Chemical Engineering Fluid Dynamics and Heat Transfer

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Lecture - 13
Exact Solution 3

So, welcome back to the lecture 13. We will continue to discuss the exact solutions.

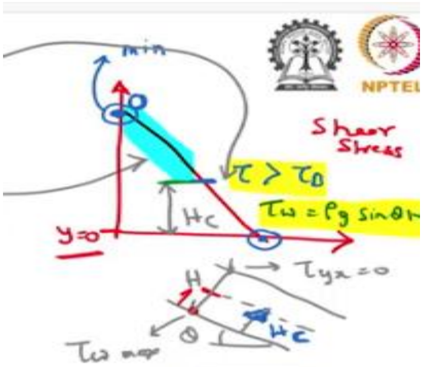
As we discussed in the last class, shear stress distribution for an inclined plane is generic and it is valid for all fluids. That means, the shear stress is minimum at the free surface its 0 and it is maximum at the wall. And as we know, the Bingham fluid behaves like Newtonian fluid, if the applied stress (τ_{yx})exceeds the yield stress(τ_B).

And if the applied stress(τ_{yx}) is below yield stress ($\tau_{B)}$ $\rightarrow \frac{\partial u}{\partial y} = 0$;there is no deformation. If the wall shear stress(τ_w) is less than that of (τ_B) ,then over the entire plate there is no flow.

If $\tau_w < \tau_B$:There will not be any flow. Once the τ_w exceeds the τ_B ,it starts to flow.

Applying the body force is basically a combination of the amount of mass that is present which translates to the film thickness and the angle of inclination. Wall shear stress is a function of H. So, higher the thickness of the layer, higher the wall shear stress and inclination will be higher. As θ increases $\sin\theta$ increases.

If $\tau_w > \tau_B$:Bingham plastic starts to flow. H_c is the critical height or transition height at which ($\tau_{xy})_{y=H_c} = \tau_B$



At $y = 0$:stress is maximum $\tau_w = \rho g \sin\theta\, H$

At $y = 0\ to\ H_c; 0 < y < H_c \rightarrow \tau_{xy} > \tau_B$

At $y = H_c\ to\ H$:If $H_C < y < H \rightarrow \tau_{xy} < \tau_B$

At $y = H$;stress will be minimum $\tau_{yx} = 0$

So, we can say that if you move away from the wall the stress is progressively reducing.

For the zone, $H_C \leq y \leq H$, $\tau_{xy} < \tau_B \rightarrow \frac{\partial u}{\partial y} = 0$

For the zone, $0 \leq y \leq H_c \rightarrow \tau_{xy} > \tau_B \rightarrow \tau_{xy} = \tau_B + \mu_B \left(\frac{\partial u}{\partial y}\right)$

If we substitute for $\tau_{xy} = \rho g \sin\theta\,(H - y)$,

$$\rho g \sin\theta\,(H - y) = \tau_B + \mu_B \left(\frac{\partial u}{\partial y}\right)$$

$$\int du = \frac{1}{\mu_B}\int \left(-\tau_B + \rho g \sin\theta\,(H - y)\right) dy$$

Upon integration,

$$u = \frac{1}{\mu_B}\left(-\tau_{By} + \rho g \sin\theta \left(H_y - \frac{y^2}{2}\right)\right) + C_4$$

To evaluate C_4,Use no slip boundary condition → at y=0, u=0: therefore C_4=0

Then

$$u = \frac{1}{\mu_B}\left(-\tau_{By} + \rho g \left(H_y - \frac{y^2}{2}\right)\sin\theta\right)$$

This is the velocity profile for Bingham plastic fluid in the zone, $0 \leq y \leq H_c$.

For the zone, $H_C \leq y \leq H$, $\tau_{xy} < \tau_B$

At this part H_C is not known, but we knew that at $y = H_c$, $\tau_{xy} = \tau_B$

$$\tau_{xy} = \rho g \sin\theta\,(H - y)$$

At y = H_c → $\tau_{xy} = \rho g \sin\theta\,(H - H_c) = \tau_B$

$\tau_B$is the material property and its value is known and the values of H and θ also known from the system. therefore, we can find out the value of H_c from the above relation.

$$H - H_c = \frac{\tau_B}{\rho g \sin\theta}$$

From this relationship, $\tau_w = \rho g \sin\theta\, H$: $H = \frac{\tau_w}{\rho g \sin\theta}$

$$\frac{H - H_c}{H} = \frac{\tau_B}{\tau_w}$$

$$1 - \frac{H_c}{H} = \frac{\tau_B}{\tau_w}$$

$$\frac{H_c}{H} = 1 - \frac{\tau_B}{\tau_w}$$

From the above equation we can find out the value of critical height.

Local velocity at H=HC, $u|_{H_C} = -\frac{\tau_B H_c}{\mu_B} + \frac{\rho g {H_c}^2}{\mu_B}\left(\frac{H}{H_c} - \frac{1}{2}\right)\sin\theta$

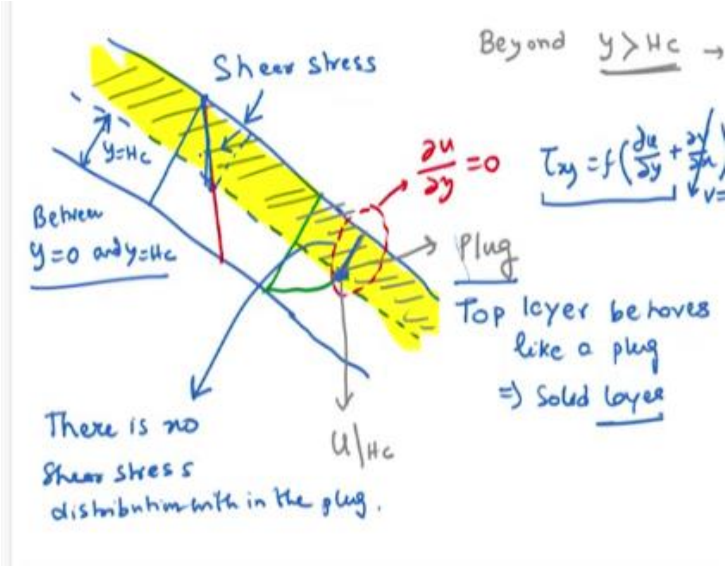
For the zone, $0 \leq y \leq H_c$ velocity profile for Bingham plastic is parabolic in nature and the shear stress profile is linear.

If we are looking into the zone, $H_C \leq y \leq H$

You know that in this zone, shear stress is low. the $\tau_{xy} < \tau_B \rightarrow \frac{\partial u}{\partial y} = 0$ that means there is no velocity gradient in that zone, and you have found out the expression for the Local velocity at the point H=HC. Now you have a non-zero velocity $u|_{H_C}$, and beyond this stage the velocity gradient is 0. That means you have a plug flow, when $y \geq H_C$.Like a solid layer float over a liquid layer below H_C.And in this zone there is no deformation, as $\frac{\partial u}{\partial y} = 0$.Shear stress τ_{xy} is a function of the velocity gradient.

$$\tau_{xy} = f\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

And for this system $v = 0\ and\ \frac{\partial u}{\partial y} = 0$,therefore the shear stress τ_{xy} becomes zero. That means, within this zone there is no shear stress distribution within the plug. Shear stress should be constant within the plug flow, but from this shear stress profile, it predicts a continuous variation. So, we can say that there is a singularity point in the shear stress distribution profile. That means there are two values of shear stress along the depth of the film.

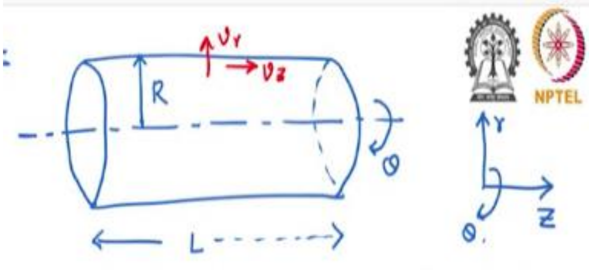


For the flow of a Bingham plastic fluid over an inclined plane, Once the τ_w exceeds the τ_B ,it starts to flow that means the closer to the wall, $0 \leq y \leq H_c$.In this zone the fluid exhibits velocity profile, because of the no slip boundary condition at the surface. Here the fluid will have a liquid like structure and above that zone which means $H_C \leq y \leq H$,shear stress lower than the critical yield stress (τ_B), and it causes no velocity profile in the zone. So, it looks like a solid plug the layer will float over a liquid like region and it is all within the same material.

<u>Flow through a tube/Pipe</u>

Pressure driven flow:

Consider a pipe of uniform cross-sectional area with the length L and the radius R with the cylindrical coordinate system and listed below are the assumptions that we are going to consider. Here we are taking cylindrical coordinate system to solve the problem r, θ,z and the component of velocities are in these three directions are v_r, v_θ, v_z respectively



Assumptions:

1.Incompressible fluid: $\rho \neq \rho(r, \theta, z, t)$

2.Steady state: All the time derivatives will be zero:$\frac{\partial}{\partial t}(\,) = 0$

3.Fully developed flow: $\left(\frac{\partial v_z}{\partial z}\right) = 0$:That means the flow in the z direction is fully developed. So, v_z does not change as a function of z.

4.system is θ symmetric:$\frac{\partial}{\partial \theta}(\,) = 0$:There is no variation in θ direction.

5.No slip condition is valid: At, r=R: v_z = 0 and v_θ=0 for all z and θ.

6.Solid impermeable wall: at, r=R; v_r =0 for all θ and z

So, how does the liquid flow through the pipe, the flow is a pressure driven flow. Next, we will consider the governing equation in cylindrical coordinate system.

Continuity equation in r- θ- z system

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Navier's equation

r- component equation:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial r} + \rho g_r - \left(\frac{1}{r}\frac{\partial (r\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z}\right)$$

θ- component equation

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \rho g_\theta - \left(\frac{1}{r^2}\frac{\partial (r^2 \tau_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z}\right)$$

So, thank you. We will continue to the next class.