Transport Phenomena
ProfessorSunandoDasgupta
Department of Chemical Engineering
Indian Institute of Technology Kharagpur
Lecture 7
Example of Shell Momentum Balance (Contd.)

So we will continue with our examples of the use of shell momentum balance and how it can be effectively used for solutions of simple problems. Simple problems I mean in which the geometry straight forward, the flow is mostly one dimensional and it's a steady flow. So for all these cases we have typically solve two problems. I would like you to work with me on one more problem then I will give you some problems to work on your own and I will provide you with the answers.

But towards the end of thispart, you yourself will start to realize that a simple shell balance approach would not suffice anymore. The geometry is getting complicated, the flowmay have multidimensional effects. So a generalized approach is necessary. But to get to that point we will first try to solve the problem which is very simple. In this case there is a two parallel plates and one of them is moving, the other is stationary.

So as the top plate let's say the top plate starts to move, it will try to drag the liquid which in intervening space. So as the top plate moves over the liquid, it will try to drag the liquid in between the two plates. So the flow start to establish and we will look at that flow when a steady state has reached. So the top plate is moving at some velocity, it imports the velocity in the liquid in here and we're looking at the steady state.

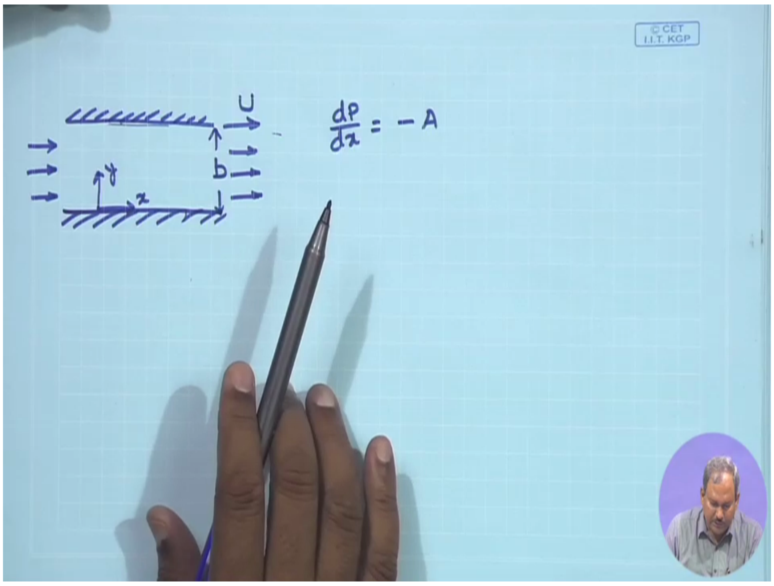
But just to make itmore interesting, what we are going to do is, we are going to assume that there exists a pressure gradient as well. So the motion in the intervening space between the two plates, one moving, the other stationary, is caused by two factors. One is the moment of the top plate and second is the pressure gradient that exits in between two points. Now if the pressure decreases in the direction of the top plate motion, if it decreases, then the fluid is going to have an additional flow due to pressure gradient from, let's say, left to right.

So the top plate is moving from left to right. The pressure gradient is in such a way that it would try to push the liquidfrom left to right. When that happenswe will say that it is favorable pressure gradient. So in order for a (favoral) favorable pressure gradient, the pressure has to decrease as we move along in the actual direction. If it is opposite that means if I move in this direction and the pressure increases, we will call it as an (unfavor) unfavorable pressure gradient.

But the problem we are going to deal with is, the flow between the two parallel plates. One plate is in motion and there is a favorable pressure gradient. So this is a problem we would like to solve using shell momentum balance and thenthere are several interesting of chutes of this problem which we will see as we move along. But right now let's draw the picture of the two plates, one stationary, one is motion and see what happens to the liquid in between.

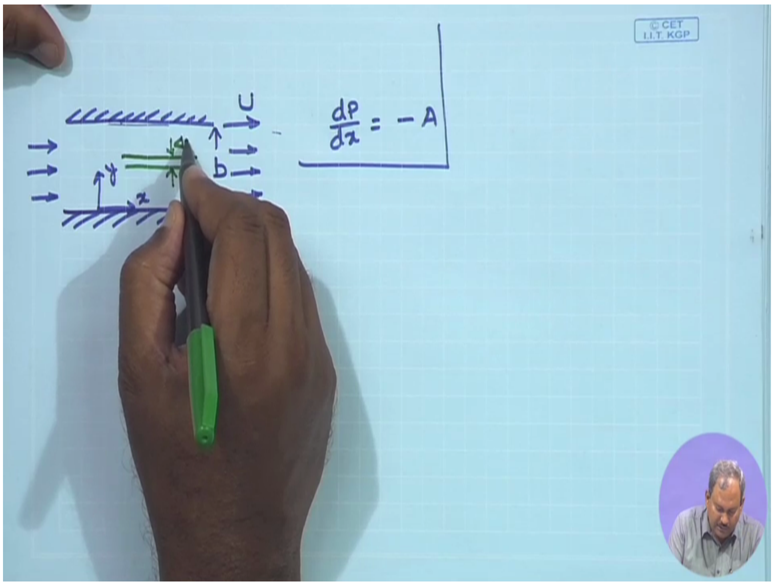
So this is my stationary plate and that's the top plate which moves towards the right with a constant velocity equal to U and let us assume that the gap in between these two plates is equal to B. In this direction its X and here it's Y and the liquid flows in this direction. So the upper plate and there is a pressure gradient. I call it as dPdX which is equal to minus A. Where A is simply a constant. So as we move in this direction, the pressure decrease. So the top plate moves, it drags the fluid and also I have a pressure gradient that forces the fluid to move from left to right.

(Refer Slide Time 04:56)



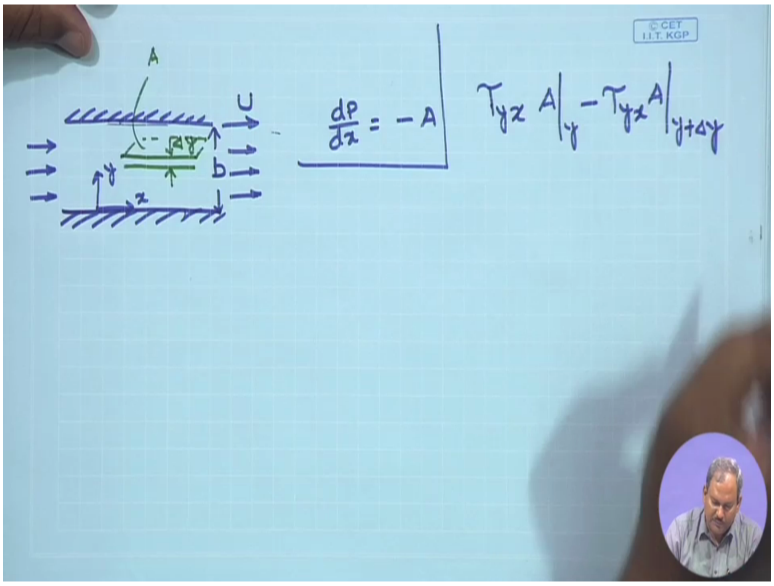
Now here I again have to find a shell, draw a cell and make the balances. The flow is in this direction but as I can see the flow changes in the Y direction. The principal direction of the motion is X, but its velocity is changing with Y. So of course my shell is going to have a thickness of delta Y in here.

(Refer Slide Time 05:27)



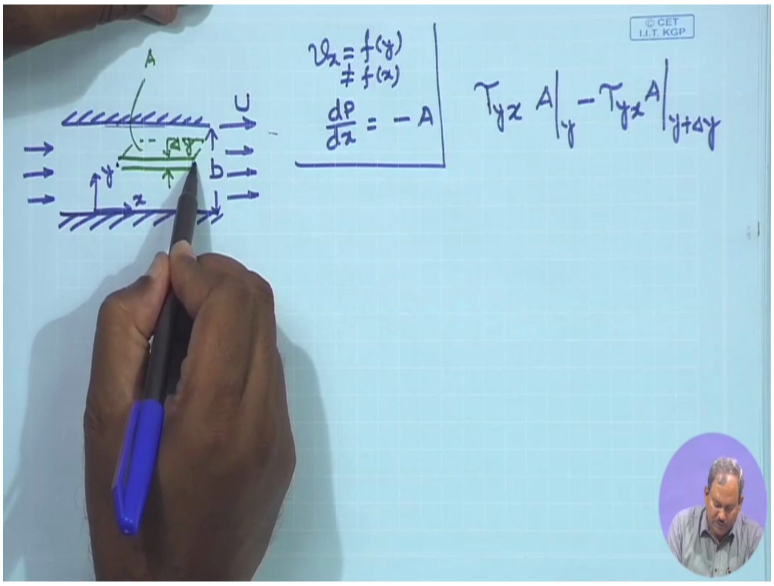
So the area of this, let's say the area of this is A. So the shear stress tau is going to have two subscripts. Principal direction of motion and the direction in which the momentum gets transported. So it's going to be tau YX. X being the direction of motion and Y is the direction in which the momentum gets transported due to viscosity. And this acts on the area A. So this is a time rate of momentum in, through the bottom face which is located at Y and the time rate of momentum out would again be the same expression but evaluated at Y plus delta Y.

(Refer Slide Time 06:20)



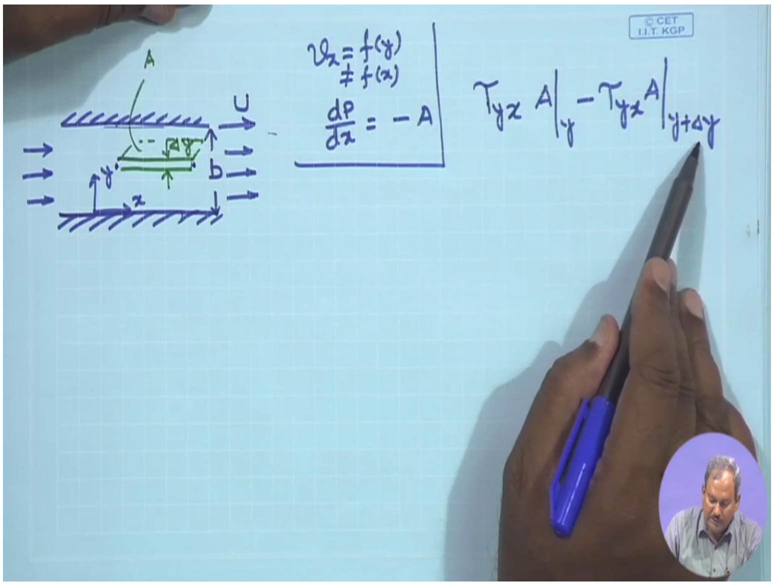
So these are the rates of momentum in and out. Obviously the convective momentum in and the convective momentum out, need not be written here, since VX is the function of Y. VX is not a function of X. So whatever be the value of velocity, at this point would be equal to the velocity of this point.

(Refer Slide Time 06:44)



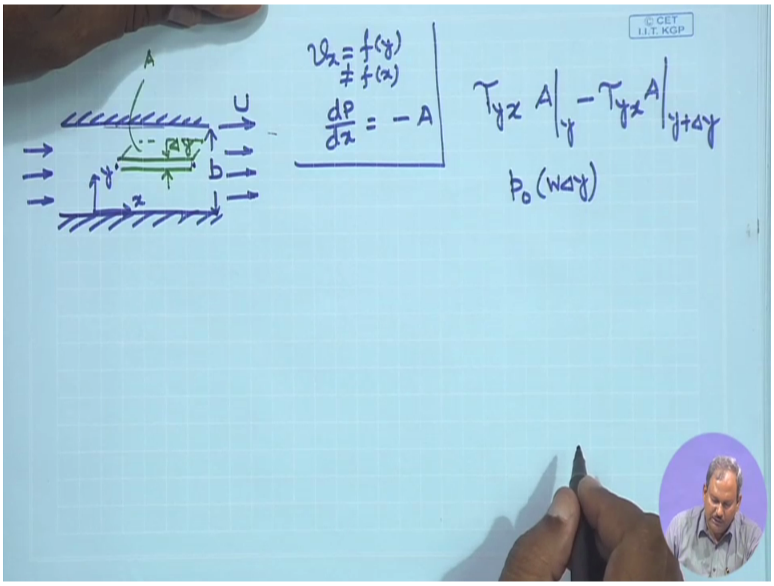
And since it is an incompressible fluid, there would not be any accumulation of mass inside the control volume and the rate of convective momentum in, would be equal and opposite to the rate of convective momentum out. So those two terms will cancel and that's why I did not write them in here. So this is a molecular transport of momentum, viscous transport of momentum which I have return. One is at Y and other is at Y plus delta y.

(Refer Slide Time 07:12)



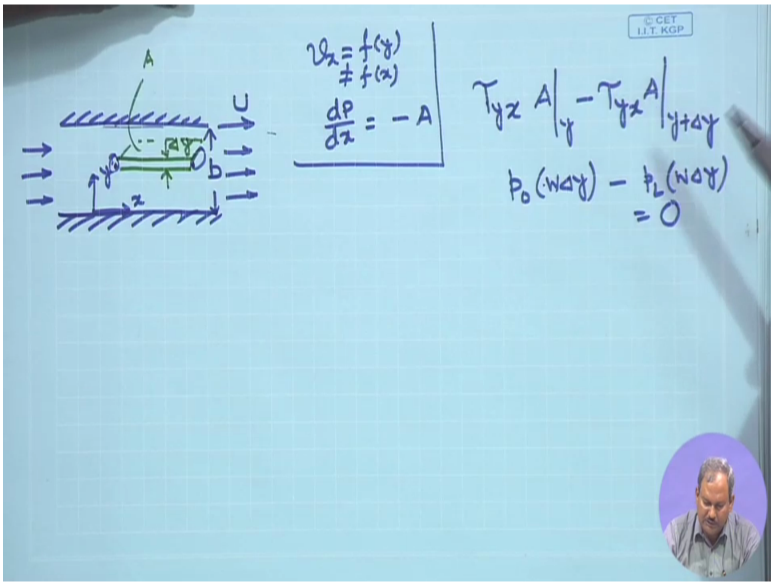
Again since the system is horizontal, there is no question of having any component of gravity in the direction of flow. So and the gravity is acting in a direction perpendicular to that of the flow. So there would not be any gravity force acting over here. On the other hand the pressure is going to play a role over here. So the pressure at, let's say, at this point is p0 which is acting on an area width W and thickness delta y.

(Refer Slide Time 07:45)



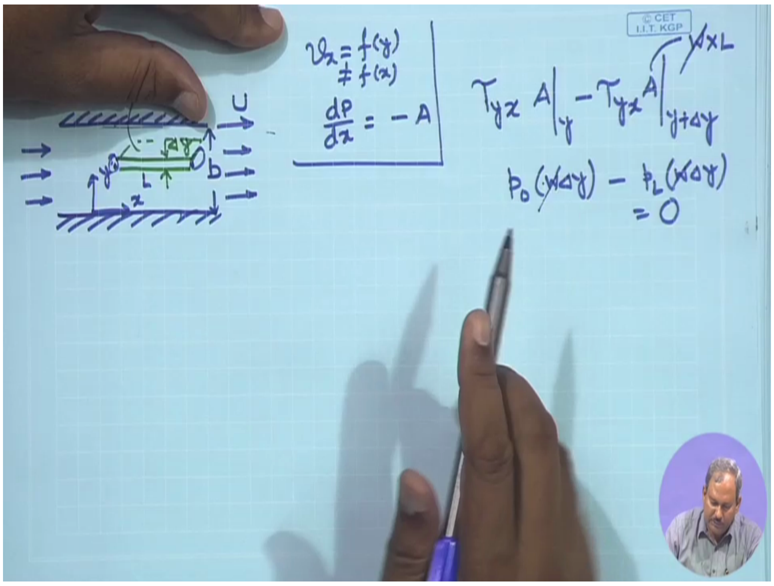
So that's the pressure which is acting on at over this surface and the pressure which is acting over this surface is minus pL times W delta y and at steady state the sum of all this must be equal to zero. So the rate of molecular momentum in, viscous momentum in, minus viscous momentum out, no conviction term, no gravity term, no body force term, the pressure term, the force due to pressure at steady state, the sum of this is going to be equal to zero.

(Refer Slide Time 08:17)



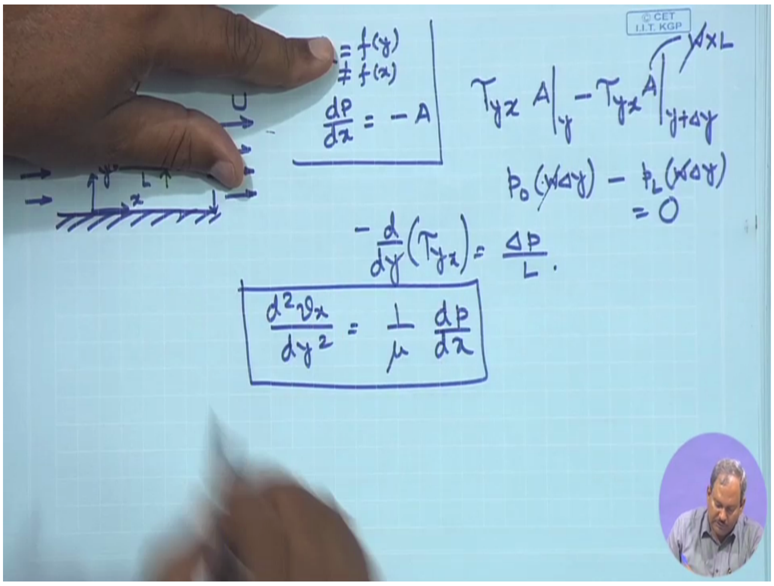
So again we divide both sides by delta y, taking the limit when delta y approach is zero. So what I would get out of these two terms is DdY of tau YX with the minus sign and essentially this area is nothing but W times L, where L is the length of the control volume. So W cancels from both side. The L is remaining which is being brought on the denominator.

(Refer Slide Time 08:52)



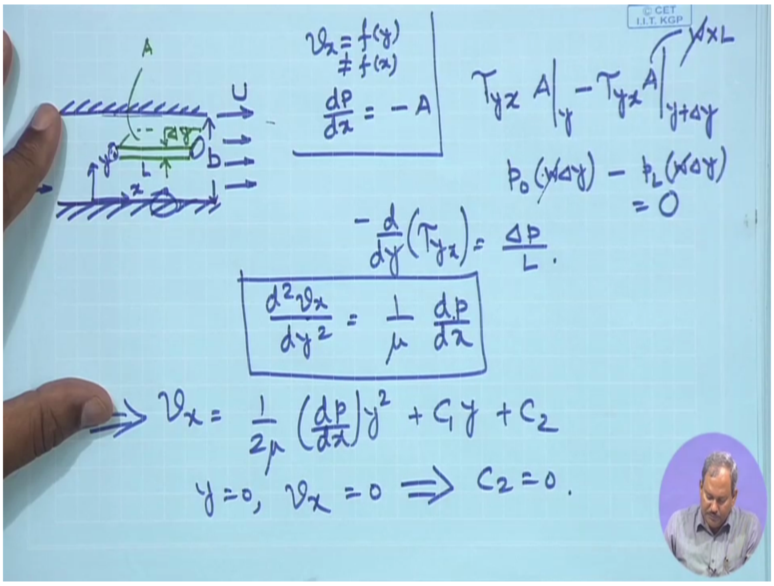
So once you do that and once you substitute tau YX, so essentially you would get is minus DdY of tau YX is delta Pby L and when you substitute and do the Newtonian fluid what youget is, D2 vX by dY square is 1 by mu dPdX. That's the governing equation for this case.Pressure gradient, viscous transport of momentum, no convection, no gravity.

(Refer Slide Time 09:24)



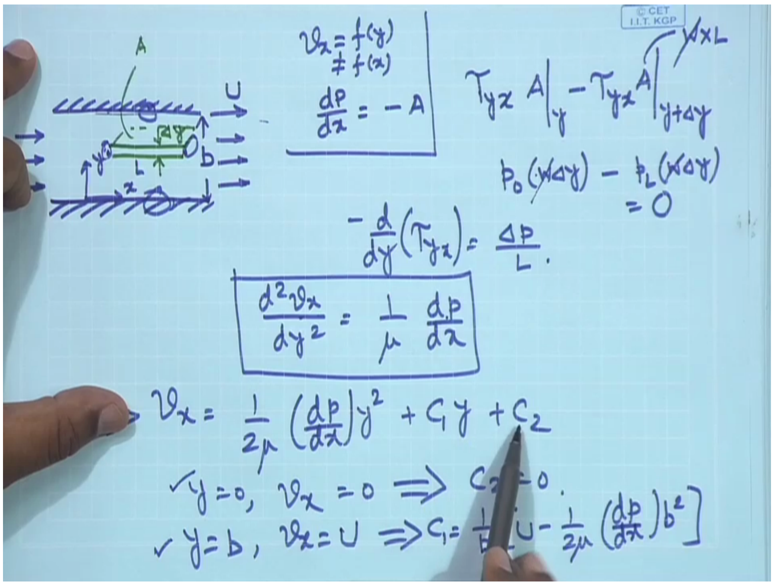
So once you integrate it,it would simply be equals to 1 by 2 mu. The pressure gradient and then I have the Y square plusC1 Y plus C2. So that's a form of the velocity distribution and I need two boundary conditions to elevate them. One is obviously at Y equals to 0, no slip condition. That is Yequals 0 is over here. VX would be equal to 0. And if you use that, it will simply tell you that C2 is equal to 0.

(Refer Slide Time 10:03)



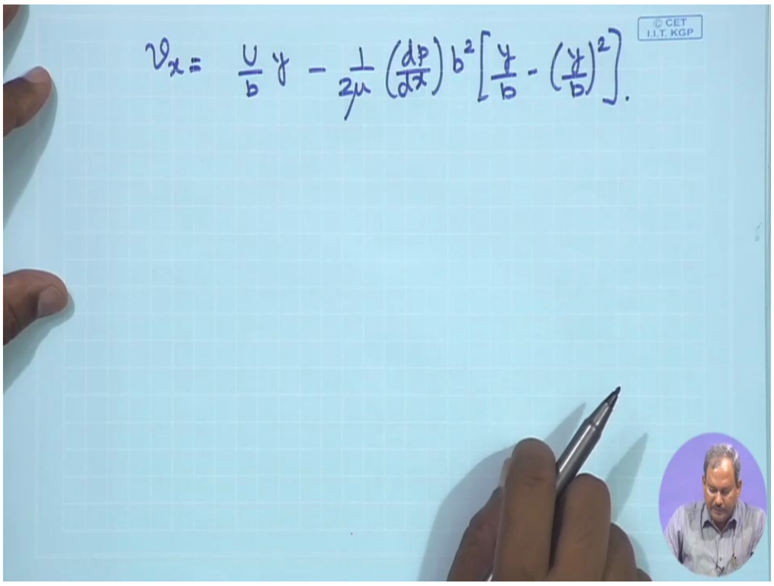
The other condition is Y equals to B, Y equal B is over here and Y equals to B, VX must be equal to the velocity of the top plate and this would give you the expression for C1 as 1 by B, U minus 1 by 2 mu dPdX times B square. So I am using no slip at the bottom and no slip at the top. No relative velocity, no related velocity at the bottom and at the top and evaluated the constant C1 and C2.

(Refer Slide Time 10:40)



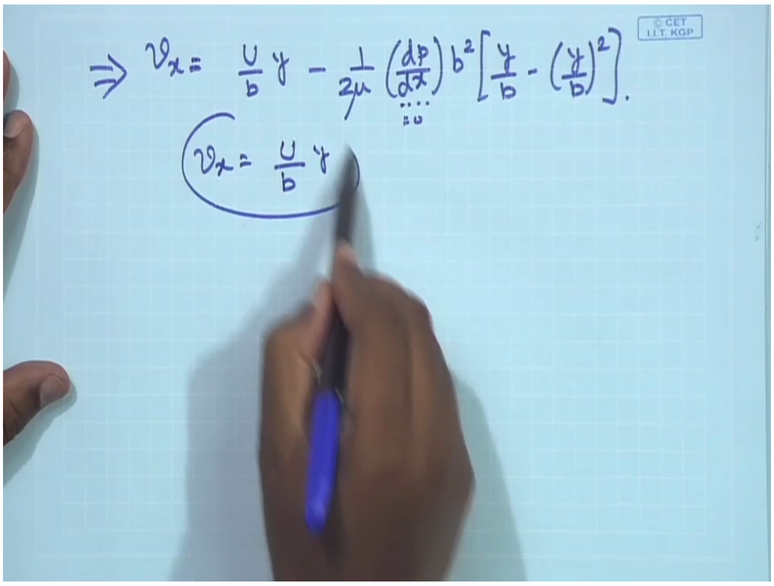
So if we substitute this in here, the final expression for velocity for a flow induced by the motion of the top plate, also sustained by the presence of pressure gradient, would simply be equal to this. The expression would be, you should try to do it on your own. That's a very straightforward case and this is what you would get.

(Refer Slide Time 11:17)



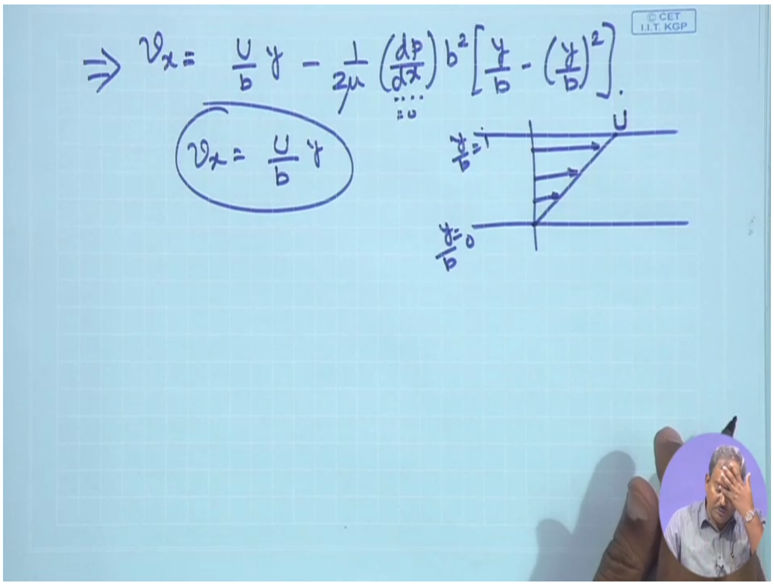
Now this combined expression is very interesting. Because it can be used toget some ideas about what kind of flow do you expect if you just have the motion of the top plate and do not have pressure gradient. If you look at the expression here, if you look at this expression, if I say there is no imposed pressure gradient, this is going to be equal to zero and if that's equal to zero vX would simply be U by B into Y.

(Refer Slide Time 11:50)



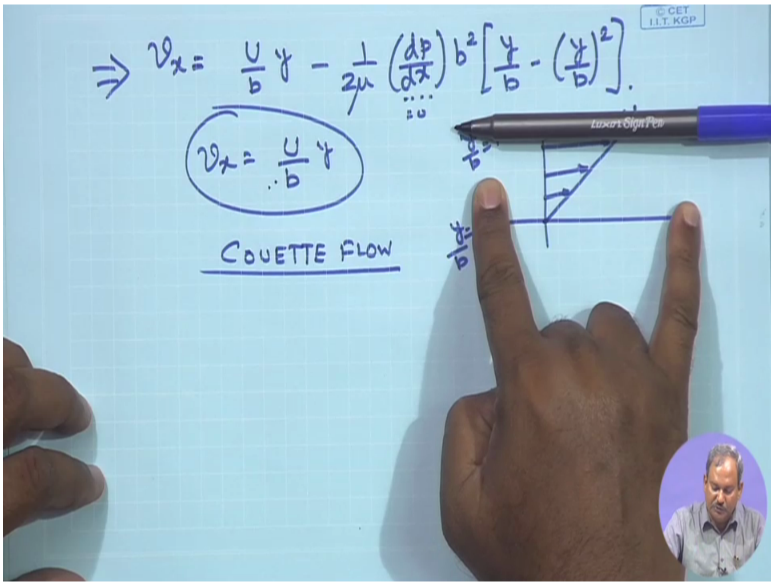
And the profile of the velocity,in the profile of the velocity in this case would simply be equal to, if this is Y equals Y by B is equal to zero and this point is Y by Bis equal to 1.The velocity simply be like this. It's a linear velocity starting at zero and the velocity is going to be equal to 1.

(Refer Slide Time 12:15)



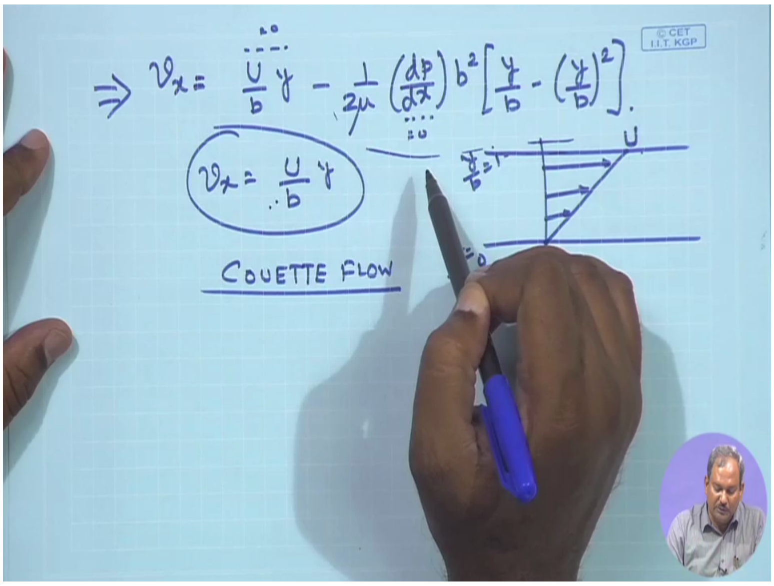
So this kind of a flow which is sustained because of the motion of one of the, let's say, the top plate, no pressure gradient, no gravity, where the velocity profile is going to be linear is known as the couette flow. The couette flow is quite common in several industrial applications where the motion of 1 plate sustains or creates a condition in which the flow takes place. And if you look at the second part of it, let's say, the top plate is stationary. The top plate is stationary, the bottom plate is also stationary, but there is a pressure gradient which is imposed in between the two.

(Refer Slide Time 13:02)



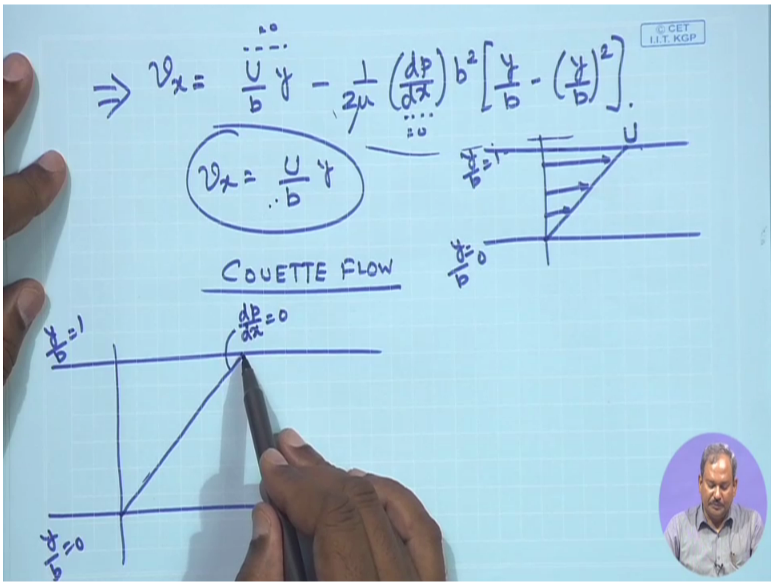
If that is the case then this part is going to be equal to zero, this is going to be non-zeroand this would be the profile when the flow is due to pressure gradient down here.

(Refer Slide Time 13:16)



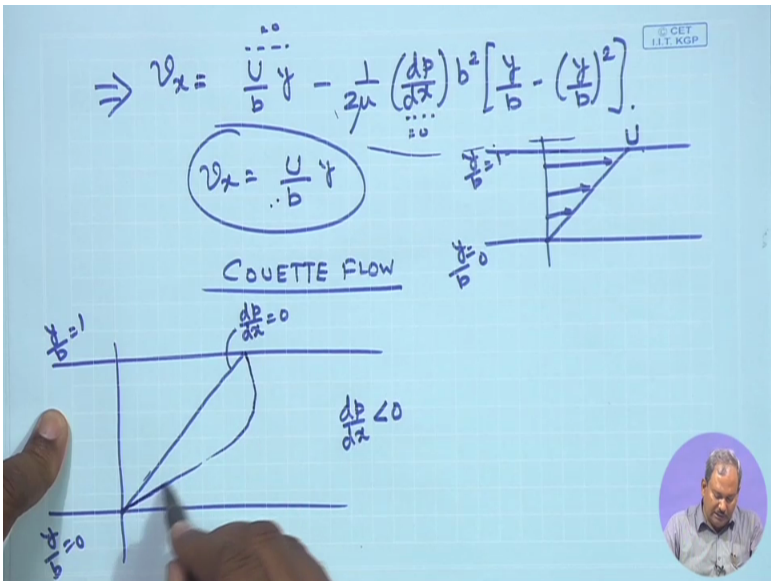
Now what would itlook like. I will go a slightly more deeper into this. Then it would clarify many of the concepts.So this is Y by B equal to zero and this is Y by B is equals to 1 and I am drawing the profile of the velocity when dPdX is zero. That means when there is no applied pressure gradient, I simply get a linear distribution.

(Refer Slide Time 13:50)



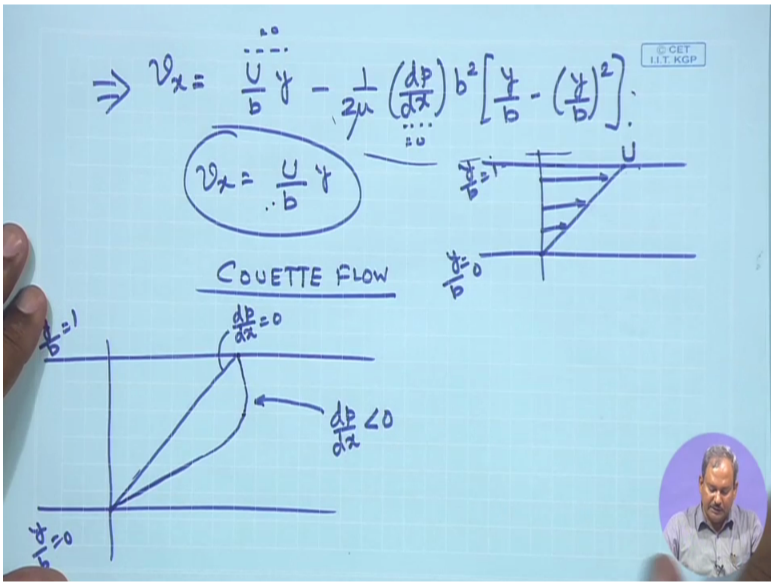
Now if I apply a pressure gradient in this case. Let's assume that the pressure on this side is more than the pressure on this side, what we call as a favorable pressure gradient. So as we move in the X direction, the pressure gradient progressively decreases or in other words dPdXis less than zero. So if dPdX is less than zero that is called a favorable pressure gradient and you would get the pressure. The condition is probably going to be something like this.

(Refer Slide Time 14:21)



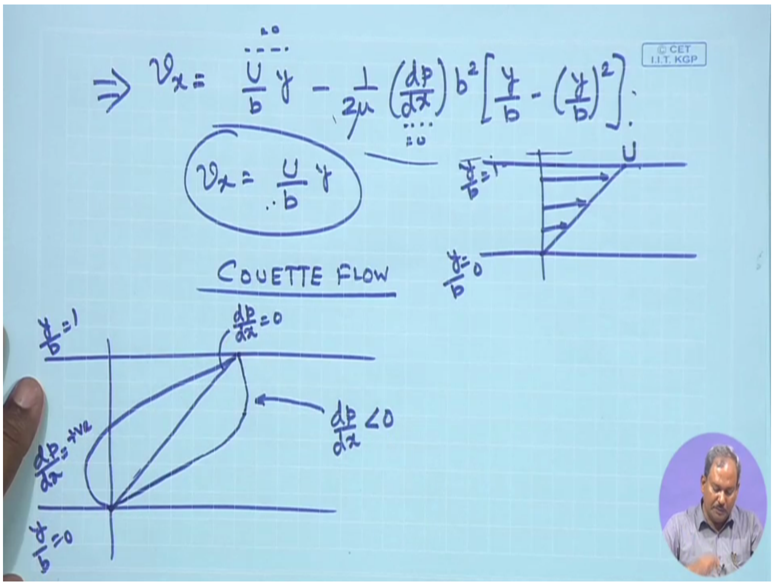
So (ha), this case is when we have a favorable pressure gradient and top plate is moving with some velocity. So this situation is unique because there is a departure from the straight line behavior of the velocity and it tries to closely resemble second part of it. So it's the (alg) it's the superimposition of the couette flow over the pressure driven flow, when the pressure gradientis going to be negative. That means favorable pressure gradient.

(Refer Slide Time 14:59)



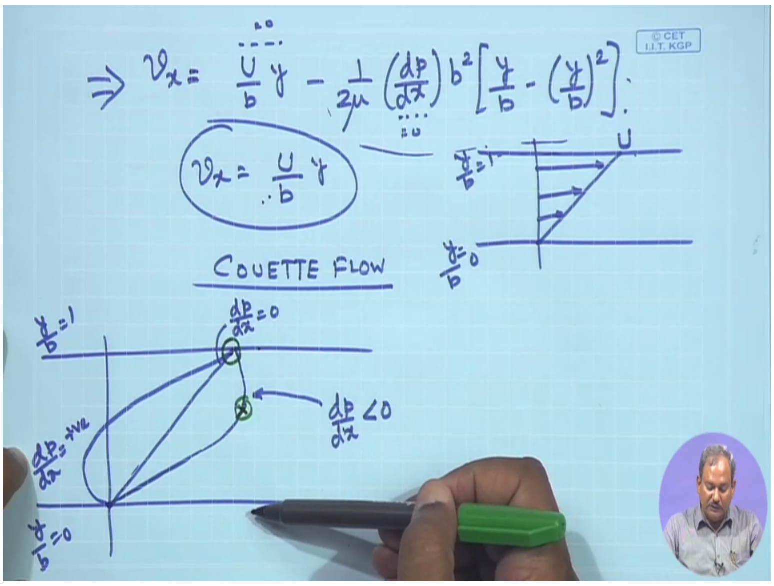
It can also happen, that the pressure on this side is going to be more than the pressure on this side or the case of an unfavorable pressure gradient. If it is an unfavorable pressure gradient, then the profile would look something like this. So this is dPdX is positive. That means as I move from this side to this side, the pressure increases. So therefore the pressure is positive,the pressure gradient is positive and it is the unfavorable pressure gradient. So on one hand you have couette flow where you get a linear distribution, favorable pressure gradient and unfavorable pressure gradient.

(Refer Slide Time 15:38)



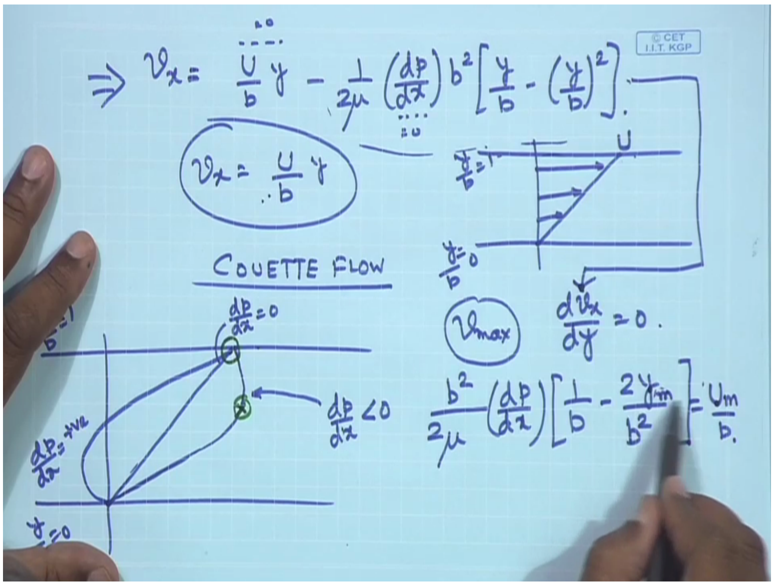
There is one more interesting thing to see here is, in couette flow, the maximum takes place at Y equals B. But if you have an unfavorable pressure gradient or a favorable pressure gradient, the location of the maximum velocity could be different. It's not going to be at Y equals B, it could be at some point in between Y equals B and Y equals 0.So if someone tells you thatwhat’s the location of thismaximum velocity?

(Refer Slide Time 16:05)



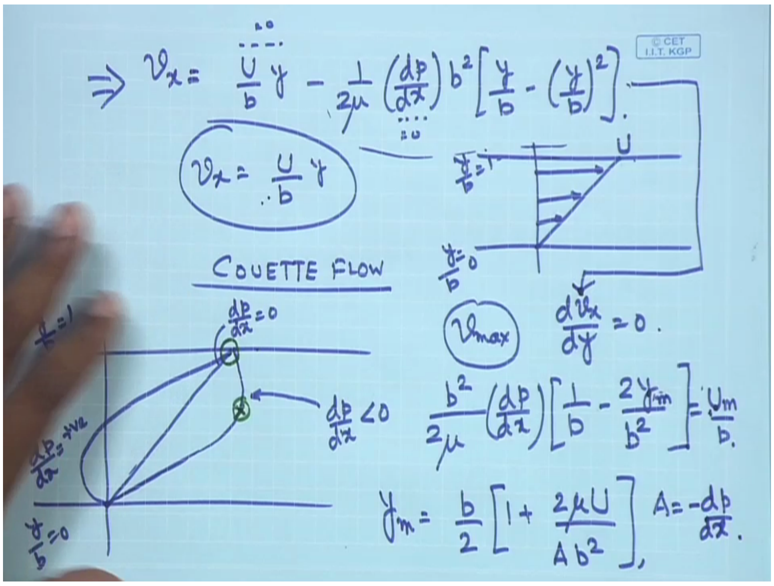
So you would like to find out what is the location of the Vmax? And we understand for Vmax, this D vX by dY must be equal to 0. So in order to get Vmax, theD vX by dY would be equal to 0 and when you substitute the expression of vX in here,differentiated with respect to Y and make it equal to 0, what you get is, this Y, I call it as yM. So this is the maximum, this is the location of Y at which U becomes uM. So this is the uMis the maximum velocity and yM is the location of the maximum velocity.

(Refer Slide Time 17:17)



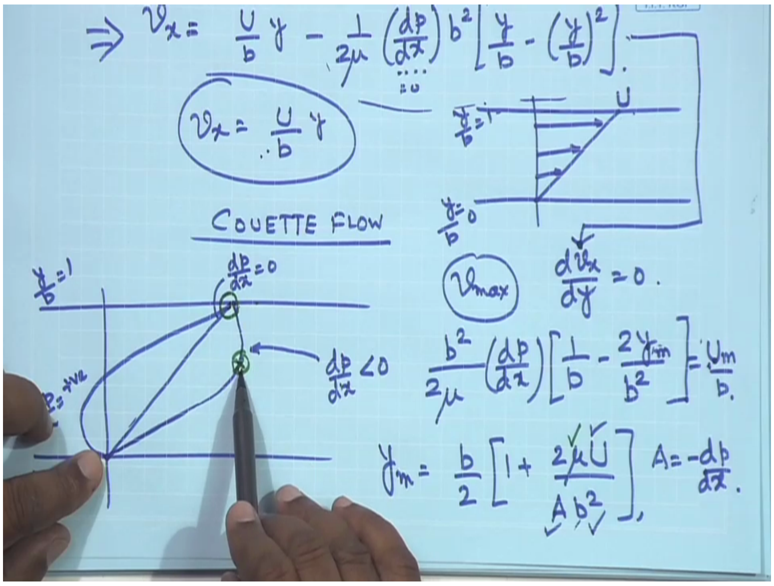
So when you simplify you would simply get yM to be equal to Bby 2, 1 plus 2 mu U by AB square. Where A,I have define simply as, minus dPdX.

(Refer Slide Time 17:36)



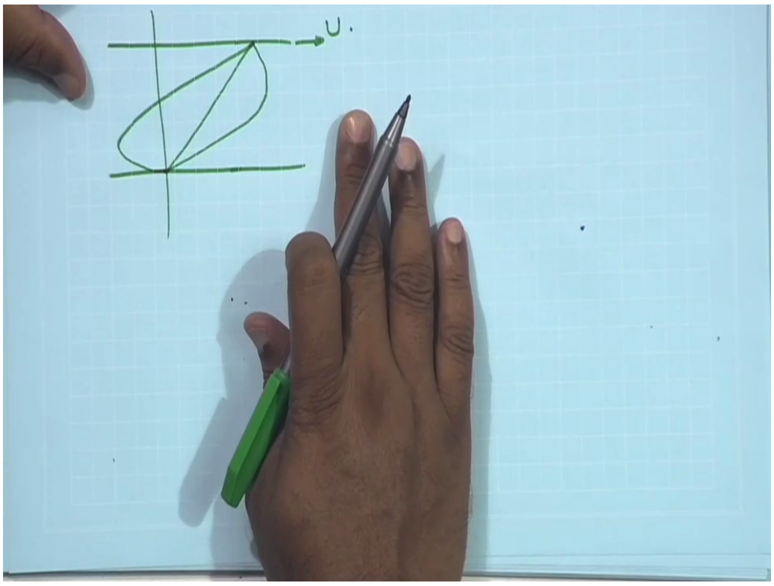
So you now have seen what is the location of the Vmax which depends on the pressure gradient, which depends on what is the top plate velocity, which depends the geometry, the separation between the two plates and thethermo-physical properties, the transport property which is mu. So the Vmax will not lie over here. V max will lie somewhere in between these two.

(Refer Slide Time 18:09)



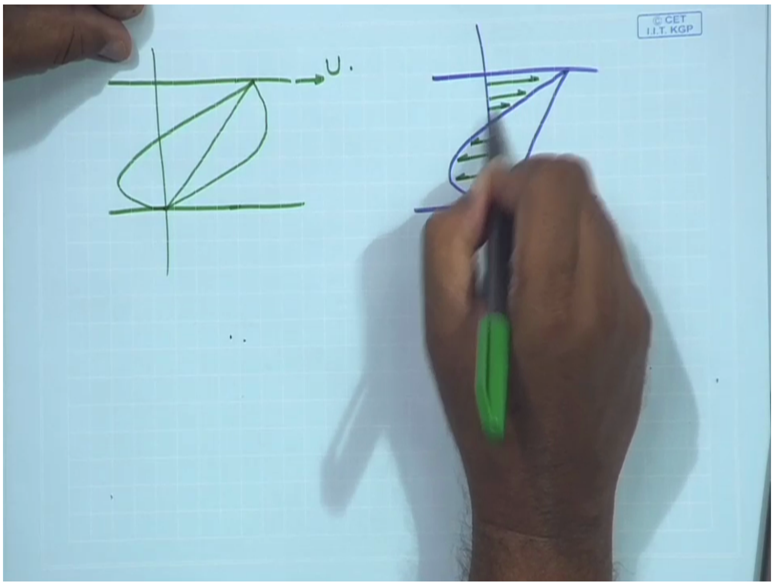
There's one more interesting thing that one can see from here is. Let's again draw this couette flow diagram where we have two plates, no pressure gradient, linear distribution, negative pressure gradient,pressure gradient and couette flow both are acting in the same direction, unfavorable pressure gradient but the pressure gradient is trying to force it from right to left. Couette flow tries to drag it from left to right. So you should be able to find out what is the negative pressure gradient.

(Refer Slide Time 18:55)



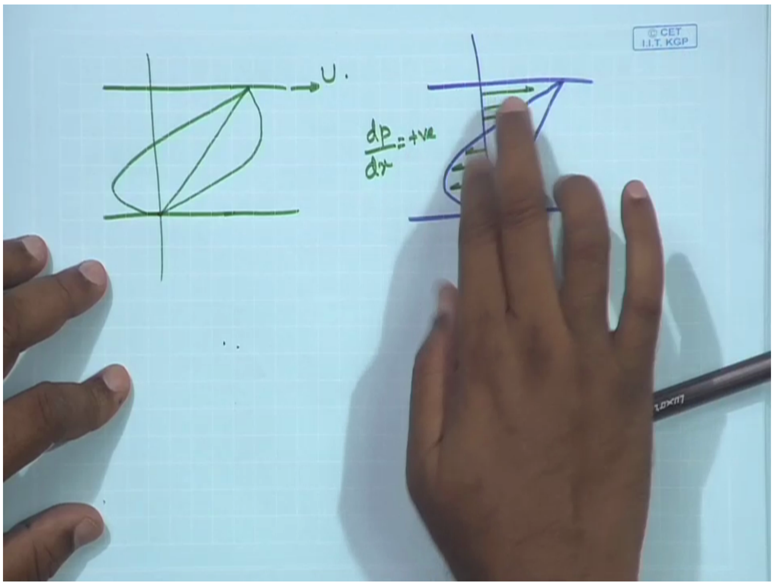
Suppose it is asked what is the pressure gradient that needs to be imposed such that there is no net flow? So what you see here is, for this case, so I amdrawing the case for couette flow and the flowwhen I have the negative pressure gradient. So if you see here, partly the flow is in this direction, in here the flow is from left to right.

(Refer Slide Time 19:30)



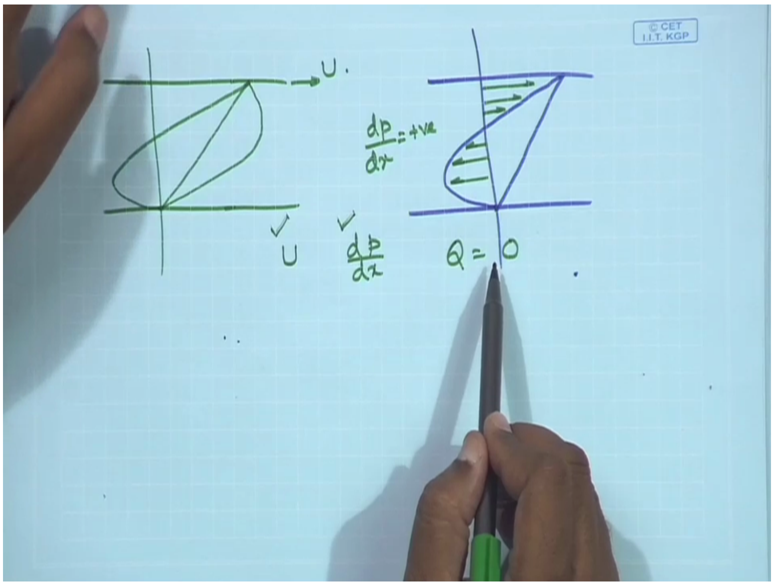
So this is a unique case where the pressure gradient is unfavorable, that means dPdX is positive and you have towards the bottom the liquid is flowing in this direction,towards the top due to the couette flow, the fluid is moving in the right direction.

(Refer Slide Time 19:49)



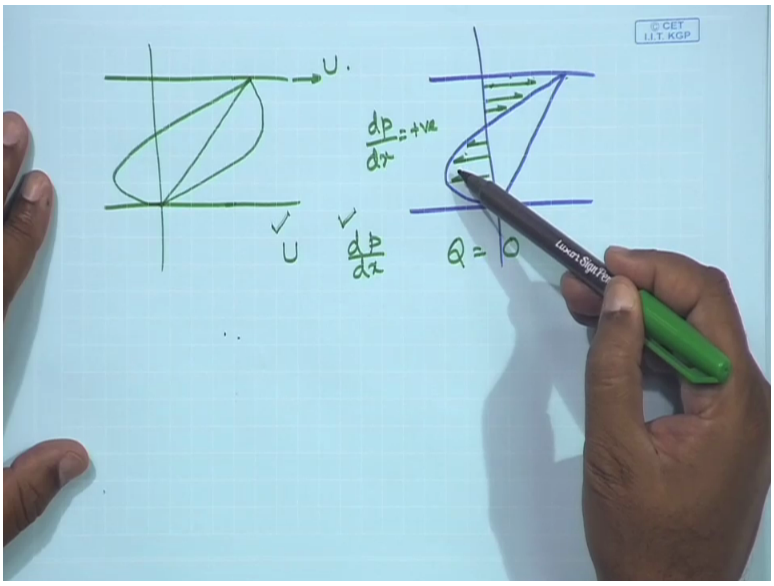
So if someone asks you, what should be the relation between U and dPdX that would give you zero netflow that is Q is equal to 0? I need to find out the relation between the velocity of the plate and the imposed pressure gradient such that there would be net flow rate to be equal to 0.

(Refer Slide Time 20:14)



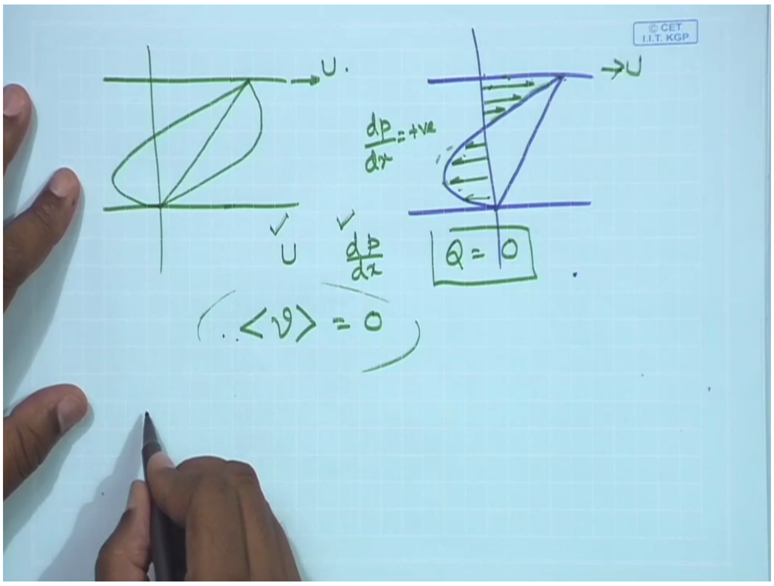
So this is really interesting thing to see and as I have drawn here the flow towards the bottom is going to be in this direction, towards the top it's going to be in this direction. But in order for Q to be 0, the netflow to be zero, the algebraic sum of the flow rate and that flow rate here and here must be equal and opposite.So the area under this curve and the area under this curve must be equal.

(Refer Slide Time 20:38)



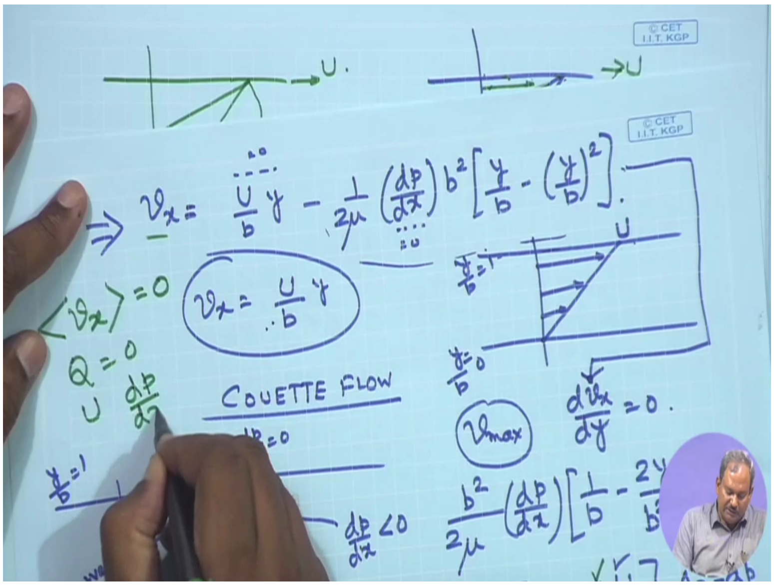
So you would be able to find out if Q is 0, this essentially tells you that the average velocity for this case be equal to 0. So the average velocity for a situation in which dPdX is positive and this is moving with velocity. The couette floor is towards from left to right, the average velocity must be equal to 0.

(Refer Slide Time 21:21)



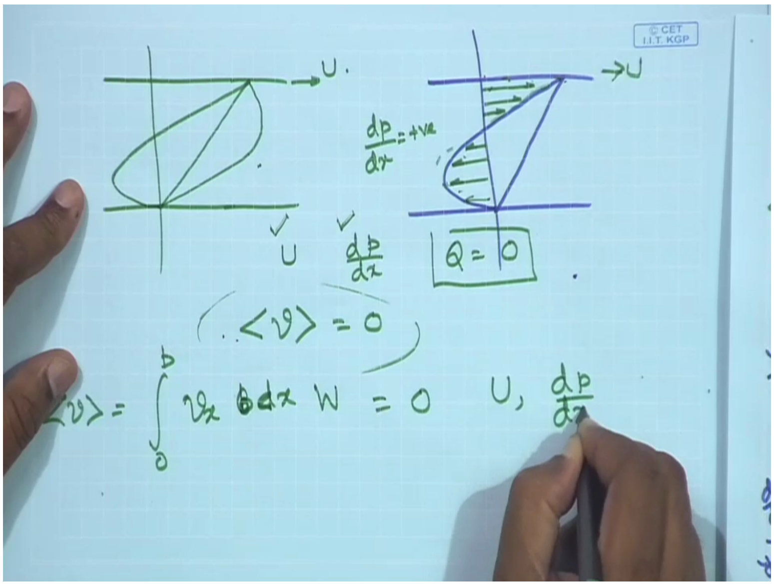
So one can find out the expression for average velocity, because the expression for velocity is known to us. The expression for velocity is known to us. From there I need to find out what is vX and this vX to be equals to be 0, which implies that Q equals 0 and this should give me a relation between U and the dPdX. I will leave that for you to work on.

(Refer Slide Time 21:39)



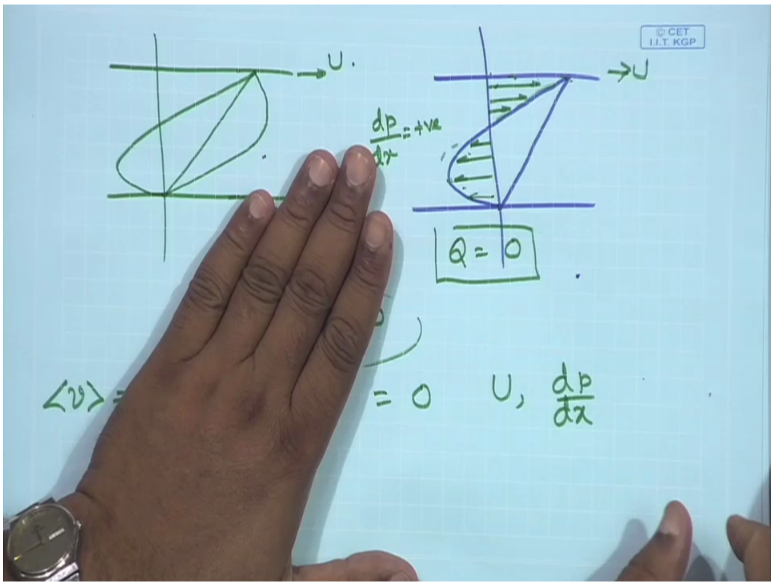
But in order to obtain this vX I need to integrate. So this V must be equal to area, which is the flow area and the flow area is going to be vX times whatever be the width times dX and X varies between 0 to B. So this is going to be W. Where W is the depth of this. So this is going to be your average velocity if you equal it to be 0,you get the relation between U and dPdX.

(Refer Slide Time 22:20)



So an unfavorable pressure gradient can give rise to a situation in which the net flow rate is 0. So the same problem gives you another interesting way to look at the same problem.

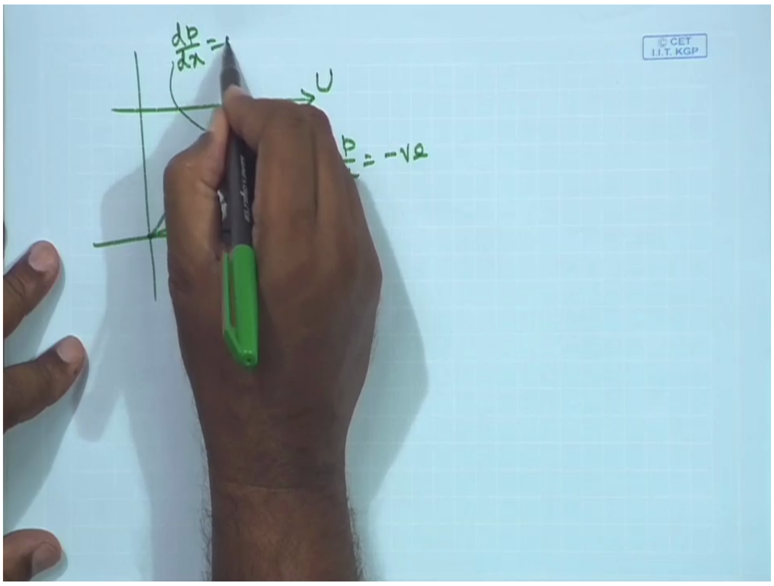
(Refer Slide Time 22:37)



The third part of this specific problem can also be thought of to provide a different type of a problem. Let's say that the top plate is moving, the bottom plate is stationery, you have a favorable pressure gradient. So initially it was a straight line couette flow and then when you start applying a favorable pressure gradient, the flow starts to deviate from its straight nature.

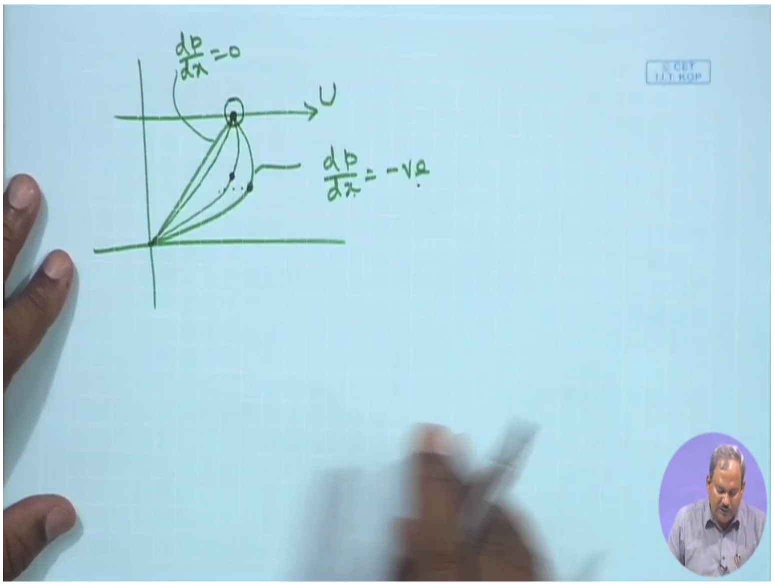
When the applied pressure gradient is large, then you simply have the pressure gradient that we have drawn it previously. With flow when you have dPdX to be negative. This is U moving with U. So this is the case where dPdX is equal to 0.

(Refer Slide Time 23:31)



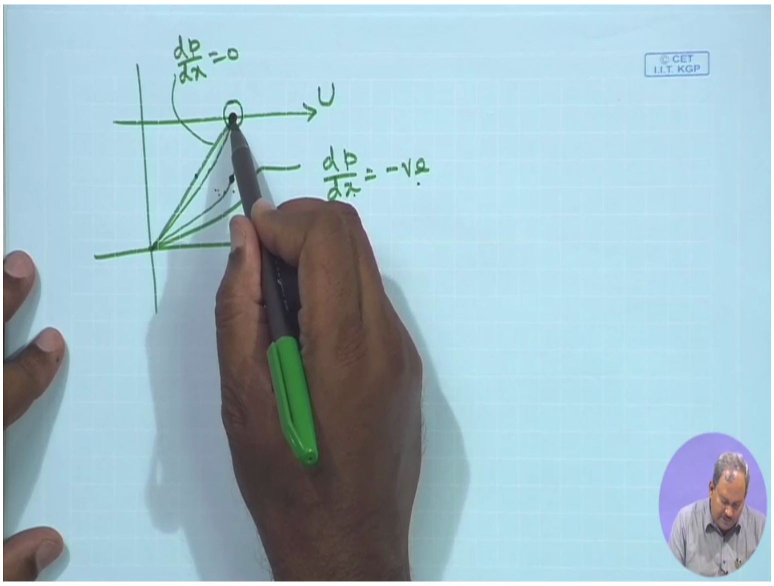
So now we are trying to compare between no pressure gradient, dPdX equals to 0 and pressure gradient which is favorable, there is dPdX is negative. The maximum is here. In the previous case the maximum was here. Now if you (pro) progressively reduce thevalue of the applied pressure gradient, the profile would look something like this and this is the location of Vmax and the point. And if you reduce it significantly then it's not a straight line. So it's not a straight line but this remains the location of the maximum velocity.

(Refer Slide Time 24:17)



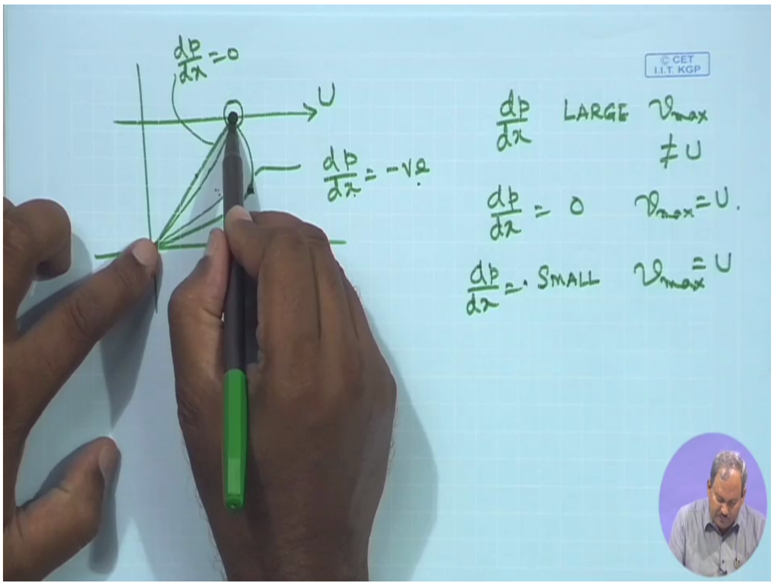
So the maximum velocity starts to move in this direction. Starts to move from right to left and when it is dPdX is 0, it's over here. But when dPdX is not 0 but small, then also you get the maximum velocity over here.

(Refer Slide Time 24:39)



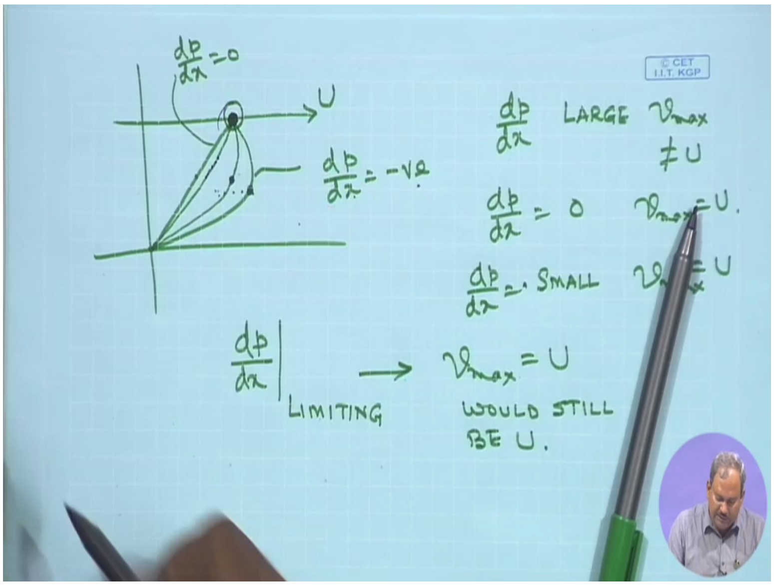
So for some value of dPdX, so that means if dpdX is large, your V max is not U. V max is something more than U. DP dX is 0, your V max is equal to U. That is over here. But when dPdX is small, it may be possible that V max is still equal to U, which is this case. DP dX is small but you still have the Vmax at the top.

(Refer Slide Time 25:19)



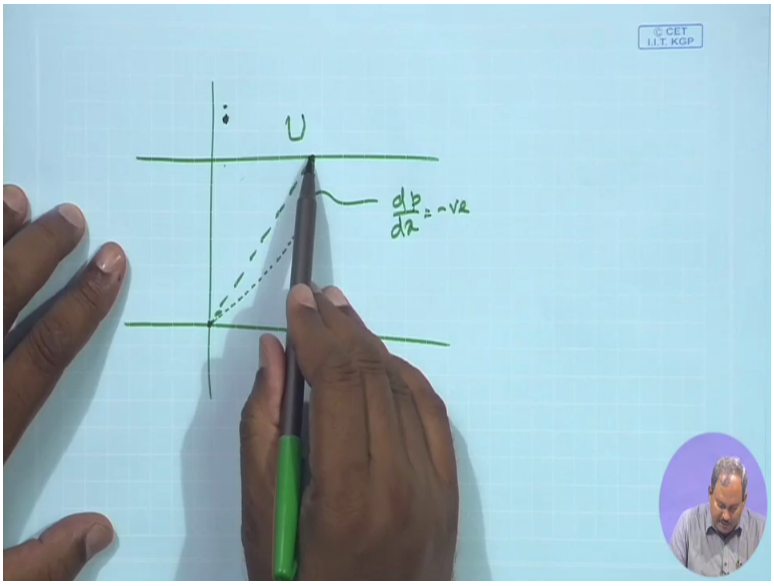
So what is the limiting value of the dPdX? Can you find out the limiting value of dPdX, such that V max would still be equal to U? So it's no longer a straight line. It deviates from a straight line. It's no longer a straight line, but the maximum is still over here.

(Refer Slide Time 25:47)



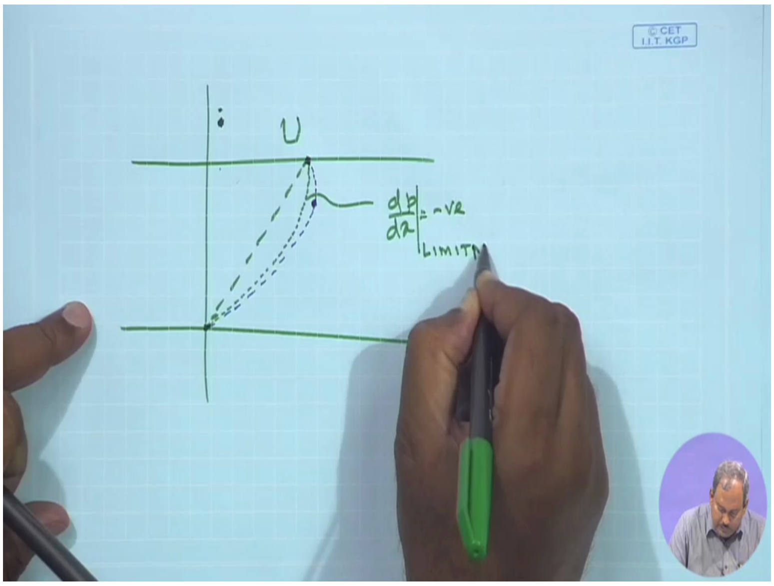
So what is the condition that one has to make, one have to impose in order to obtain the limiting value of dPdX such that the Vmax is still equal to U? I'll leave that problem for you to think about. I will just give you a pointer that how would the profile look like when you have. This is my straight line, the velocity is equal to U and this is the limiting case. This is a limiting case where the maximum velocity dPdXis negative. This is dPdX is negative. But the maximum is still over here.

(Refer Slide Time 26:40)



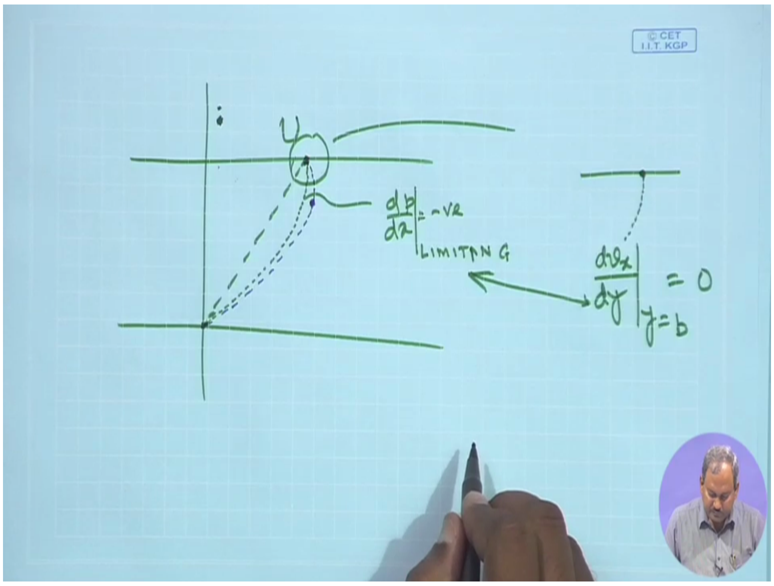
Any increase in the dPdX and what you are going to get is, the maximum is going to be somewhere here and not the value of U max. So this is your dPdX limiting.

(Refer Slide Time 27:01)



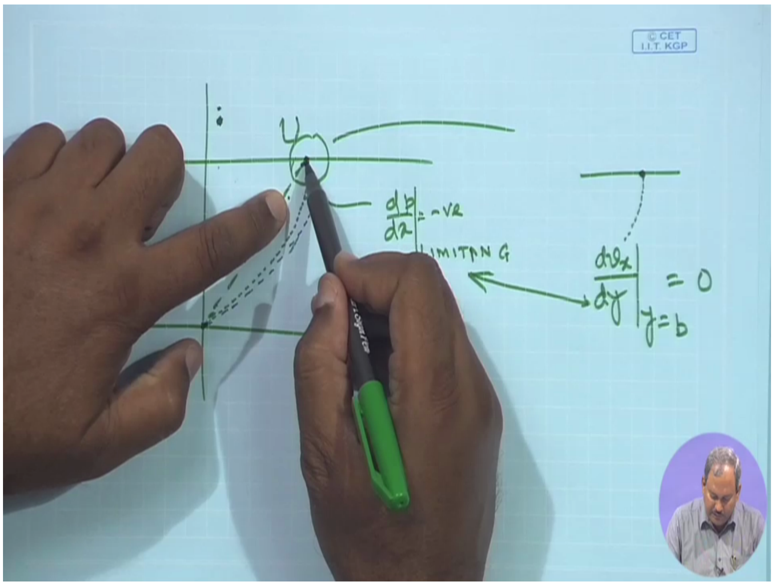
So how do you find out what is the value of the dPdX negative? Look at the profile over here. If I expand this, the limiting value can be obtained by looking at the profile. That means the slope of the curve, when it reaches the top is going to be equal to 0. Or in other words D vXdY at Y equals B is 0 for the limiting case.

(Refer Slide Time 27:34)



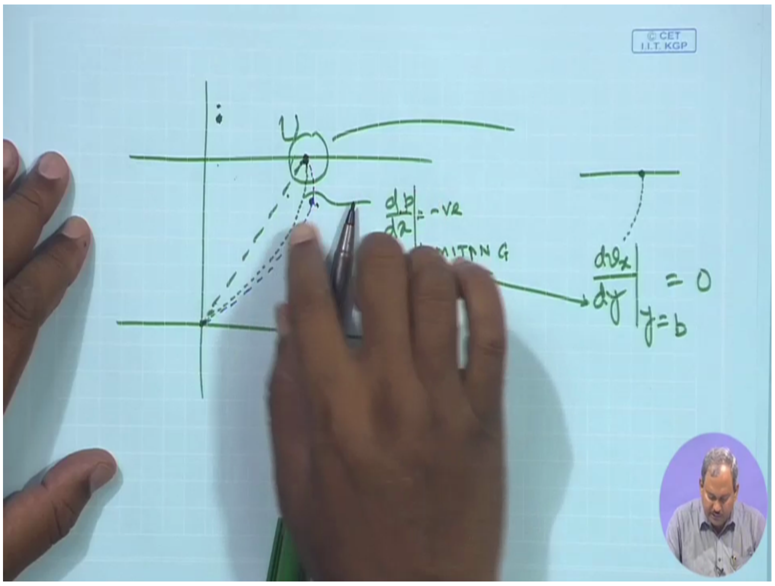
Is that clear now? I have a linear profile and this is the case when dPdX is large. I progressively slowly start to reduce the value of dPdX. If it is too small, then it's no longer linear. But the maximum is still over here.

(Refer Slide Time 27:52)



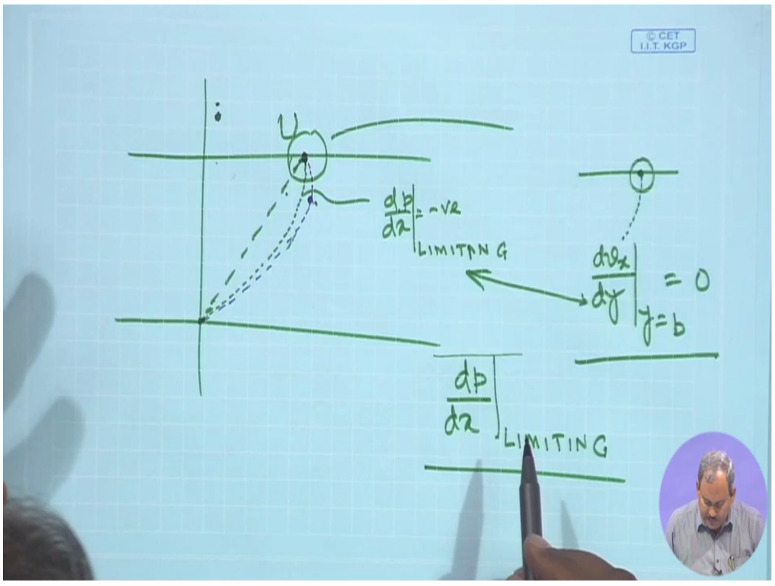
If it is large, is not linear but the maximum is not over here. So this is the limiting case which is at the junction where the maximum velocity is over here and any slight increase in dPdX tilts it from left to right and you can get new maximum.

(Refer Slide Time 28:13)



So how does it look like for the limiting case? It's simply approaches the wall with a zero slope. If it approaches the wall with a zero slope, then this is the condition which you can use to obtain what is the expression for dPdX limiting. This is something which I think you should do it on your ownand it would give you very interesting result.

(Refer Slide Time 28:44)

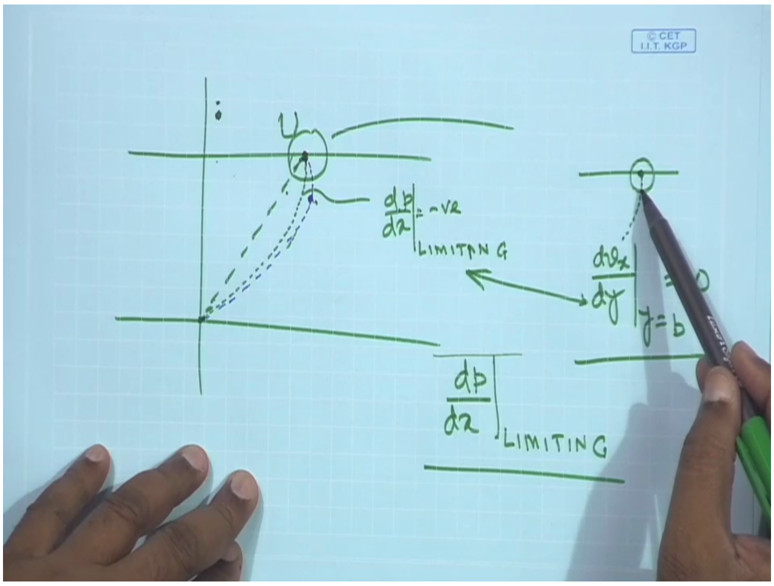


I will just point out feature of this problem of this limiting dPdX. It can be asked like, in order to move the top plate, you need to pull it. You need to apply some force in order to allow it to move at a constant velocity over a film of liquid which has a finite depth. What would be the force which needs to be applied when you pull the top plate? What is the force needed at the limiting condition, limiting dPdX? The one that we have just described.

So the limiting dPdX, what we define as the value of the pressure gradient, if you go beyond that the maximum of fluid velocity will not lie at the top plate. It will be somewhere in between the top plate and the bottom plate. But dPdX limiting is that value which is the maximum pressure gradient that you can apply while maintaining the velocity at maximum at the top plate. And we have seen how the profile look like.

So looking at the profile I think I have given you a pointer about what is the force needed to pull that plate in that case. if you see the force is simply the viscous forces. The viscous force is given as mu times the velocity gradient. Now if you look at the profile that we have drawn over here, the value of D vXdY is equal to 0 for under the limiting condition.

(Refer Slide Time 30:33)



So if this value is equal to 0 under the limiting condition then what you get is an interesting result. That under this limiting pressure gradient, you do not need any force to sustain the motion of the top plate. So the top plate would perpetually move if you can maintain the limiting condition as it is. So it's an interesting problem the one we have discussed. First of all it gives you an idea of couette flow. Only the motion of the plate can drag fluid which is known as couette flow.

You can also impose the pressure difference either assisting the couette flow or opposing the couette flow. And in both cases you get the profiles. In the case where the pressure gradient is opposing the couette flow, then near the top plate the fluid will still be moving towards the right. Near the bottom plate because of the pressure gradient, the flow will be towards the left maintaining the no slip condition at the bottom plate. And you can think of a condition in which you have an opposing pressure gradient.

With flow towards the right the combination is such that there will be zero netflow. The trick to do that problem is to find out what is the average velocity. Cross-sectional average velocity equated to zero and you get what is the opposing pressure that you need to apply in order to obtain a zero net flow. And the last part of the problem what we have seen is that the maximum velocity in couette flow will always be at the top plate which is moving with some velocity.

If you start applying a favorable pressure gradient, the lenient nature of the curve will start to deviate. If we apply too higher pressure gradient, then the maximais going to be somewhere between the top plate and the bottom plate. But if you apply just the right pressure, there would be a maximum pressure gradient in which the velocity maxima will still be at the top plate. Any slight increase and location of the maxima will come down. That is what we call as a limiting pressure gradient.

And under that limiting pressure gradient condition, there would be no force required to pull the plate over the film. So a single problem would clarify the concepts of pressure gradient flow, no slip condition, no (flo) netflow and so on. What I would do in the next class is to give you some of the problem to work on. So I will introduce the problem explain a bit of it and provide you with the answer. And I would expect you to do it on your own and come up and check with the solutions.