interest for me is to check whether the joint energy density coming out from the short-time Fourier transform is non-negative. We have defined yesterday the spectrogram which gives me the joint energy density, and we know why the spectrogram qualifies to be the energy density. By definition, the spectrogram is non-negative. So, there is not much to discuss here. It is fairly obvious that the joint energy density is going to be non-negative in the entire time frequency plane. Again here I would like to make an observation here. Strictly speaking the joint energy density should have a factor of 1 over 2 pi here, but I am just following the convention that, that is followed in Mallat's Book, Strictly speaking, it should be A1 over 2 pi. Because I am working with angular frequency here, this chi has units of angular frequency. But no worries as long as you remember this fact that the spectrogram is a squared magnitude of the short-time Fourier transform, you are safe.

(Refer Slide Time: 09:06)



So, let us examine the next important property which is the property of marginality condition or whether the spectrogram meets the marginality requirement. Recall what we mean by marginality requirement is, if I integrate the joint energy density in one variable along one-dimension, then I should recover the energy density along another dimension. That is exactly what I am doing here. Let us say I want to integrate the spectrogram along the frequency axis. Then if the spectrogram satisfies a marginality requirement, what I should recover is the signals energy density in time, because I am walking along frequency axis, right. This 1 over 2 pi again comes here, because of the change in notation.

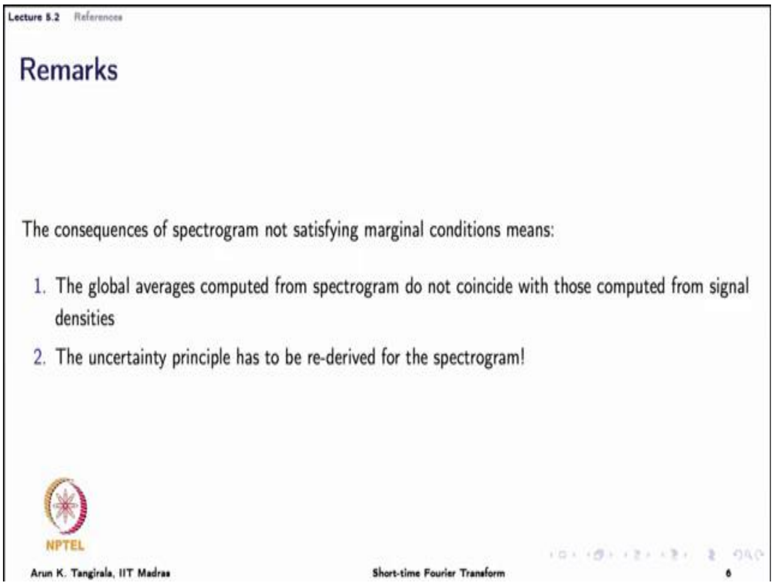
Now, what I have done here is, I have indicated this with s hat of tau. S hat of tau is energy density as a function of tau which has units of time t and as I just said if everything comes out all right, that is if this spectrogram satisfies the marginality requirement, then this integral should evaluate to mod x of tau square. Let us see if it does. What I have done here is now the substituted for the spectrogram the definition itself, where I have written spectrogram as x of t times w of t minus tau times x star of t prime times w star of t prime minus tau times this factor here. The reason these conjugates appear is, because spectrogram if you see, it is a modulus square, but I make as well rewrite this modular magnitude square as the product of this complex number times is conjugated itself, and then I use the definition of a short-time Fourier transforms.

So, that is all I have done. There is nothing new here. The only point here is I have now a triple integral, but no worries we will be able to simplify fairly easily. Now, all we are doing here is we are rewriting here. There is a d c that is missing here. We should make note of them. Sorry, the d chi is missing here, because I have evaluated this exponential along the chi dimension. In fact, what should not be present here is this e to j chi times t prime minus t. So, that should be taken out. That does not appear. So, when I integrate e to the j chi t prime minus t along the chi, then I get dirac delta function. This is one of the definitions of dirac delta function. Let me write that for you. Integral minus infinity to infinity e to the j chi, let us say some variable y and let some evaluate this from minus infinity to infinity for in terms of d chi. Then, this is simply delta, the dirac delta also. This is one of the definitions of the dirac delta function. You may refer to the literature.

As similar result, we had used not exactly this result, but a similar result we had used in deriving the analytic associate of a given signal. There the integral ran from 0 to infinity, but now the integral run from minus infinity to infinity. So, that is why there is change in result. If you go back to the derivation of the analytic signal, you will see that this integral, where run from 0 to infinity and then you have an extra term here, all right. So, keep that in mind. We have just used this property and replaced this integral, one integral of e to the j chi prime minus t times d chi with the dirac delta function itself. Therefore, this factor is e to the j chi, the second equation should vanish. It should not be there and then using the property of the dirac delta function, the double integral simply reduces to as single integral, line integral, and all are done here have been substituted for x of t prime and w of t in terms of their representations that we had used here in equation 1, right, because I have evaluated the magnitude. The phases do not participate. The final result that I have here is that I do not get the marginal equal to mod of x of tau square which is what I should have ideally if the spectrogram satisfies the marginality requirement.

Now, what that means is if I walk along the frequency axis and add up the energy densities, I do not get the energy density at that particular time frame whatever time you pick, you will not recover the energy density. However, the total energy requirement is satisfied as we will talk about at shortly and we have seen that yesterday there in the previous lecture as well. So, you can follow a similar derivation here, and prove that the frequency marginal evaluated by integrating the spectrogram along the time dimension is not equal to the true or the signals frequency marginal. That is again the story that we have here as expected, and therefore you have to be careful when you are working with these spectrogram, and you are fixing your position at particular time t, and you are summing up the energy density along all frequencies or even over a band of frequencies.

(Refer Slide Time: 15:05)

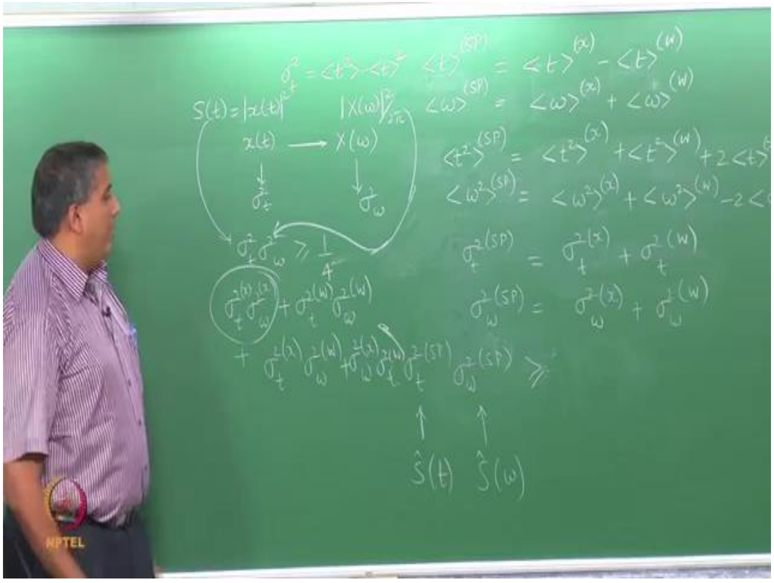


Let us ask what is a consequence of this violation of the marginality requirement, because it is not immediately obvious, why this is such an important result? Well, there are two, at least two consequences. There are many, but of the many there are two important consequences. One is that the global average computed from this spectrogram does not coincide with the once that you compute from the signal. Earlier we had defined the global averages, namely the average time t computed from the spectrogram, and the average frequency here computed from spectrogram in equation 3A, these quantities here will not equal. For example, once that I compute from the signal, for instance I have here, the average time t computed from the spectrogram that will not equal to the average time that I compute from the signal itself. Why because there is a window that is standing between you and the signal when you take the short-time Fourier transform.

And in fact, I will give you the exact expression for the average time computed from the spectrogram in terms of the averages of the signal and window, likewise for the frequencies as well. I give that to you very soon, but let me point out more important consequences that arise out of this lack or the violation of the marginality requirement condition. Most important consequence is the uncertainty principal has to be re-derived for the spectrogram. We just mention that a global averages computed from the spectrogram do not coincide. Not only do the global averages not coincide, but also this, sorry the spreads that I compute from the spectrogram.

What I mean by spread is suppose I want to compute the duration of the signal from the spectrogram rather than from the signal, then they will not match. Likewise if I want to compute the bandwidth of the signal from the spectrogram rather than from the signal, they will also not coincide, right. Now, remember that we derive the duration bandwidth principle or the uncertainty principle for the signal, but now we are saying that I cannot use that same duration bandwidth principle for this spectrogram. So, let me actually discuss that a bit more in detail now.

(Refer Slide Time: 17:46)



So, what I have first of all is the time average computed from this spectrogram is in fact, you can show starting with the definitions that we had in the equations 3A, 3B and 3C. In fact, 3A you can show that the global average that you compute from the spectrogram is related to the time average of the signal and the window by this expression and likewise, when I choose a real and symmetric window, then the average of the window will go to 0. That way I can ensure that these two coincide, right. However, when I look at this spreads which is more serious issue, in fact before jumping into the spreads, let me give you the expression for the average of t square, that is remember the reason for giving this is, because I know that sigma square t for a signal is given by this expression here. Already we are giving the expression for this. In order to compute sigma square sp, that is the duration of the signal if I want to compute from the spectrogram, then I need an expression for this and I already have an expression for this. Therefore, I can evaluate.

So, what I have here is, this is the expression that I have for the average of t square computed from the spectrogram and likewise, the average of the square frequency shares a relation with those of the signal and the window by this expression. Now, I can compute the sigma square t square, sigma square t for the spectrogram. In fact, if you use this expression, you apply this expression to the case of computing from the spectrogram and for that of the signal and for that of the window itself, you can show just working out the algebra, this would turn out to be sigma square t x plus sigma square t w. Assuming that I have chosen the real and symmetric window, right which means the averages of the window in time and frequency are 0 and likewise for the bandwidth as well. So, now, I have this result, the duration. Let us say this is the square duration, but I will just with some use of terminology, I will call this as duration. The duration here computed from the spectrogram, computed from the signal plus duration computed from the duration of the window itself. Likewise, the bandwidth of this is very interesting result. The bandwidth that I compute from the spectrogram is a bandwidth of the signal plus the bandwidth of the window.

Now, this bandwidth of the window can never go to 0. It cannot, because I am going to use a finite window that is a window of finite length. Its bandwidth cannot be 0. It has to be a non-zero value. Therefore, this is never equal to this and likewise, this is never going to be equal to 0 unless I choose an impulse like window, right. So, in general the duration and bandwidth of the spectrogram is going to be different from that of the signal, these are not local duration and bandwidths. The difference between local and global bandwidth is that a local duration or bandwidth is when you are analyzing in a local time frequency cell.

Now, what we mean by deriving, re-deriving the duration bandwidth principle for this spectrogram is that this product here cannot be expected to be bounded below by 1 over 4. This is not possible. Why is that? It is because the duration bandwidth principle says any signal or any function, let say denoted by x of t and whose Fourier transform is denoted by x of omega, let say the duration of the signal in time is sigma square t and computed likewise, the bandwidth computed from x of omega, they share or rather they satisfy this duration bandwidth principle which means the sigma square t and sigma square omega have to be computed from the signal and its Fourier transform, right, but here that is not true. What we mean by this is that these are not computed from marginal, in the sense how may computing from the original marginal. These I am computing this from what is giving me, this. This is this global duration from the spectrogram s hat of t that we had seen earlier, that is when I compute the marginal from the spectrogram, and then I use that to compute the duration. That is a global duration.

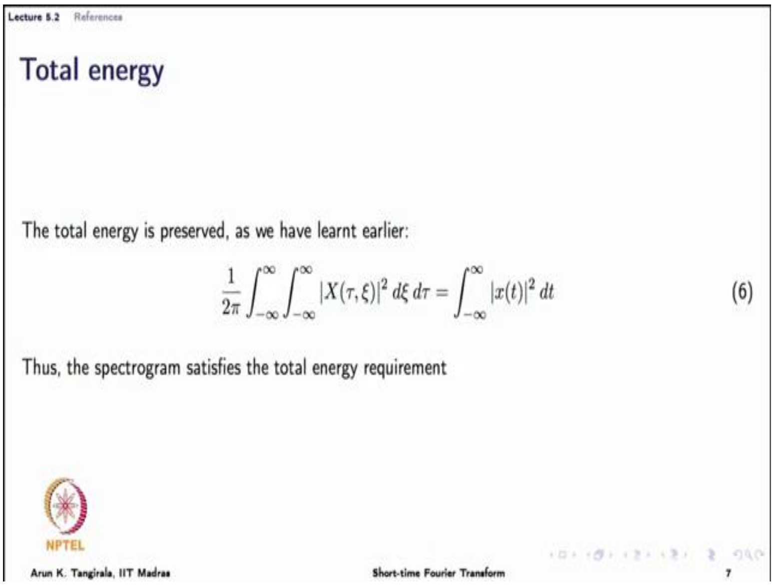
Likewise here to compute the global bandwidth from the spectrogram, I have to evaluate the marginal from the spectrogram. This is different from the energy density that I compute directly from the signal. So, sigma square t is actually coming s of t and sigma square omega A, strictly speaking over 2 pi is going to give me this bandwidth, and these energy densities themselves are coming from a Fourier transform relation, but s hat of t and s hat of omega are not falling out of a fourier relation. That is because the spectrogram does not satisfy the marginality property. Therefore, this bound is not right, nevertheless these still satisfy a product like relation. The question is what this bound is just, because 1 over 4 is not the right answer, it does not mean that the global duration and bandwidth computed from the spectrogram does not satisfy a lower bound.

Now, how do I determine this lower bound? Well, there has been some work done already, but then a quick calculation here that when I compute this product, I am actually computing the product of these terms and I have four terms essentially. One of the first terms being, this actually evaluates to sigma square t times sigma square omega that is of the signal plus these terms, the product of the duration bandwidth of the window, and then the two cross terms that I have. This itself the least value of this is 1 over 4. From the duration bandwidth principle, I know already that the least value for this is 1 over 4. The least value for this is 1 over 4 and then I have two more, but do not expect these are also bounded below by 1 over 4, but if you think that way, coincidentally the bound for this turns out to be 1.

So, the global duration and the global bandwidth that I compute from the spectrogram actually bounded below by 1. Of course, a question is how does it matter? Well, rather than looking upon this as a lower bound, the way it can be looked upon is it tells me how fast the bandwidth changes when the duration changes, right. Here for this signal or for any function, the bandwidth actually changes according to this 1 over 4. So, if the duration is increased by a certain factor, then the bandwidth is more or less going to increase two times the duration, but here the story is different and essentially what we are trying to say here is that the global duration and bandwidth that you compute from the spectrogram are related, but they are related in a different way and the spectrogram make things worse than that of the signal when it comes to global properties. It should be expected, because as far as global properties are concerned, nothing can beat the Fourier transform. Locally you may have an improvement and to evaluate what is happening locally, one has to actually work with the product of sigma square t given omega time sigma square omega given t.

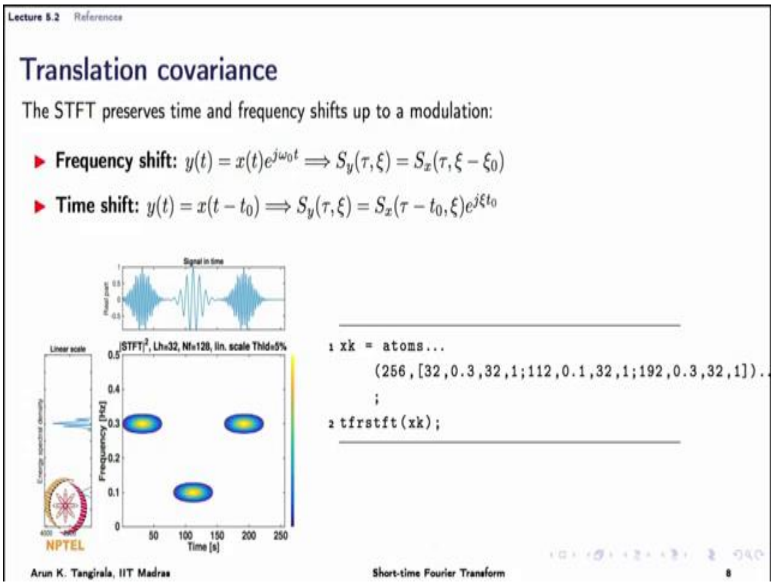
So, this also turns out as a certain lower bound. In fact, the first thing that we can say is that it has an upper bound which is that of the value of the global product itself. That means local properties cannot be worse than what you see globally. That is what it means and you can show that it is also bounded below by the sigma square t sigma omega of the signal. What essentially these results tell me is globally the spectrogram is not such a useful tool as expected which means if I know that there is a certain frequency component that is spread all over time, I should not be using spectrogram. I should be using Fourier analysis, but locally that is a local properties are going to be much better than what I have than global behavior. Definitely it is going to be better than what the Fourier transform can give. That is the summary of these observations. So, the marginality property has an important consequence.

(Refer Slide Time: 30:15)



Moving on, we remarked earlier and even yesterday we have seen this result. The area under the spectrogram gives me the energy of the signal itself. Therefore, this spectrogram satisfies the total energy requirement which is good, because even if this is not satisfied, then it is not a useful tool.

(Refer Slide Time: 30:38)

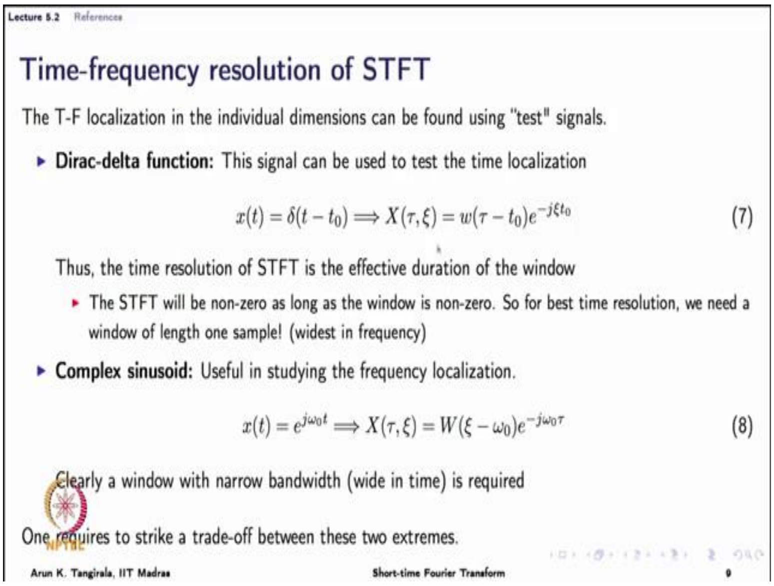


So, let us ask if the short-time Fourier transform is sensitive to translation covariance, in the sense if there is a translation of that signal in time, is it reflected appropriately in the energy density and likewise, if there is a frequency modulation, does this spectrogram reflect the same. Mathematically yes. If I have a frequency shift, typically if a frequency shifts in terms of modulations, then the spectrogram is sensitive to that. That means you can see that the center of the energy density now shifts to whatever frequency is being given here. It should be omega naught here and if I have a time shift, then the spectrogram is actually sensitive to the, in fact this is not just a spectrogram, this is actually the short-time Fourier transform itself that I have written here. The spectrogram would be invariant to this factor here. Sorry make that small correction later on.

Here is an illustration of these properties here. What I have done is, I have taken a Gaussian modulated signal here and I generate this signal through the atoms routine in time frequency toolbox, the syntax for which you can look up. I am generating 256 observations of this. What I am doing here is there is Gaussian modulated frequency wave, amplitude modulated frequency wave of frequency 0.3. It diced on here exponentially and then stays IV for a while, and then I have a frequency lower frequency 0.1. Once again Gaussian modulated. Then, that diced on and the same frequency reappears. In other words, this feature has actually translated itself thing of it as a translation. The joint energy density does show that and you can think of this change in frequency as a frequency shifted from 0.3 from 0.1, and the spectrogram is able to nicely pick those frequencies and time shifts.

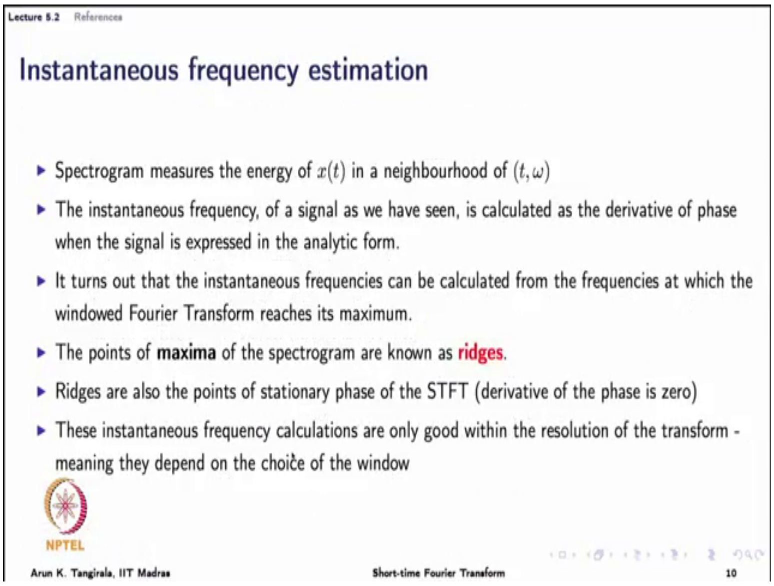
Of course, what you will find here is that it has also nicely picked up the time localization of these features, and that is a coincidence, because I have chosen the window width exactly equal to the width of this feature. If I choose a different window width and you should try this out, you will see that it may not be able to pick exactly the time localization, but whatever it sees, you will see the shifts, right. Of course, as usual we have the spectrum here showing frequencies 0.1 and 0.3. So, just a note here the atoms routine will return an analytic signal. So, you do not have to actual take an analytic representation of x k. It can directly pass that to the t f r s t f t.

(Refer Slide Time: 33:33)



This is something that we have discussed yesterday, the time frequency resolution of short-time Fourier transform. Essentially the time resolution of the, that is time localization of the short-time Fourier transform is at best the window itself. You cannot get a finer resolution or localization of the energy than the width of the window as expected, and you cannot get better frequency localization than the spread of the window in the frequency. That is your window is really limiting your ability to see through the signal, it is like you are wearing some colored glasses and looking at the signal.

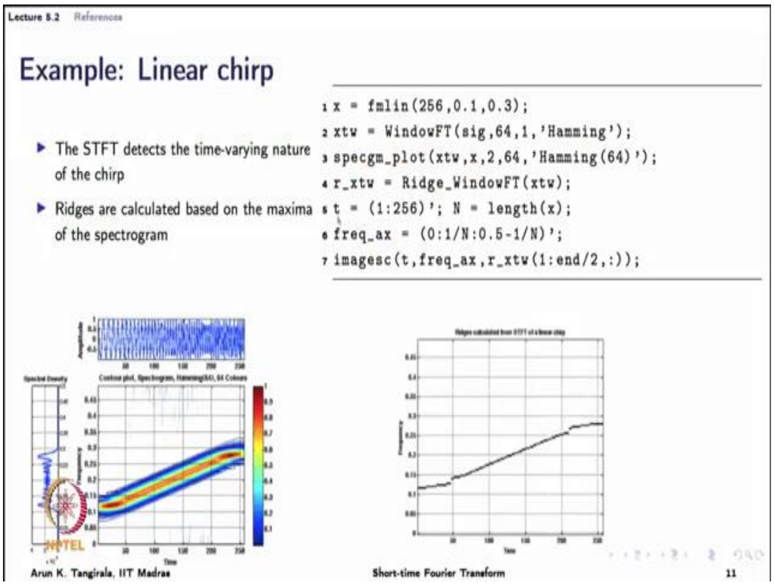
(Refer Slide Time: 34:11)



Finally, we will talk about instantaneous frequency estimation. I gave you an example yesterday of a linear chirp, and what we noted is that the maximum, that is the local maximum of the spectrogram recovery instantaneous frequency for that example, and it was a coincidence in the previous lecture. Now, I am giving you the general result. I am avoiding the theoretical proof, it is fairly involved. I refer you to Mallat's book chapter 4. There is a derivation, and a theorem, and a huge long derivation for the proof. The final result as far as practicalities is concerned, utility is concerned; that the points of maxima of the spectrogram which are known as ridges, the frequencies at which this maxima occur locally in time will give me estimates of the instantaneous frequency. That is the essence of that result.

But remember these instantaneous frequency calculations are only good within the resolution of the transform. If you remember we said a signal can have multiple frequencies at a given time. If these multiple frequencies are spaced less than the frequency localization ability of the window, then you will have a problem. We just now said at best you can resolve the frequencies dependent on the window properties. The windows frequency spread will determine your ability to do well.

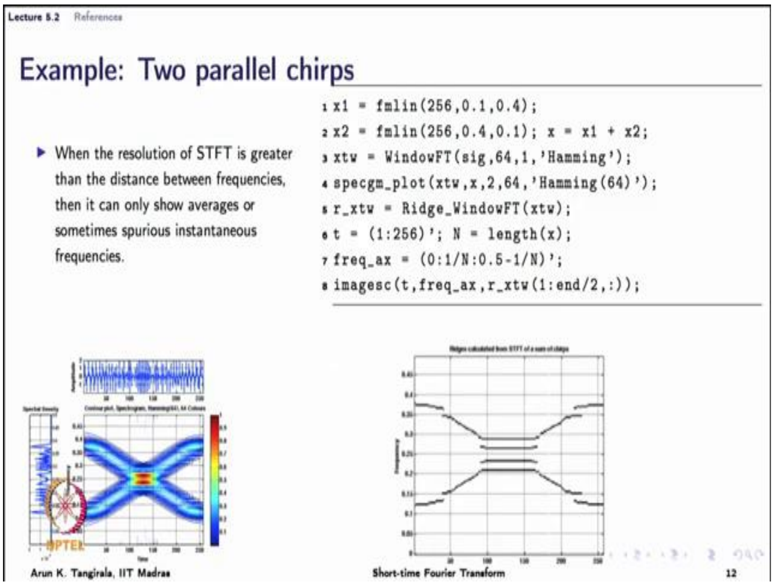
(Refer Slide Time: 35:42)



So, let us look at this example of the linear chirp that we used yesterday, that is for which we derived the theoretical expression. Here I am using the wave lab, not the time frequency tool box. I have here, the… Well, I am using both time frequency tool box and wave lab. Time frequency tool box and for generating the signal, and generating a linear chirp and what I am doing here is using the window f t which computes a short-time Fourier transform in wave lab to pass it on to routine that I have written called spec gram underscore plot which essentially plots the figure that you see here at the left bottom. On the top you have the chirp, on the left panel you have the spectrum, and here you have the spectrogram. It nicely follows the frequency changing behavior, but of course the frequency localization or the energy localization the frequency domain is spread, because I know that at any instant that is there should be a single frequency, but I cannot get that with the short-time Fourier transform, because of the window. The windows bandwidth is the one that is masking that true frequency.

What you see on the right here is I have actually seen through kind of unmasked what is happening as a consequence of the windowing, and picked up the local maxima and I have the instantaneous frequencies here. So, it is very nice. So, this is a very nice result and this is extensively used in calculating local frequencies also used in image analysis and so on. These are called ridges. We will see a similar result even with wavelet transforms, where we follow these scalograms instead of spectrograms.

(Refer Slide Time: 37:34)



Another example I have of two parallel chirps. Both these examples are taken from Mallat's book, and once again here I generate two chirps and add them up. So, one starts to chirp up and other starts to chirp down, they both meet at the certain point here at this point in the center. Again I follow similar procedure as far as mat lab is concerned. You should try to run all these codes by yourself, and check if you indeed get these results. The ridge underscore window as you have seen in the previous code is the one that gets me, that gets you the instantaneous frequency estimates, and as I mentioned earlier at this point when the frequencies of both chirps have come very close at the point of meeting, they have come so close that they are beyond the resolvability of the window itself, and there you see some meaningless results. You see that there are four frequencies and so on.

And that it is expected, because you are working with the mathematical definition of instantaneous frequency, but outside this time zone I have the instantaneous frequencies nicely resolved. So which is very good, so the spectrogram not only gives me the joint energy density, it not only gets me the local behavior, but also facilitates computation of instantaneous frequencies. So, that is a summary. So, with that we will close this lecture and these are some of the references again that we have used in developing the lecture material. Please go through a lecture once again, because this lecture is fairly theoretical, particularly the derivation of the lower bound for the product of duration and bandwidth, global duration bandwidth that I compute from the spectrogram and so on.

Rather than worrying just about the theoretical result in fact, you should be asking what is a consequence is of each of this result in practicality or with respect to a certain application. Some of these properties may not be relevant to a particular application that you are looking at, but given the academic nature of this course, we discuss all the properties. In the next lecture, we are going to study the effect of windows, we have seen to a certain extent, but we will exclusively devote the lecture to two things. One is how the choice of window length and the choice of window can affect the computation or the joint energy density itself. And two, how is short-time Fourier transform computed in practice, that is a discrete version of it. Until now we have studied the theoretical property. So, see you in the next lecture.

Thank you.