Optimization in Chemical Engineering
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Lecture - 05
Introduction to Optimization (Contd.)

Welcome to lecture 5 this is the last lecture of week 1; so we will continue our discussion on review of linear algebra.

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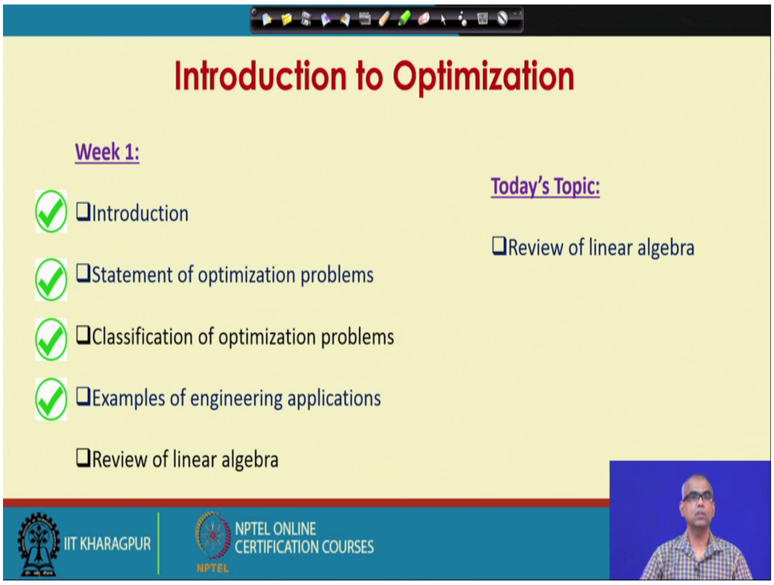

In our previous lecture we started our discussion on review of linear algebra and we talked about vectors; today we will talk about matrices.

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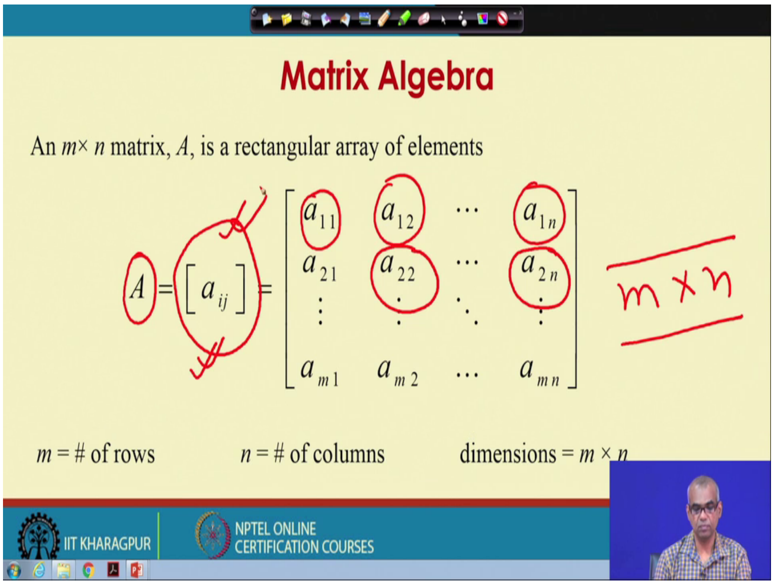

An m by n matrix A is a rectangular array of elements; so the matrix A can also be represented as shown; so if you have a rectangular array where you have m rows and n columns we say this rectangular array of m n elements as m by n matrix.

So, the dimension of the matrix is m by n. So it can also be represented as this where small a ij represents each element of the matrix; usual notation is i-th row j-th column. So a 2 2 a is the element in the second row second column, a 2 n is the element in second row n-th column so on and so forth. So normally we will express matrix by this capital letter or also by small letter with subscript ij within square bracket.

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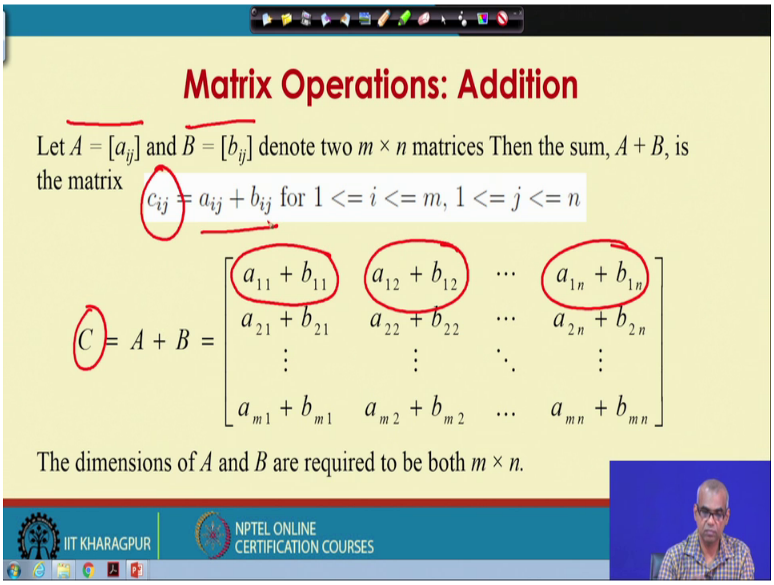

Now, we will talk about matrix operations first let us talk about matrix addition. Matrix addition is defined between two matrices when the dimensions of both are same. So let us consider A is a matrix of dimension m by n lets consider another matrix B with same dimension m by n. So then the sum A plus B will be the sum of the corresponding elements. So the first element of the matrix that is a sum of A and B will be the a 1 1 which is the first element of matrix a first row first column and the same first row first element of the matrix B. So A 1 1 plus B 1 1 will be the first row first column element of matrix C; similarly all other terms.

In general this can be expressed as if A is represented as a ij, B is represented as b ij both has same dimensions then the elements of C c ij will be a ij plus b ij for i equal to 1 to m and j equal to 1 to n; note that the dimensions of both A and B must be same.

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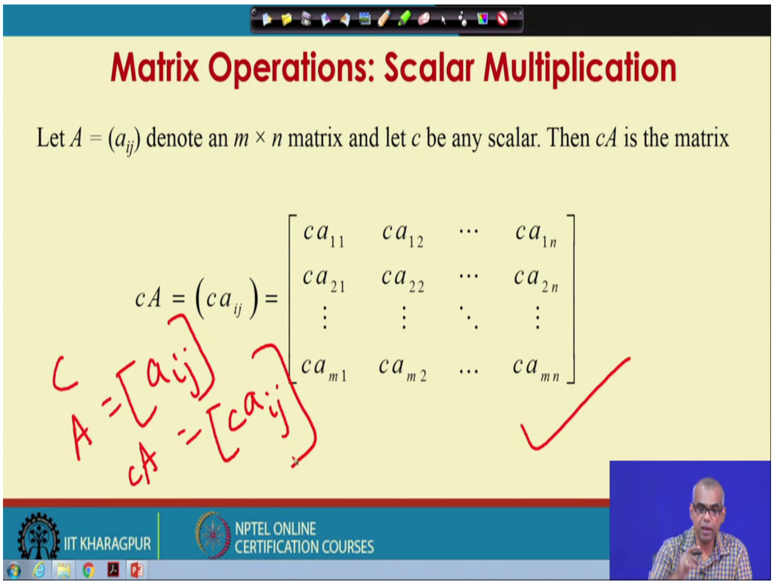

Scalar multiplication; if A equal to a ij denote m by n matrix and let C be any scalar then c A matrix will be a matrix with the elements of a, but multiplied each element multiplied by the scalar c as you can see in this matrix.

So, c is a scalar A is a ij so c a will be c a ij so each element of a will be multiplied by this scalar c.

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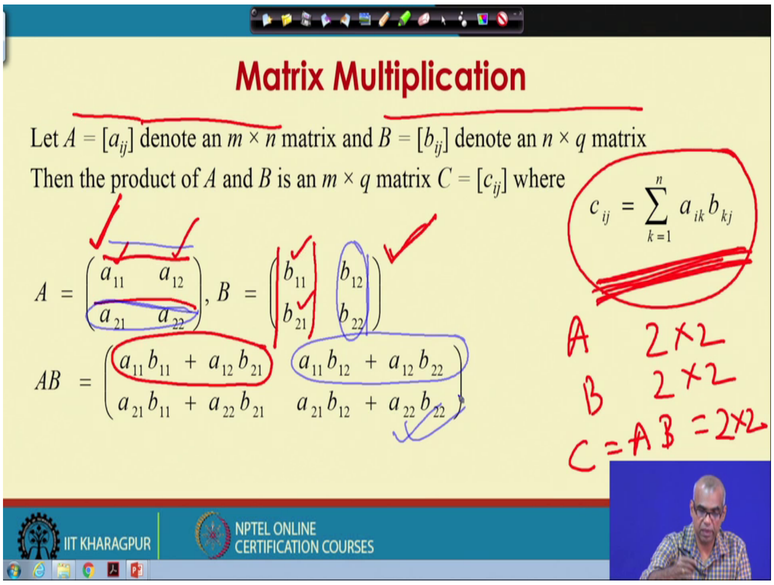

Matrix multiplication if the matrix A is m by n matrix and B is n by q matrix then the product between A and B is defined. So the product between 2 matrix will be defined when the column of first matrix is same as is equal to the number of column in the first matrix will be equal to number of rows in the second matrix. So m by n matrix and n by q matrix can be multiplied so the product of A and B will be m by q matrix and the element c ij of the product matrix c can be computed as sigma k equal to 1 to n a ik b kj.

So, let us take this example matrix A a 2 dimensional matrix a 1 1 a 1 2 a 2 1 a 2 2 similarly matrix b so a is 2 by 2 matrix b is also 2 by 2 matrix; so the multiplication is defined and the product matrix c will also be 2 by 2 matrix. So what will be the elements of the matrix c that will come from this formula c ij equal to a ik b kj where k runs from 1 to n note here i is the number of rows in the matrix A, j is the number of column in matrix B. So if the matrix A has these elements and matrix B has these elements so the product matrix A B can be found out by this multiplication.

So, what you do is you take a 1 multiply with b 1 plus you take a 1 2 you multiply with b 2 1 that becomes the first element of the product matrix; so a 1 1 b 1 1 plus a 1 2 b 2 1 so that becomes 1 1 element of the product matrix a b so what will be the 1 2 element; that means, the first row second element so that will be you take this first row and consider the second column of B; so a 1 1 plus b 1 2 plus a 1 2 into b 2 2 so that becomes a 1 2 element of the product matrix. Similarly a 2 1 will be obtained as this multiplied by this and a and the 2 2 element will be this multiplied by this so it will be this so this way you can perform matrix multiplication.

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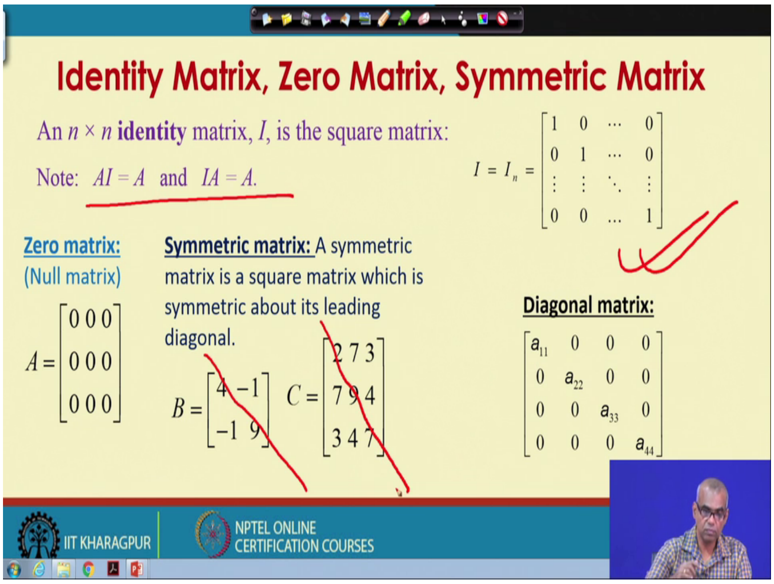

Let us now define certain common matrices such as identity matrix, zero matrix, symmetric matrix etcetera an n by n matrix an n by n identity matrix is a square matrix as shown where all the elements in the principle diagonal are unity. So the identity matrix n by n is a square matrix with n rows and n columns and all the elements in the principle diagonal are unity all other elements are 0. If we post multiply or pre multiply identity matrix with any matrix A the product becomes same A. So A multiplied by identity matrix I is A identity matrix I multiplied by any matrix A equal to A of course, the matrix multiplication must be defined; that means, the number of columns in the first matrix will be equal to number of rows in the second matrix; zero matrix also known as null matrix is a matrix with all elements equal to 0.

A symmetric matrix is a square matrix which is symmetric about its leading diagonal so if you consider this leading diagonal with elements 4 and 9 you see the matrix is symmetric around this diagonal the other half diagonal elements are both minus 1; similarly this matrix is also symmetric the diagonal matrix is the matrix where all the elements are 0 except the elements in the principle diagonal. Note that identity matrix is also a diagonal matrix, but there the elements in the price principle diagonal are all unity.

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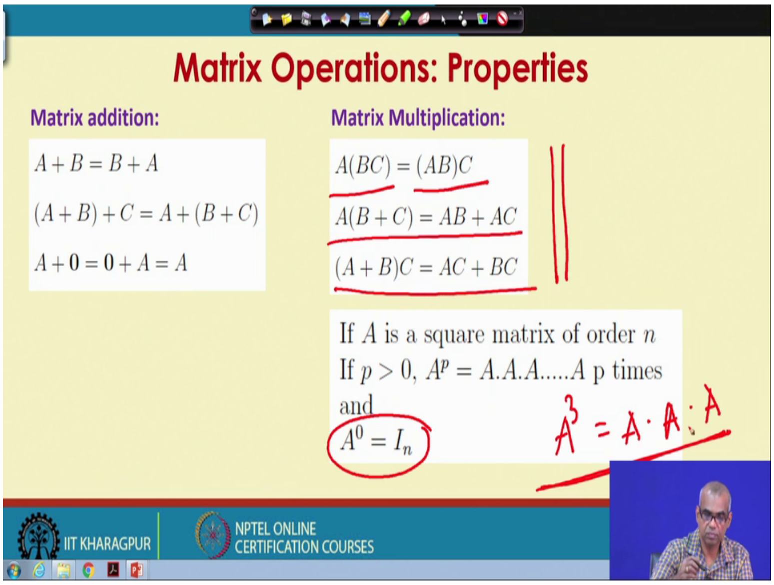

Here are some matrix and operations matrix operations properties, matrix addition properties A plus B equal to B plus A, A plus B plus C equal to A plus B plus C, A plus 0 equal to 0 plus A equal to A.

Matrix multiplication follows this properties A into B C is same as A B into C. Similarly this similarly this if A is a square matrix of order n and if p greater than 0 A to the power p is A times A times A up to p times. So A to the power 3 is A times A times A; so A to the power 0 is the identity matrix. So if A is a square matrix of order n A to the power 0 will be identity matrix of order n. So here the matrix is a square matrix.

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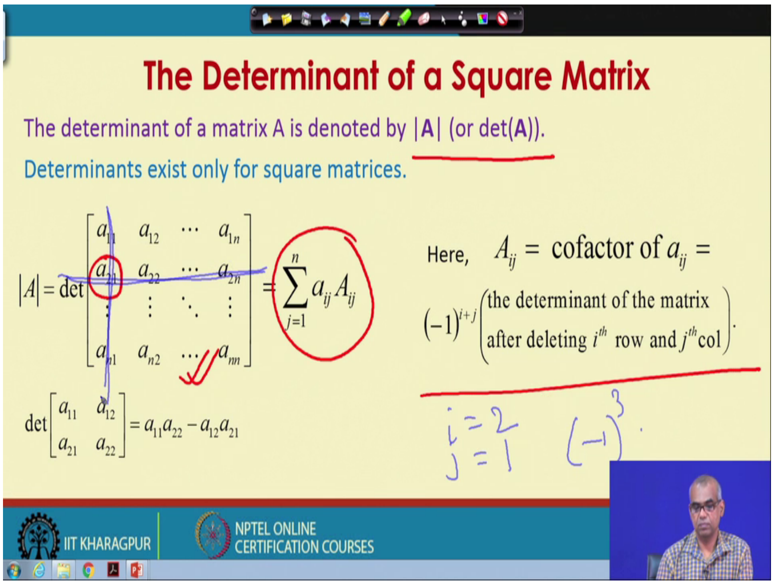

The determinant of a square matrix; the determinant of a matrix A is denoted like this it is also denoted as det of A, determinants exist only for square matrices for non square matrices determinants do not exist. So determinant of a matrix n by n which is shown here as follows can be obtained by this here small a ij represents the elements of the matrix a capital A ij represents the cofactor of the elements in the matrix. So the cofactor is defined as say cofactor a ij is defined as minus 1 to the power i plus j into the determinant of the matrix after deleting i-th row and j-th column.

So let us consider I want to find out the cofactor of the element a 2 1; so what I will do is I will delete the row and column containing elements a 2 1. Then the part of the matrix that is remaining you form a determinant and then the sign of that will be minus 1 to the power i plus j here i is second row to j is first column 1 so minus 1 to the power 3 into the determinant that exist after deleting second row and the first column. So that is how the determinant can be obtained for any square matrix of order n by n.

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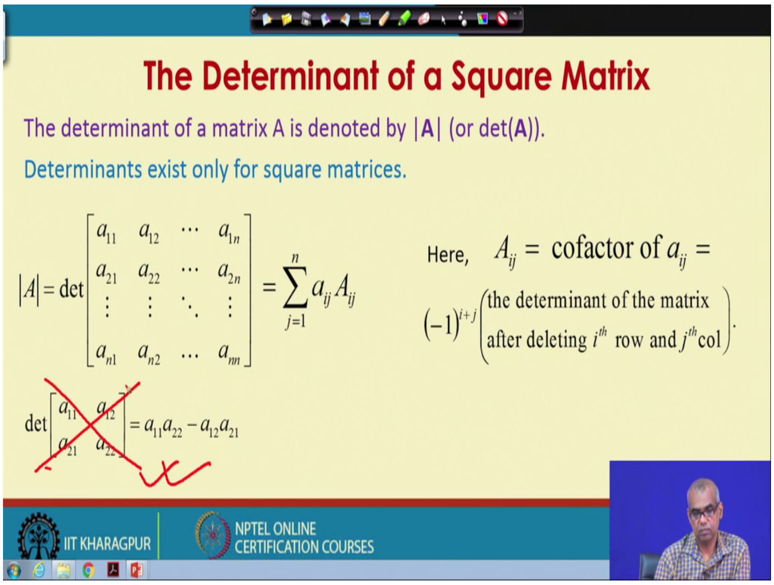

If you look at a 2 by 2 matrix so the determinant of a 2 by 2 matrix is a 1 1 into a 1 2 a 1 1 into a 2 2 minus a 1 2 into a 2 1. So basically you first multiply this and then minus multiply this two; so that is how you can obtain the determinant of a square matrix.

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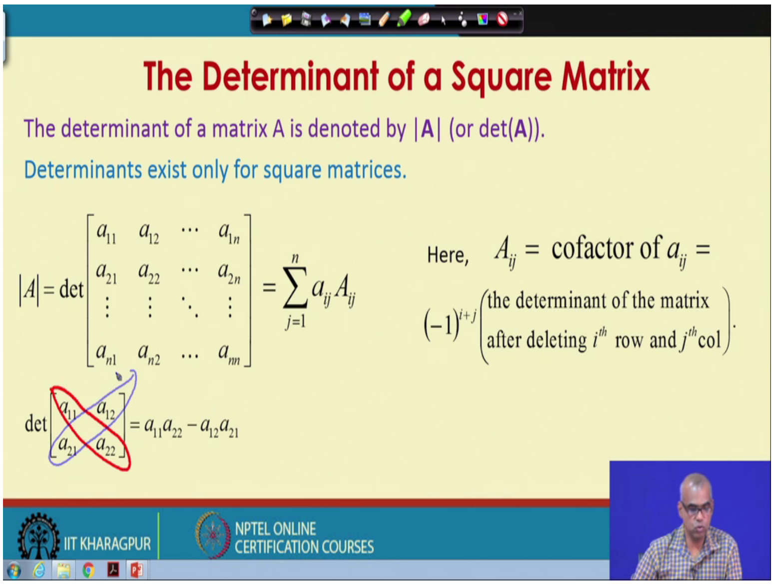

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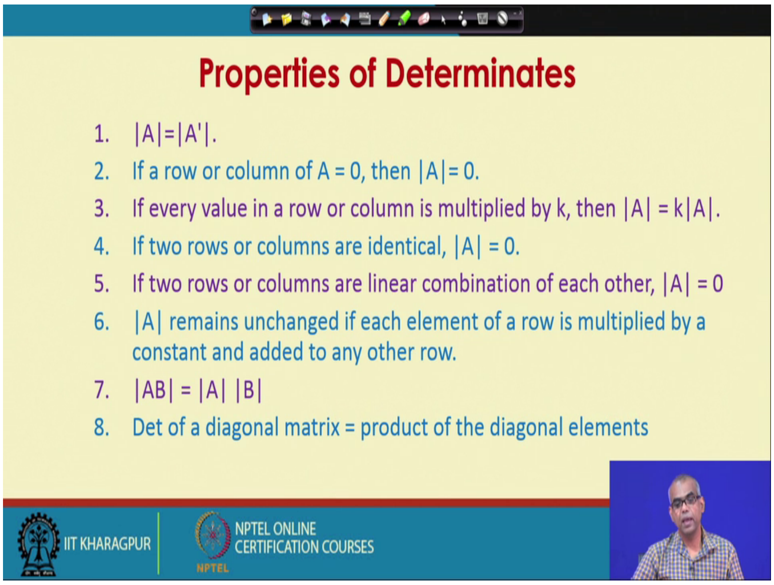

Here are some properties of determinants the determinant of matrix A and the determinant of transpose of A are same we will see later what you mean by transpose of a matrix. You basically by transpose of a matrix you change the rows into columns and columns into rows. If a row or column or matrix a equal to 0 then the determinant of a equal to 0 if every value in a row or column is multiplied by say scalar k then the determinant of A equal to the scalar k times the determinant of A.

If two rows or columns are identical the determinant of a equal to 0 if two rows or columns are linear combination of each other determinant of a equal to 0; determinant of a remains unchanged, if each element of a row is multiplied by a constant and added to any other row determinant of the product A B is same as the product of determinant of A and determinant of B; determinant of a diagonal matrix equal to product of the diagonal elements.

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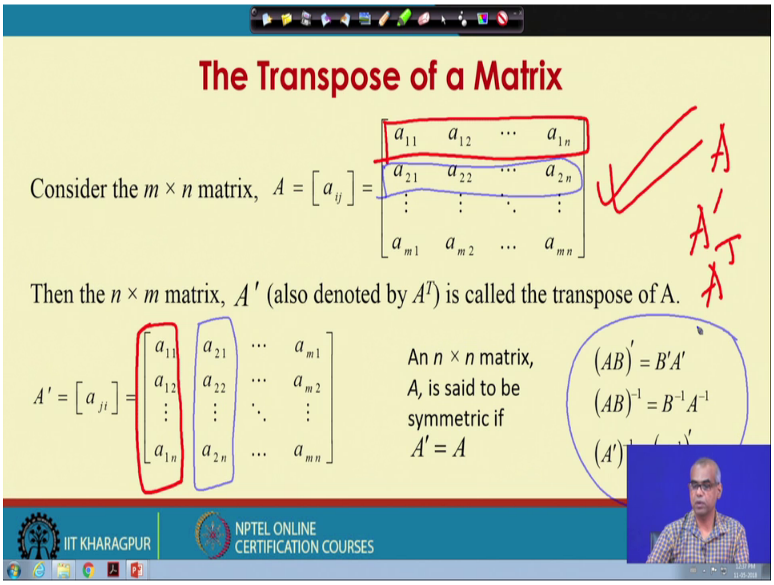

Consider the m by matrix consider m by n matrix as shown; so if I want to take the transpose of this matrix what I will do is the rows of this matrix will be converted to the column of the transpose matrix; if the matrix is represented as A the transpose of the matrix will represent as this or like this A prime or A to the power T or transpose of A.

So, the matrix A has this as first row in the transpose matrix look at here this becomes the first column; similarly you have this as the second row in matrix A; so in the transpose matrix this becomes the second column so on and so forth. So the rows of matrix A becomes columns in the transpose matrix an n by n matrix A is said to be symmetric if A transpose equal to A. So for a square matrix if the transpose of the matrix is same as the original matrix the matrix becomes symmetric or we call this matrix as symmetric.

Remember this properties transpose of product A B is equal to B transpose into A transpose A B inverse equal to B inverse into A inverse A transpose to the power inverse equal to A inverse to the power transpose.

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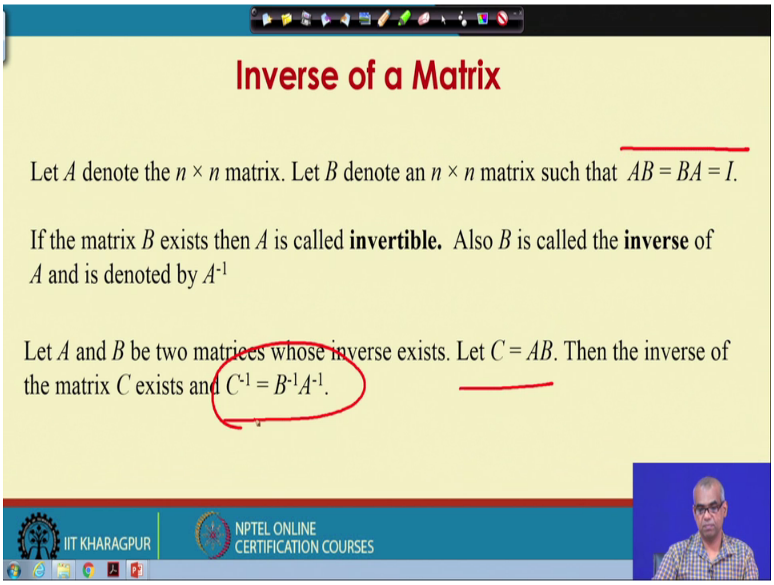

Inverse of a matrix; let A denote n by n matrix let us also consider another n by a matrix B such that the product A B equal to the product B A equal to the identity matrix I. So we have a matrix A which has dimension n by n; let us consider another matrix B with the same dimension n by n and if A B equal to B A equal to identity matrix I then if such matrix B exist we call matrix A is invertible and the matrix B is called as inverse of A and we represent the inverse of A as A to the power minus 1. Let A and B be two matrices whose inverse exists and let C equal to A B then the inverse of the matrix C exists and C inverse equal to B inverse into A inverse.

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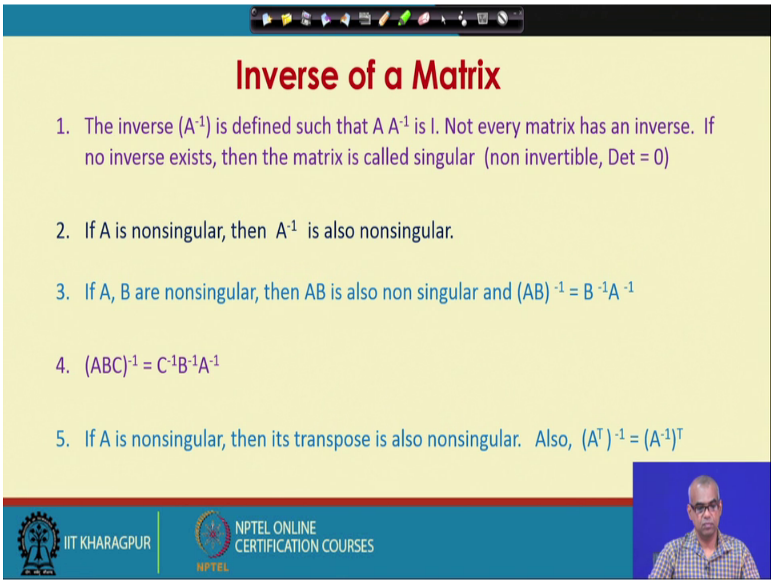

Here are some properties of inverse of a matrix the inverse A inverse is defined such that A into A inverse is equal to identity matrix I; not every matrix has an inverse if no inverse exists then the matrix is called singular matrix. It is non invertible the determinant of that matrix will be equal to 0. If A is non singular then A inverse is also non singular if A and B are non singular then A B is also non singular and A B inverse equal to B inverse into A inverse. A B C inverse equal to C inverse into B inverse into a inverse. Please note the reverse order if a is non singular then its transpose is also non singular also A transpose inverse equal to A inverse transpose.

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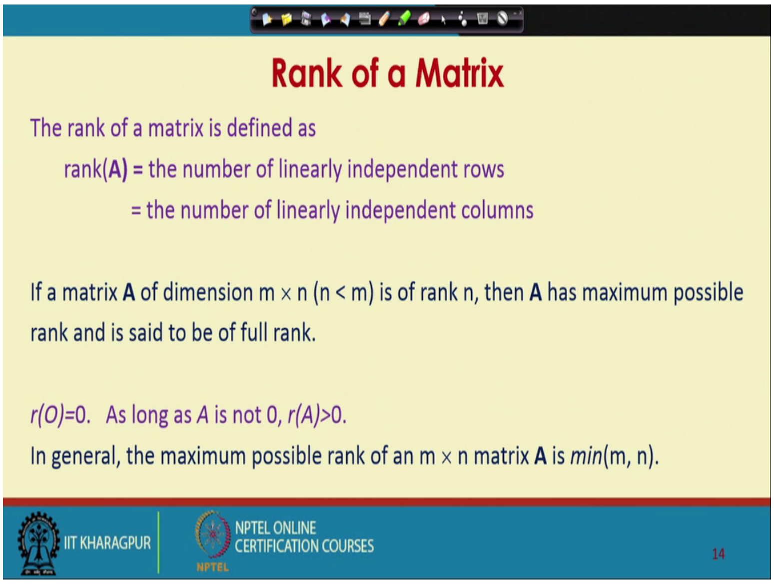

Rank of a matrix; the rank of a matrix is defined as the number of linearly independent rows; it can also be defined as number of linearly independent columns. So the rank of a matrix is equal to number of linearly independent rows or number of linearly independent columns. If a matrix A of dimension m by n where n is less than m is of rank n then A has maximum possible rank and is said to be of full rank; rank of 0 matrix equal to 0 as long as A is not a 0 matrix the rank of A will be greater than 0 in general the maximum possible rank of m by n matrix is minimum of m and n. So if you have m rows and n columns the minimum between the two is usually the maximum possible rank of the matrix.

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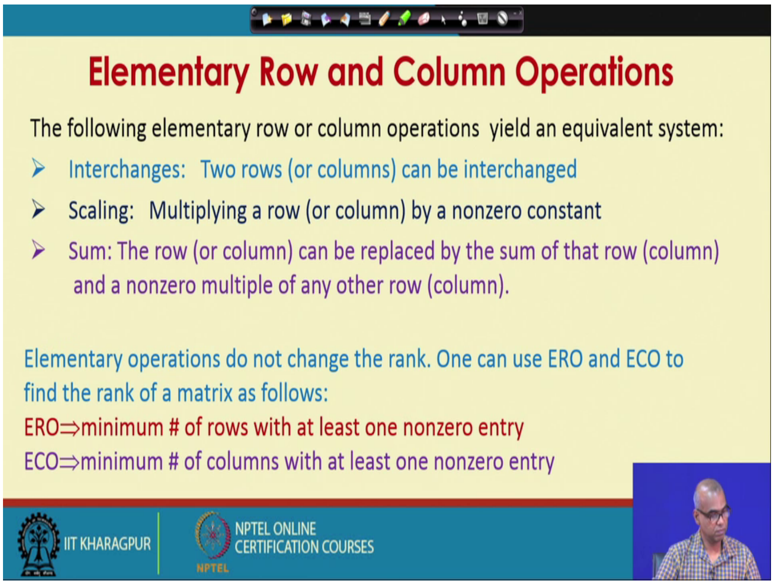

There are certain operations which we call elementary row and elementary column operations; when you perform the elementary row or elementary column operations you get an equivalent system there are 3 elementary row operations or elementary column operations which you can perform on a matrix and that will and equivalent matrix. They are interchanges scaling and sum first 2 rows can be interchanged, 2 columns can also be interchanged. So if 2 rows or 2 columns are interchanged you get equivalent matrix nothing changes multiplying a row by a non zero constant you get equivalent system you also get equivalent system by multiplying a column by non zero constant third the row can be replaced by the sum of that row and a nonzero multiple of any other row.

We can also perform column operations so the column can be replaced by the sum of that column and a non zero multiple of any other column. So these 3 operations are known as elementary operations and by performing elementary operations you will get an equivalent matrix or equivalent system. Elementary row operations do not change the rank; one can use elementary row operation or elementary column operation to find the rank of a matrix as follows when you do elementary row operations minimum number of rows with at least one nonzero entry is the rank of that matrix. Similarly when you perform elementary column operation minimum number of columns with at least one nonzero entry is the rank of the matrix.

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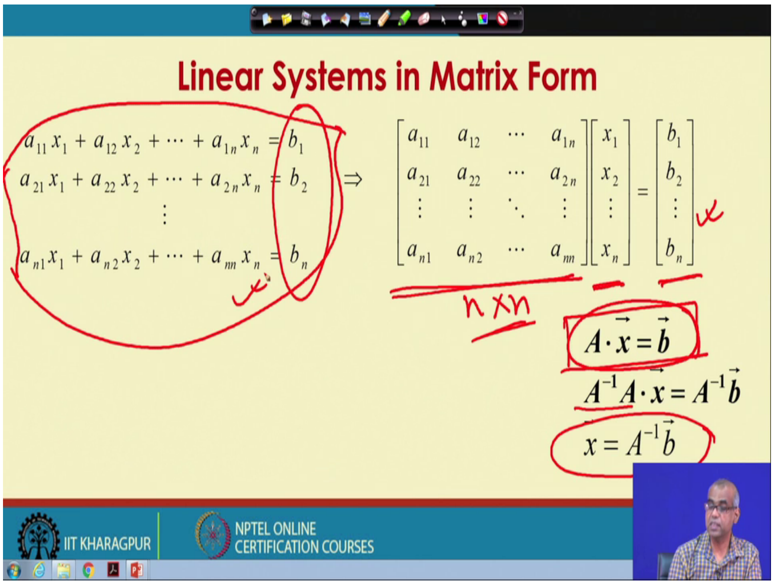

Let us now talk about linear systems in matrix form; so basically how do I express a linear system of equations in matrix form; so you have n equations in n variables they are all linear equations and they are written as shown a 1 1 x 1 plus a 1 2 x 2 up to a 1 n x n equal to b 1 and so on and so forth. So this can be retained by this coefficient matrix which is an n by n matrix this column vector which is x 1 x 2 up to x n and this right hand side vector which is these vectors on the right hand side.

So, using matrix notation you can very neatly write it as A into x equal to B; where A is the coefficient matrix which is order n by n you have n equations n variables x is a n column vector and B is this right hand side vector again n vector. Now this n linear equations in n variables can be compactly written in matrix notation as x equal to B; so these equation allows us to solve for x also pre multiply this equation by A so a inverse pre multiply by A inverse; so A inverse A x equal to A inverse b A inverse A is identity matrix so we can get x equal to A inverse B; so you can solve for this linear system of equation as x equal to A inverse B.

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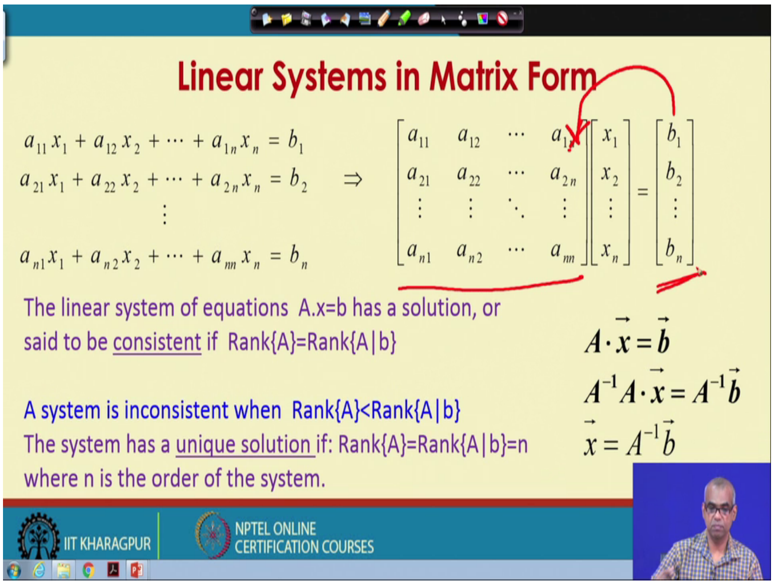

The linear system of equations x equal to B has a solution or said to be consistent if rank of A is equal to rank of augmented matrix A B; that means, you have this matrix as a and then put this, this column vector a here so another column in the matrix a comes from this vector b so that is known as augmented matrix.

So, the linear system of equations x equal to B has a solution or said to be consistent if rank of A is equal to rank of the augmented matrix A B; a system is inconsistent when rank of matrix A is less than rank of augmented matrix A B. The system has a unique solution if rank of A is equal to rank of augmented matrix A B equal to n where n is the order of the system.

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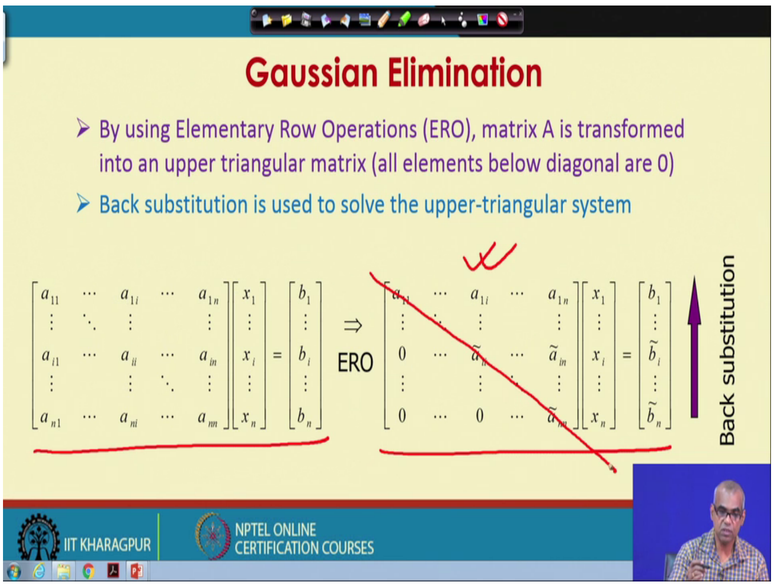

Next we will talk about Gaussian elimination; which is useful for solving set of linear equations by using elementary row operations a matrix A is first transformed into an upper triangular matrix; upper triangular matrix means all elements below diagonal will be equal to 0 then you can easily perform back substitution to solve the upper triangular matrix.

So, Gaussian elements, Gaussian elimination is an efficient way of solving set of linear equations the first step is to perform elementary row operate matrix operations and convert the matrix A to an upper triangular matrix so we get an upper triangular matrix where all the elements below the diagonal are 0 then perform back substitution to solve the upper triangular system. So this is shown in this figure so you have a x equal to b as the system which is converted as x equal to b again, but since I have 1 the elementary row operations to convert the matrix a into an upper triangular matrix see that below this principle diagonal all elements are 0. And then it becomes easy to easy for back substitution because you get x n is equal to b n by a n n.

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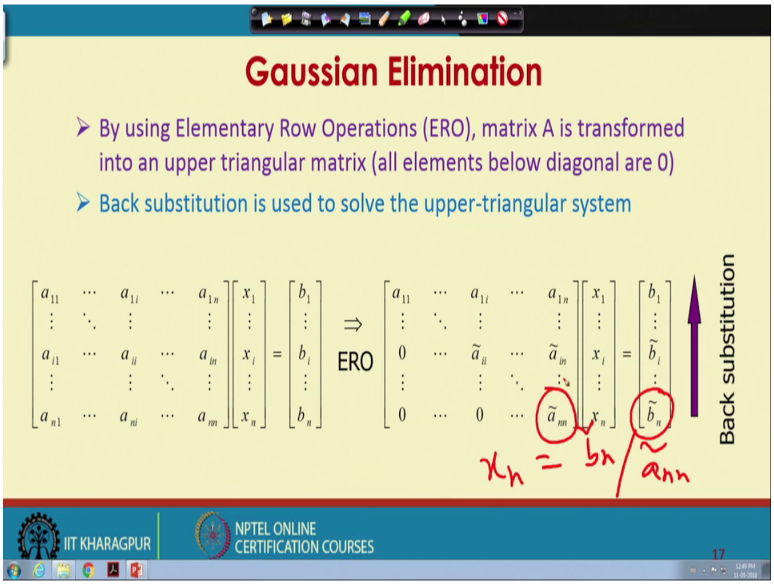

Once you get this you put this value in the preceding rows or preceding equations you get the value for that variable and so on and so forth.

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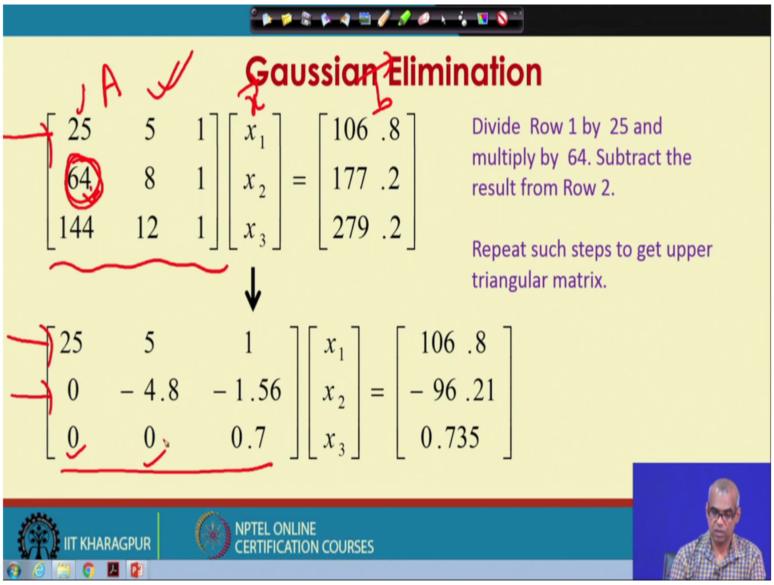

Here is a simple example of Gaussian elimination you have this matrix A coefficient matrix and x 1 x 2 x 3 so 3 by 3 matrix coefficient matrix a x equal to b this is the b vector this is the x vector and this is the matrix A; so you perform Gaussian elimination to convert this A matrix into an upper triangular matrix so we have done it here see all the elements below the principle diagonal are 0. So this remains unchanged the first row let us look at the second row 2 1 element must be 0. So this has to be converted to 0; so what you do you perform elementary row operations you divide row 1 by 25; so this becomes 1 multiply that by 64 so it becomes 64 and subtract the result from row 2 so these becomes 0.

So, that way you convert this element this element is also 0 by performing row operations so finally you get this.

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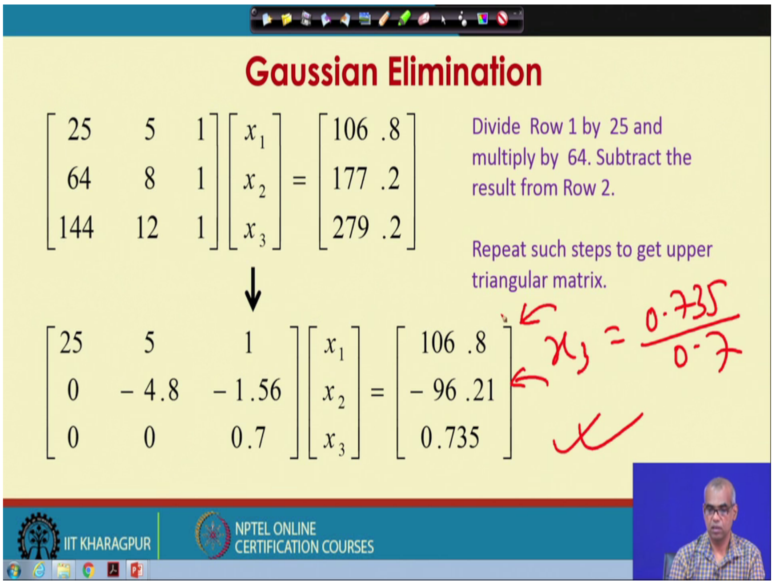

So, from here you see x 3 equal to 0.735 divided by 0.7; so once you get x 3 you put it in this equation you will get x 2 you put here in this equation you will get x 1 so this way you can solve a system of linear equations by Gaussian elimination.

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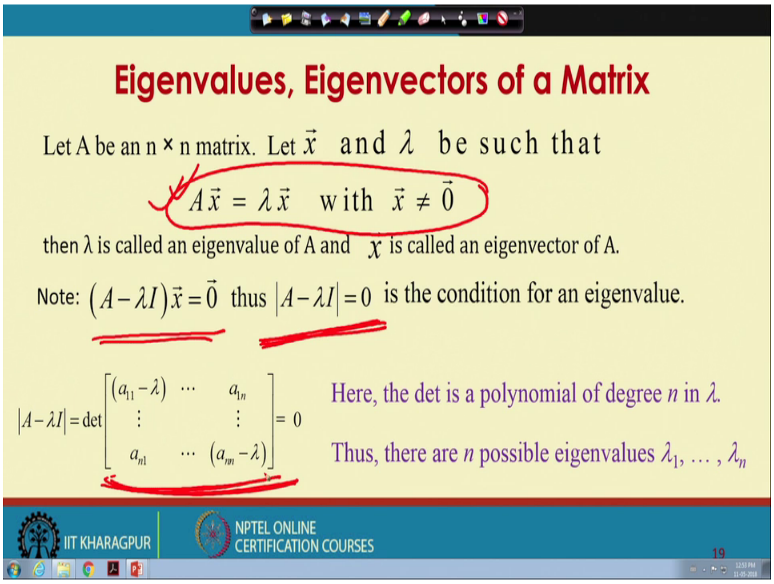

Finally let us briefly talk about Eigen values, Eigen vectors of a matrix. Let A be an n by n matrix let x be a vector and lambda be a scalar such that A x equal to lambda x so I have a vector A I have a vector I have a vector x and matrix A and a scalar lambda such that A x equal to lambda x with vector x is not equal to 0. Then lambda is called an Eigen value of matrix A and x is called an Eigen vector of A.

Look at here A minus lambda I; I is identity matrix into x equal to 0 so the determinant A minus lambda I equal to 0 is the condition for an Eigen value; so to find the Eigen values we have to solve A minus lambda I equal to 0 you have to look at this determinant you have to solve this equation. So A minus lambda I, A minus lambda I equal to 0 is basically this determinant equal to 0. So this determinant is a polynomial of degree n in lambda; so if you solve this there are n possible Eigen values lambda 1 lambda 2 up to lambda n. Why; because the determinant A minus lambda I or this one will become a polynomial of degree n in lambda so a polynomial of degree n in lambda will have n roots so lambda 1 lambda 2 up to lambda n are the n possible Eigen values of the matrix A with dimension n by n.

So with this I stop our discussion on review of linear algebra as well as the discussion on week 1.