Transport Phenomena
Prof. Sunando Dasgupta
Department of Chemical Engineering
Indian Institute of Technology, Kharagpur
Lecture Number 20
Boundary Layers (Cont.)

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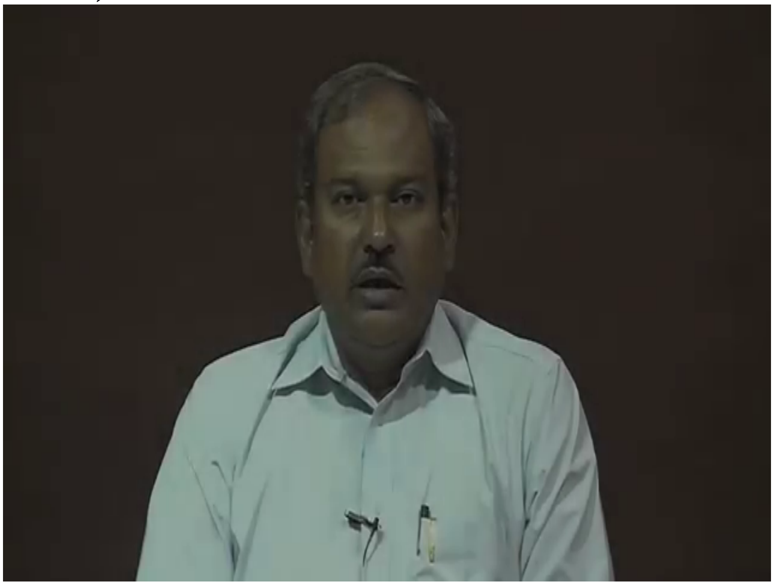

So in this class, we would see some of the, we would try to solve a few problems which would clarify our concepts of displacement thickness, the growth of the boundary layer, the flow outside of a boundary layer using very simple straightforward example. And we would also solve another problem which would give us slightly more involved ideas about how the growth of the boundary layer takes place in a specific flow. But let's first talk about the first problem. In this case, we have a wind tunnel which is square in cross-section. So at the beginning I have the wind coming in and the cross-section at this point is a square. The dimension of this square, the entry point is provided and so the flow takes place through the wind tunnel which is of constant cross-section.

And you can understand that the, the growth of the boundary layer is going to be along all sides of the, of the bound, of the wind tunnel, so the boundary layer thickness if you are, let's say x distance from the entry point is something when you go to some other distance, let's 3 x, the thickness of the boundary layer would be even more, Ok? So the boundary layer thickness progressively increases from a value equal to zero at the beginning and it will keep on continue, it will keep on increasing. So what is going to happen to the core flow? The core flow which is taking place outside of this boundary layer, the area available for this core flow will keep on decreasing since the boundary layer is going to grow and it will become thicker and thicker. And we understand that inside the boundary layer the flow is viscous.

Since the flow is viscous, the flow velocity would be less than that of outside of the boundary layer that means inside the boundary layer, the fluid will move at a lower velocity as compared to the velocity of the fluid outside of the boundary layer. So the flow inside the boundary layer is slower as compared to the flow outside of the boundary layer. Now since the thickness of the boundary layer keeps on increasing as we move along the flow, the area available for the core flow where the effect of viscosity, viscous forces are unimportant, that region will keep contracting. So you would expect that the flow, since the flow area reduces in, in and the equation of continuity has to be obeyed at all times so the flow velocity outside of the boundary layer will keep on increasing as we move in the x direction.

Because more and more area is now under the boundary layer or the slow flow condition. So in order to compensate for the slower flow inside the boundary layer, the flow outside the boundary layer must increase in order to satisfy equation of continuity. So this problem, the one that we are going to do is an use of an extension of this concept. Now whenever we talk about inviscid flow and viscous flow and if you would like to transform the viscous flow which is there in boundary layers to an inviscid flow situation, we must use the concept of displacement thickness. What is displacement thickness? It is the distance by which the boundary has to be moved inwards in this specific case so as to obtain the same reduction in mass flow rate as in the case of viscous flow. So let us say the boundary was here, Ok.

This was, this was the square cross-section and I have some thickness of boundary layer on all four sides so I have slow flow inside the boundary layer, faster flow through the core and I would like to, in order to use Bernoulli's equation which is true only for inviscid flow, truly speaking, so we have to convert this flow with boundary layer and core flow to a situation in which there is only one velocity, only one core flow and it is flow of an inviscid fluid. So in terms of flow rate I am changing the, changing the boundary layer and core flow to a case where it is only core flow with some velocity. So how is it done?

It is by reducing the size of a channel by a distance which is called displacement thickness which takes into account the reduction in mass flow rate inside the boundary layer that would have happened in an inviscid flow if you restrict the flow geometry. So due to flow due to the viscosity the flow is reduced. How much it is reduced? I am trying to find an equivalent of that in inviscid flow. How do I reduce the flow in the inviscid flow case? Simply reduce the area. So I reduce the area by an amount which is exactly equal, which would give rise to a reduction in mass flow rate due to the presence the viscosity, the viscous boundary layer in the first case. That is the concept of displacement thickness. So you would see the concept of displacement thickness being applied for this specific case and it should give us better understanding of the utility of the concept of displacement thickness and therefore the problem that we are going to see here

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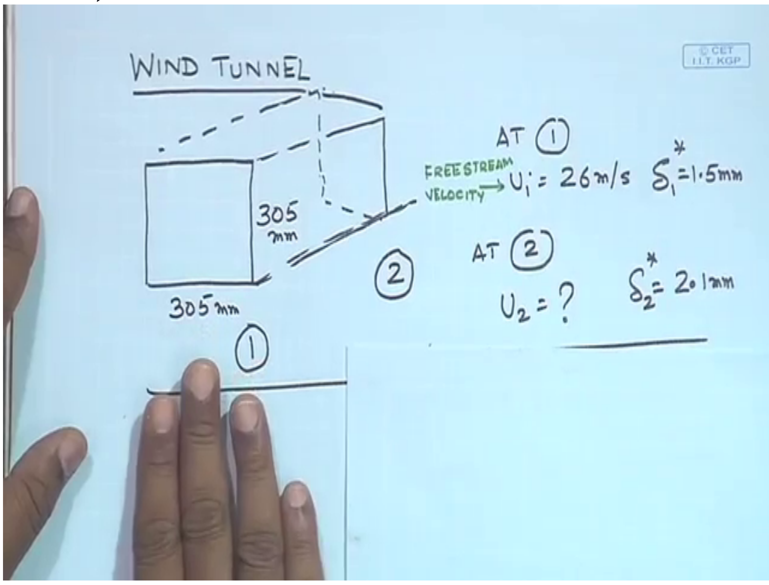

is wind tunnel which is 305 millimeter by 305 millimeter and let's assume that the 1 and 2 are two stations in it where the velocities are measured at 1, where the freestream velocity is found to be 26 meters per second and the corresponding value of displacement thickness is measured to be 1 point 5 millimeter. At some point downstream, at another station called 2, the boundary layer thickness has been calculated; the displacement thickness has been calculated to be equal to 2 point 1 millimeter. What is required is calculate the

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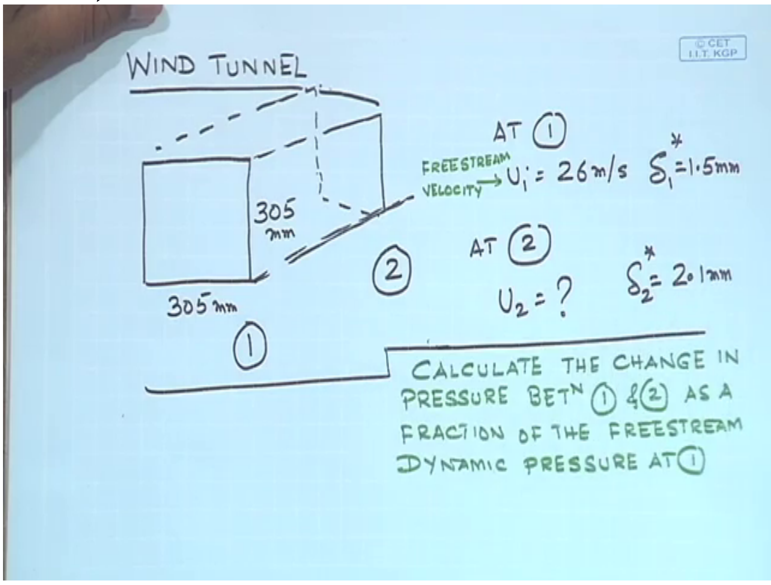

change in pressure between 1 and 2 as fraction of freestream dynamic pressure at 1. I will repeat it once again. Calculate the change in pressure between 1 and 2 as a fraction of the freestream dynamic pressure at 1. What is the freestream dynamic pressure at 1? By definition, we know that the freestream dynamic pressure is nothing but equal to half rho u 1 square. So u 1

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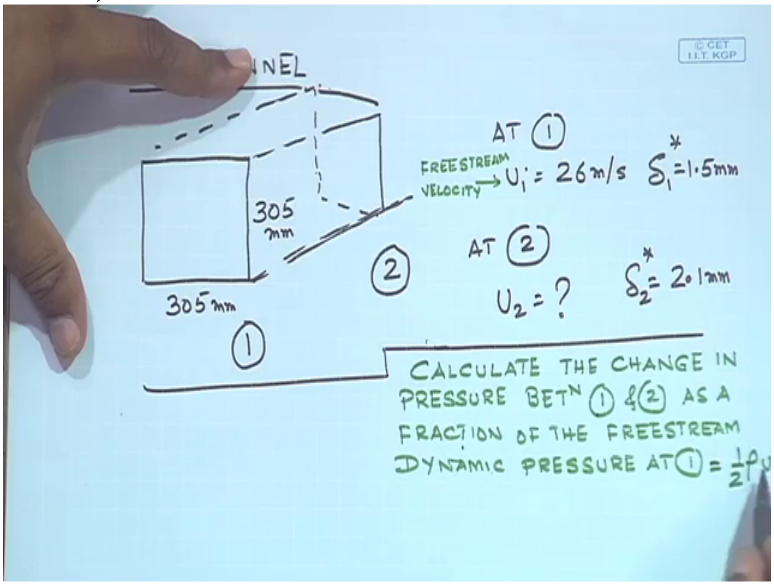

is the freestream velocity at 1, and half u 1 square is the dynamic pressure. So I would like to find out what is p 1 minus p 2 divided by half rho u 1 square. This quantity has to be evaluated. Based

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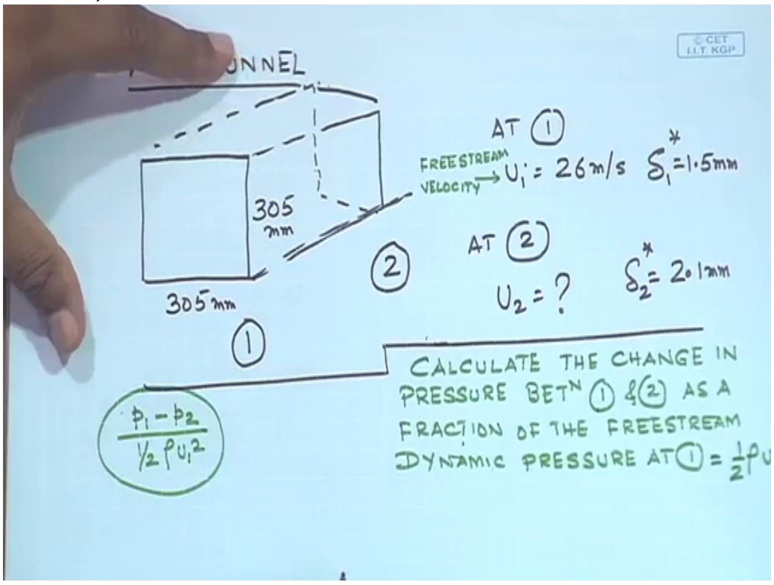

on the, equations based on the, on the numbers provided, that is u 1 is 26 meter per second, del 1 star, del 2 star is given, u 2 star is not provided. So the first thing one has to do is, whenever we are trying to find out what is the difference in pressure between one point and the other, the first equation that comes to our mind is use of Bernoulli's equation. But while using Bernoulli's equation we have to make sure that this is an inviscid flow case. Since Bernoulli's equation is ideally applicable for the case of inviscid flow. Had it not been the

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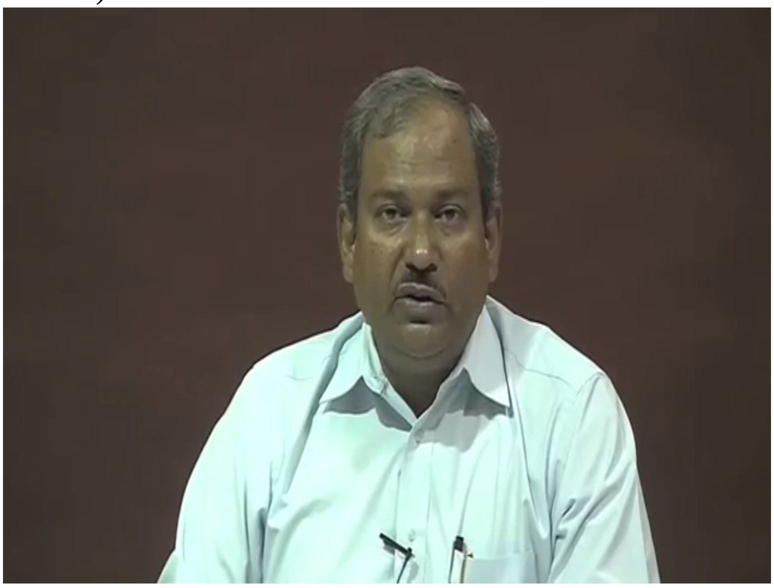

case then the head loss or the frictional loss factors would have to be incorporated in the Bernoulli’s equation of which we do not have any idea at this moment , what would be the friction factor and so on. So in order to use the true form, the, the ideal form of the Bernoulli's equation, I need to write the Bernoulli's equation between station 1 and station 2 where the points for which, for the streamline for which I am writing the Bernoulli's equation must lie outside of the boundary layer. Because outside of the boundary layer, the flow is inviscid and therefore the use of Bernoulli's equation is justified.

So I am writing Bernoulli's equation between station 1 and station 2, when both station 1 and station 2 are either located outside the boundary layer or the flow situation in 1 or and 2 are replaced by the corresponding inviscid flow case. And in order to convert viscous flow, that means with boundary layers to inviscid flow, one as per our previous discussion, one has to invoke the concept of displacement thickness. So displacement thickness allows me to convert a viscous flow to an inviscid flow by raising the platform by certain amount which is the displacement thickness and then treating that everything on that platform is moving as, as if it's an inviscid flow, Ok.

So that is the concept of, that is the concept of displacement thickness. So in this specific problem, the sides are 3 0 5 ok, 3 0 5 millimeter by 3 0 5 millimeter, the values of displacement thickness, thicknesses are provided. So I can, I can safely say that I am going to raise this, if I raise the platform by that displacement thickness and bring in this side also by the displacement thickness then the smaller square that I have, Ok, this side raised by delta 1 star, this side reduced by delta 1 star, so whatever area that I have, I can, I can safely say since these two are displacement thicknesses, this square, through this square, only inviscid flow is taking place. So what was taking place in the original 3 0 5 square, 3 0 5 millimeter by 3 0 5 millimeter can, in terms of mass flow rate, will be identical to, if I reduce 3 0 5 by delta 1 star and 3 0 5 by delta 1 star Ok and I am going to bring this down by delta 1 star, raise this up by delta 1 star, bring it in by delta 1 star and this side in by delta 1 star.

So my square where everything is inviscid flow right now has a dimension equal to 3 0 5 minus two delta 1 star, Ok. This was 3 0 5 by 3 0 5. I raise this up, bring this down, so my this dimension is 3 0 5 minus two del star, this sides I bring in both sides by del star so this, the this dimension now becomes 3 0 5 minus two del 1 star. So the square area which I have right now, in which only inviscid flow takes place by the definition of displacement thickness has a dimension equal to 3 0 5 minus two del 1 star on all four sides and then equation of continuity can be used. Bernoulli's equation can be used. And that's what we are going to do in this problem. So what I have in here is then, I am going to write the Bernoulli's equation between location 1 and 2. I would assume that these are horizontal,

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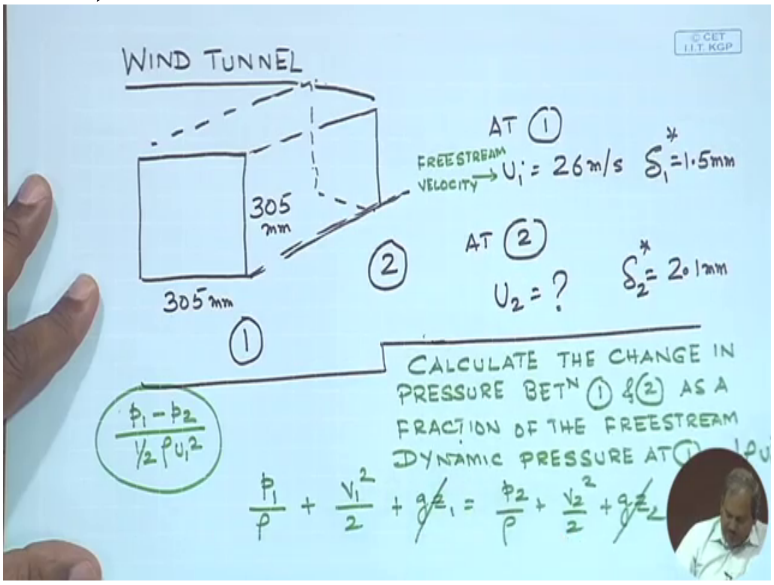

the wind tunnel is horizontal and therefore this p 1 minus p 2, p 1

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minus p 2 would simply be equal to half of rho v 2 square minus v 1 square. Remember that v 2 and v 1 will both have to be the velocity in the freestream or the inviscid flow, inviscid flow situations. So p 1 minus

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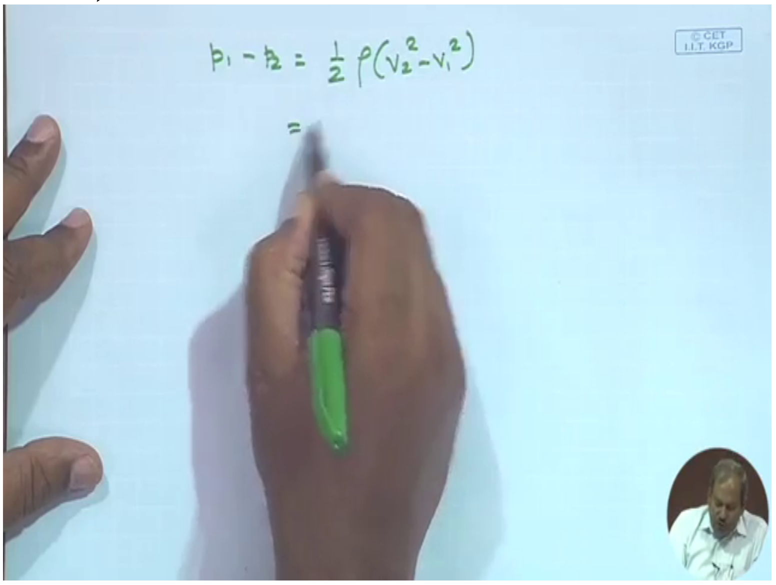

p 2 would simply become half u 2 square minus u 1 square where u 2 and u 1 are the freestream velocities. So I will bring this as half rho u 1 square and therefore this would become u 2 by u 1 whole square minus 1 and

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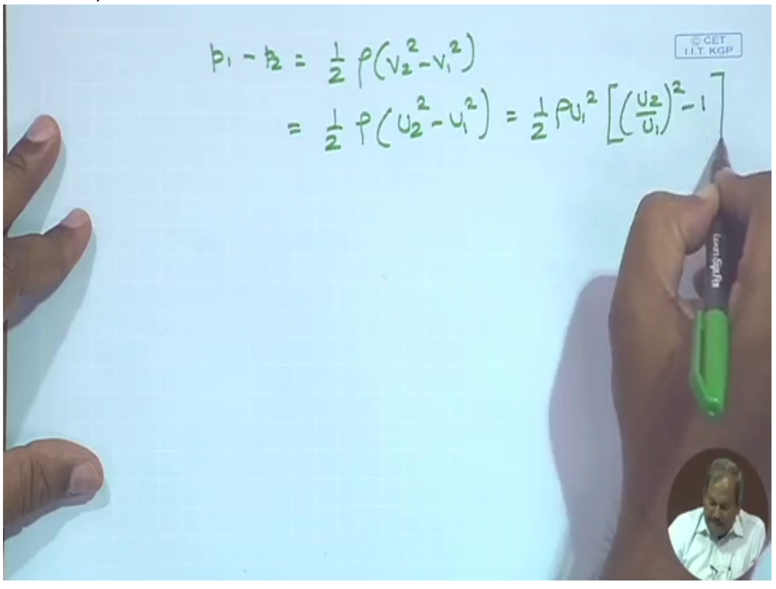

this is nothing but, nothing but the dynamic pressure at 1, and therefore

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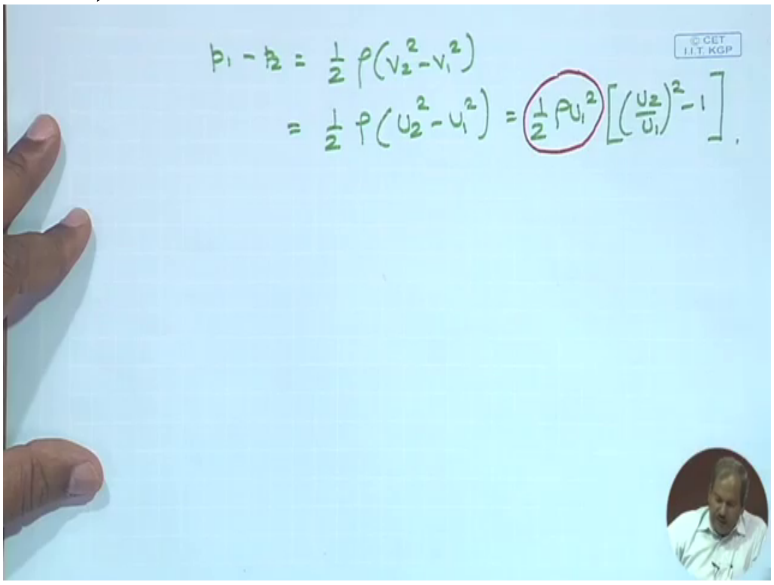

p 1 minus p 2 by half rho u 1 square, that is the difference in pressure as a fraction of the freestream dynamic pressure at 1, the quantity that needs to be calculated would simply be equal to u 2 by u 1 square minus 1.

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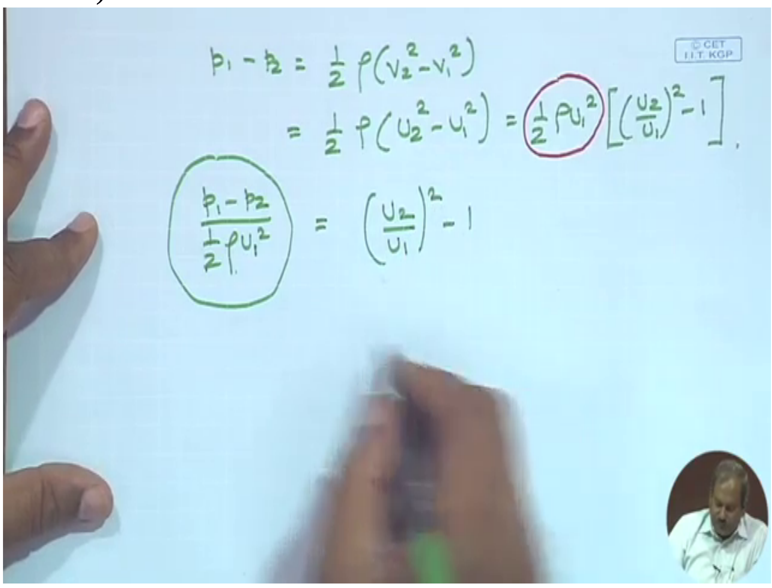

And from equation of continuity, we also know that u 1 A 1 must be equal to u 2 A 2, rhos are the same at every point

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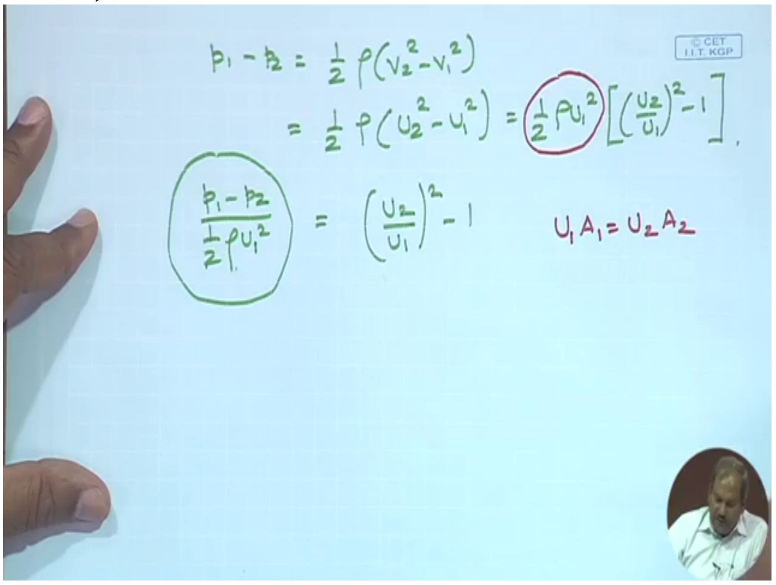

and therefore u 2 by u 1 would simply be equals A 1 by A 2.

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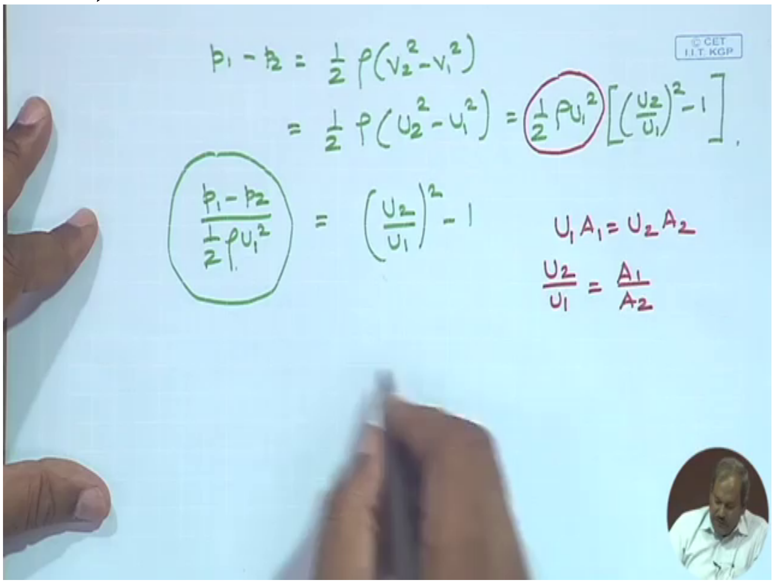

And u 2 by u 1, the trick comes in this A 1, it is simply going to be l minus two delta, sorry. As I mentioned

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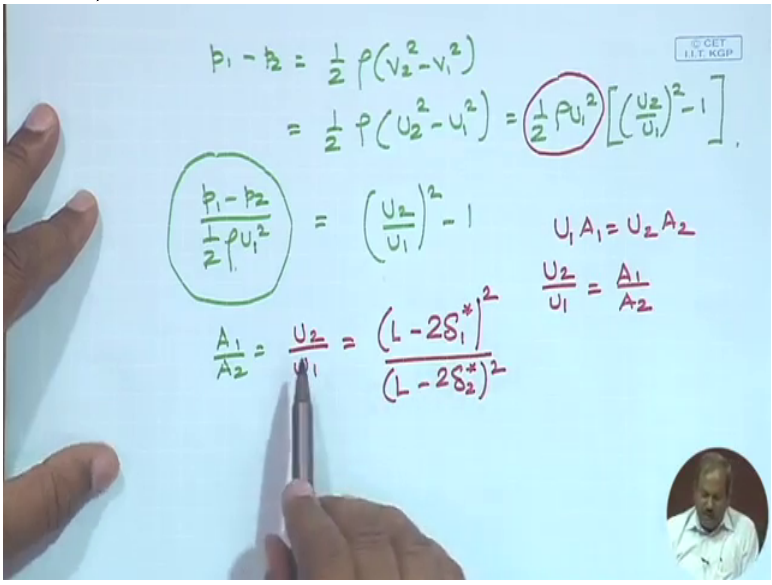

before the area, the flow area available for inviscid flow at station 1 and at station 2 by, through the use of the concept of displacement thickness would simply be reduced by L minus 2 delta 1 star and L minus 2 delta 2 star, because everything is, everything is reduced by delta 1 star

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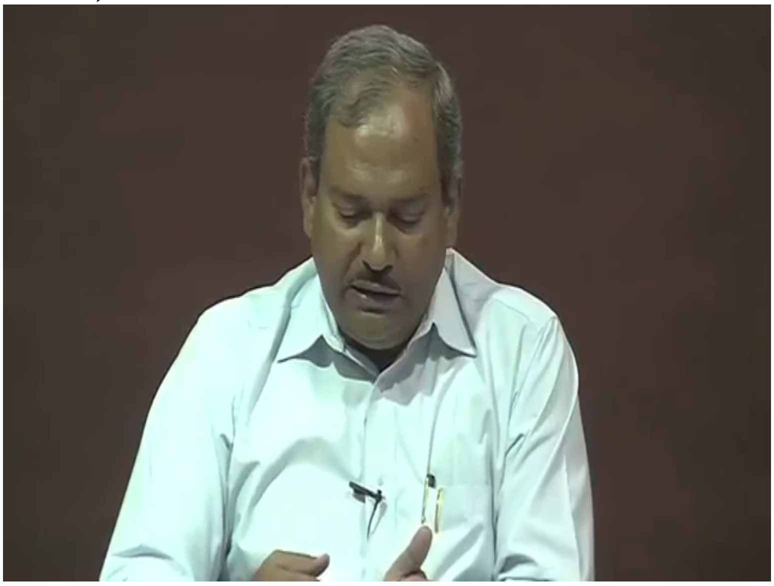

star from all four sides, either delta 1 star or delta 2 star from all four sides, so therefore I am simply writing half rho u 1 square would simply be equals to,

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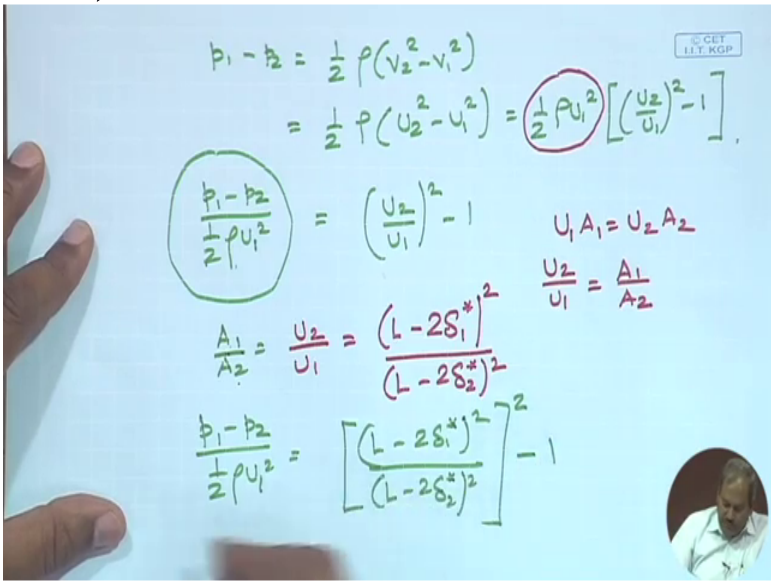

so this is going to be the pressure change between locations 1 and 2 as the fraction of the dynamic pressure at location 1 and what you would get is, this is going to be equal to 0 point 0 1 6 1 or

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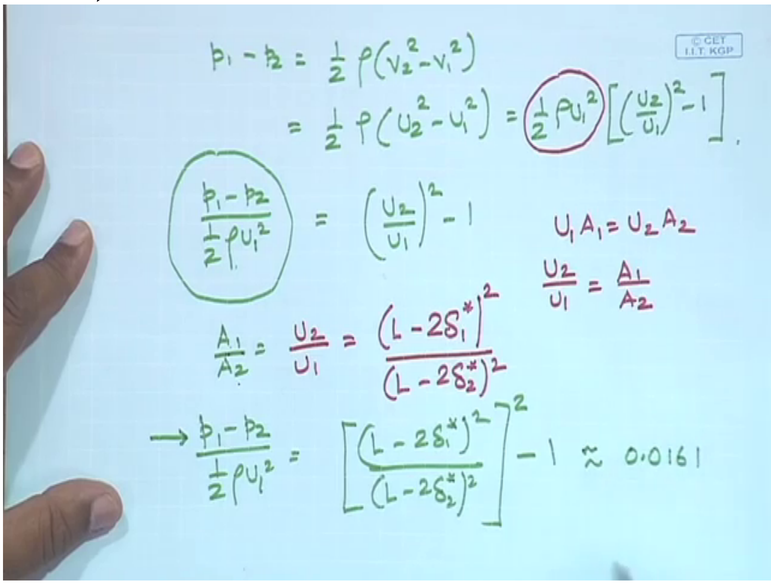

1 point 6 percent. So this is a

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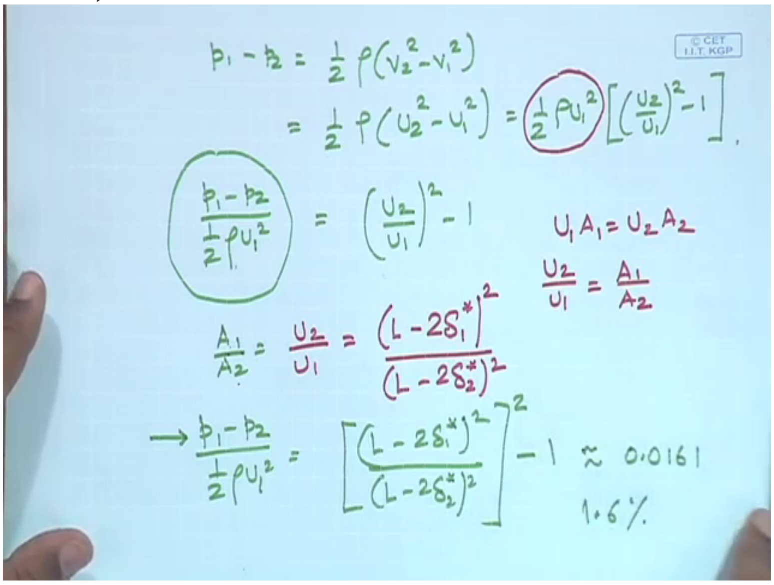

nice example, this is a nice example where you would, where you would get an idea of what is the use of the, use of the displacement thickness in order to obtain, what would be the freestream velocity, what would be new area and the freestream velocity when we transform, when we replace the viscous flow, that is flow with the boundary layer to a inviscid flow where all, at all points the fluid is moving with the same velocity. So it would give you an idea, it would give you a nice idea about how to the pressure difference between two points through the use of boundary layers and through the concept of the, concept of the displacement thickness.

So we will move on to our next problem now.

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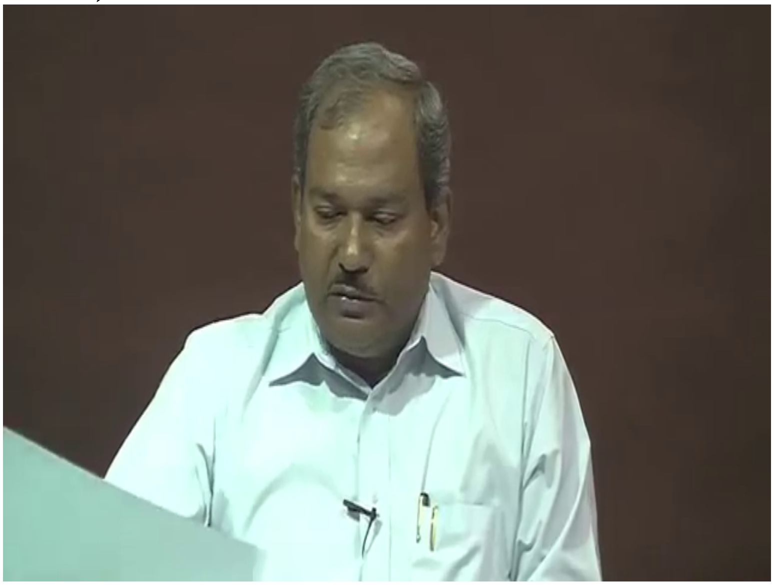

And this problem tells us that the numerical results that we have obtained, this is the numerical results of Howarth,

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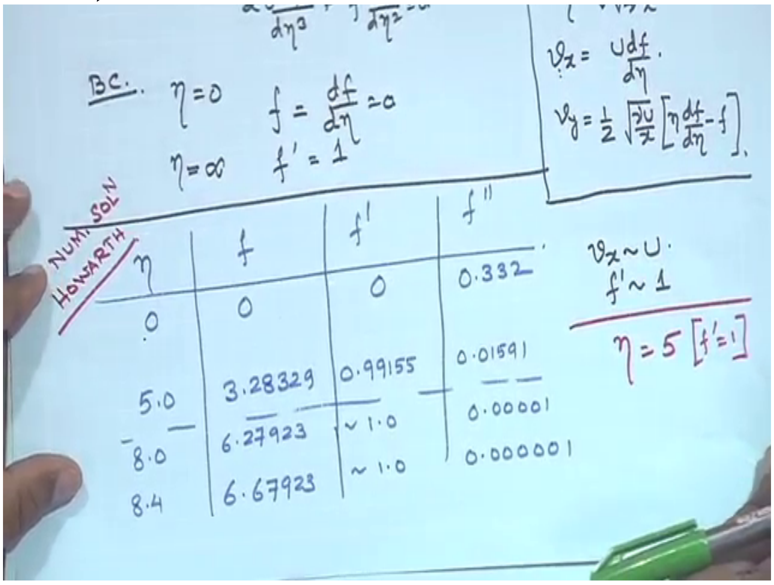

that we have obtained which contains the value of eta. For any value of eta the corresponding values of f, f prime and f double prime were provided and we also know that what is the, what is the expression for v x, the expression for v y and so on and through the use of this table, we have seen that a closed form solution, I mean a complete expression for the velocity, for the, for the growth of boundary layer which is this and

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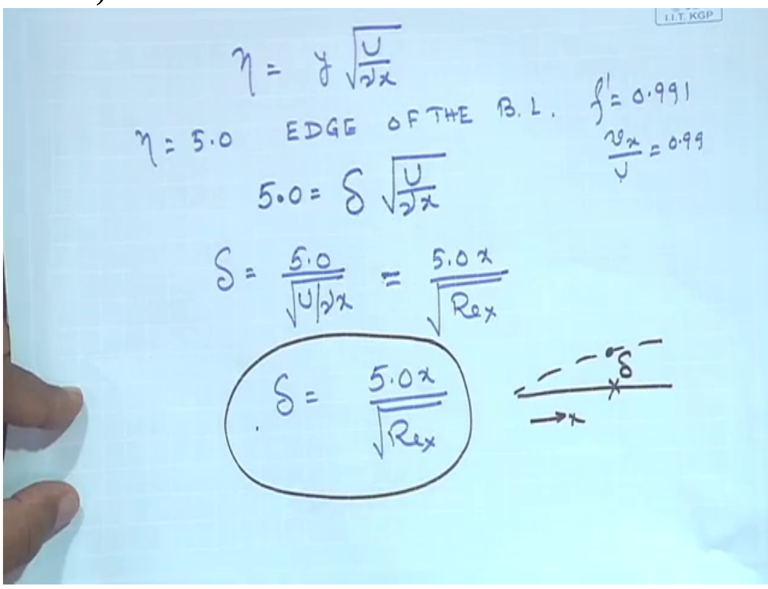

the wall shear stress which is this was

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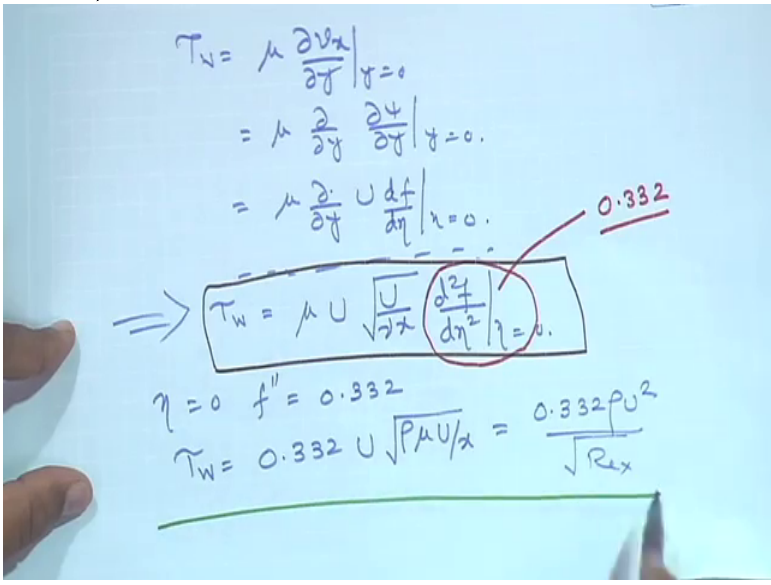

obtained. So these expressions for tau w and the expressions for delta were obtained through the use of the values presented

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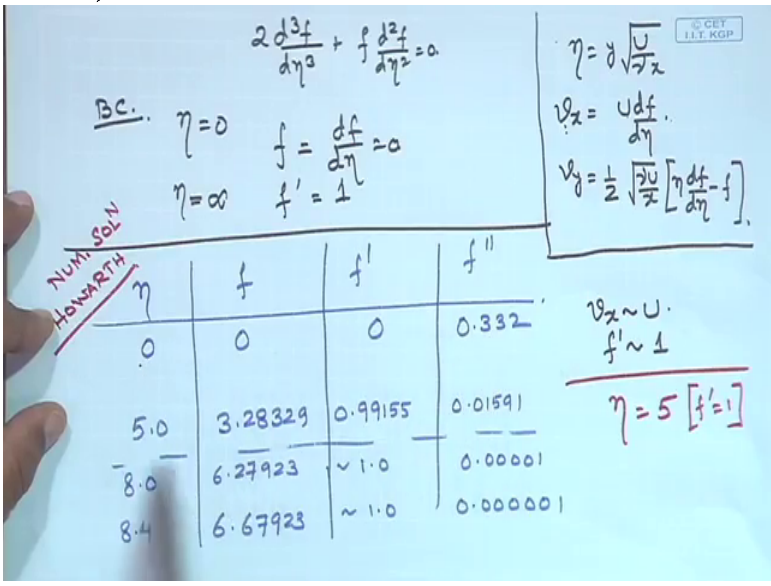

in this table. Because at eta equals 5, we see that f prime, which is, which is nothing but v x by u, so v x reaches 99 percent of the freestream velocity and so on and we have obtained this expression. The problem that we have right now tells us that use the numerical results of Howarth to evaluate the following quantities. This is laminar boundary layer flat plate, so therefore the Howarth solution, numerical solution of Howarth is available.

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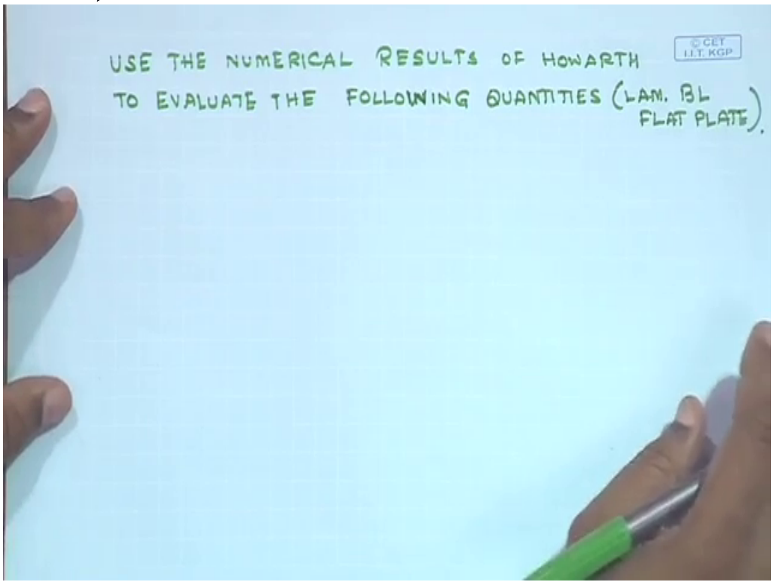

The first thing that you have to find out is what is delta star by delta and this you have to evaluate at eta equals 5 and when eta tends to infinity. We will come to the

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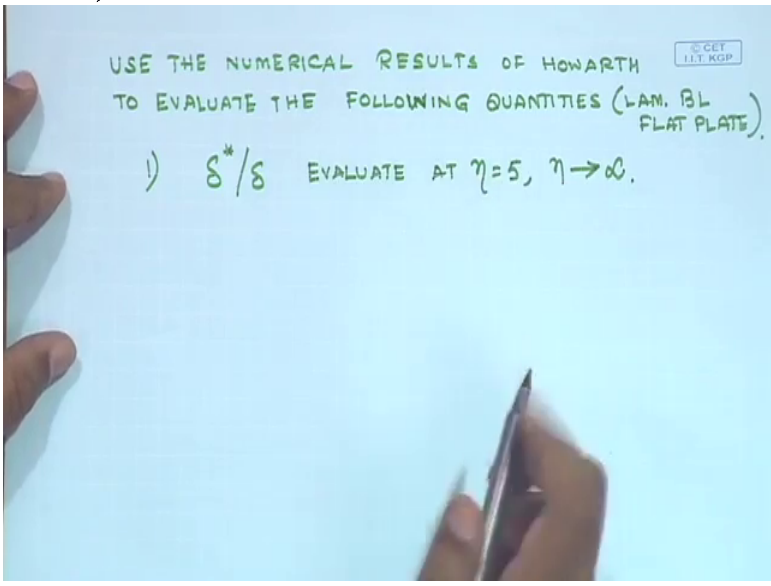

second part later. So let's first start with what is the expression for, the definition of delta star, the displacement thickness which is by definition x 0 to delta 1 minus v x by u times d y ok this we have derived in the,

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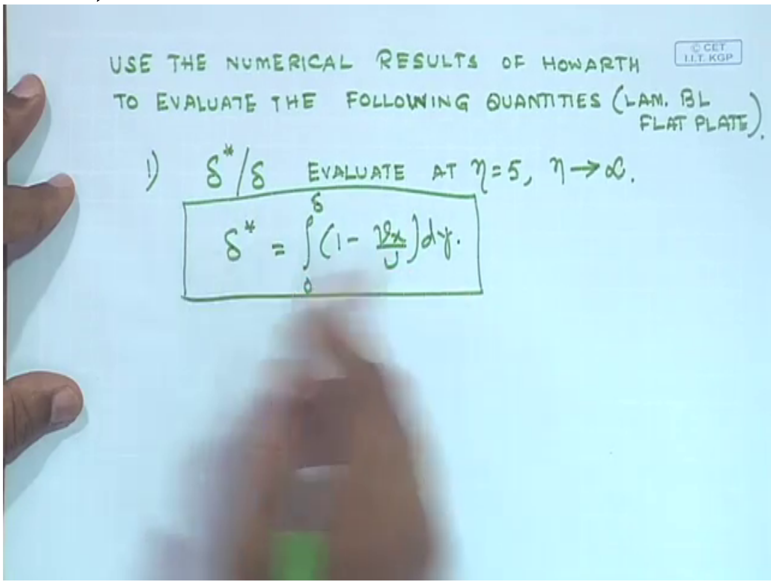

in one of the previous classes based on our analysis. And we also know that eta is the combination variable which is, which combines y and x and therefore one can write y is equal to eta times nu x by u and therefore d y is equals d eta root over nu x by u.

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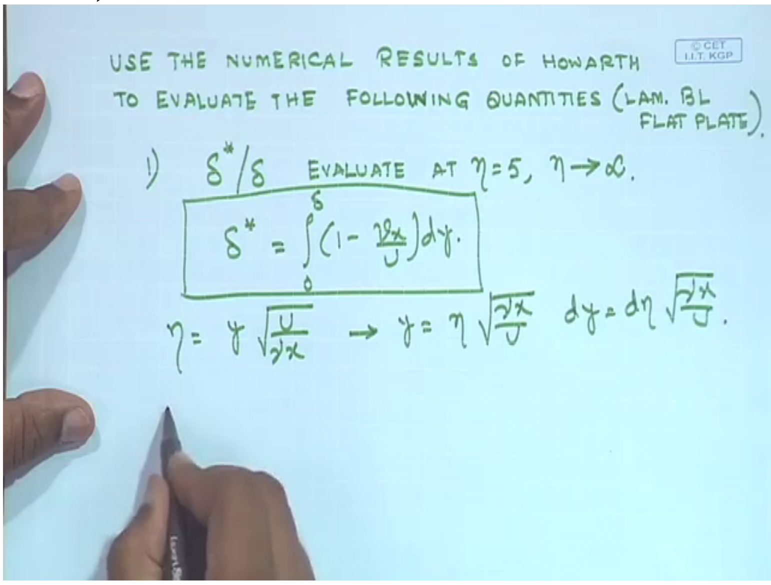

So delta star in this, if I convert this y to eta it changes from zero to some value of delta, that value of delta could be, could be different at different cases. In one minus v x by u but instead of d y I bring in this which is nu x by u times d eta ok and delta star

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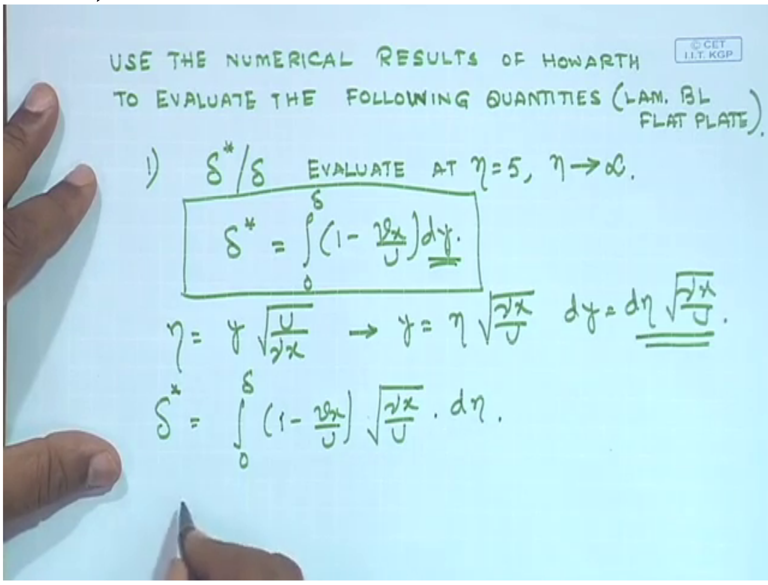

would then be equals, I take this outside, since the integration is on eta, nu x by u zero and 1 minus v x by u, we have seen in our, from our this analysis, v x

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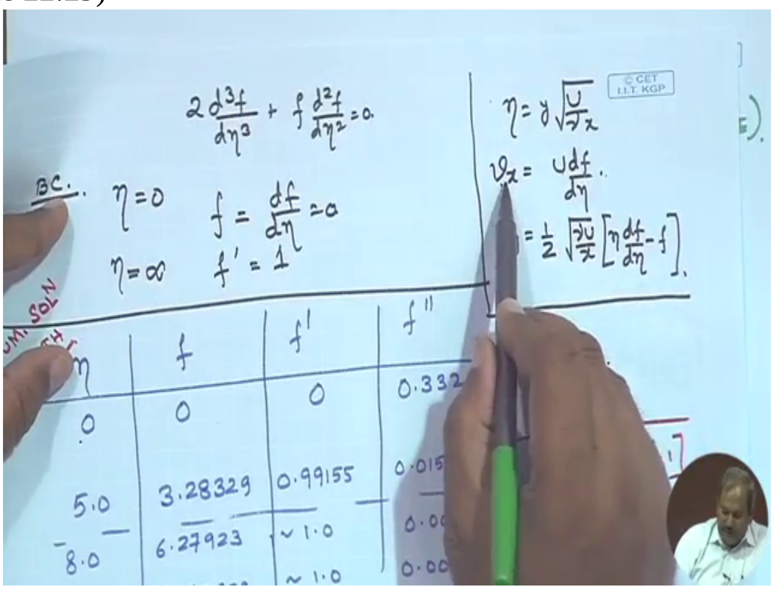

by u is simply equals d f d eta, so I will change this v x by u as d f d eta this is d eta and it is zero to eta now. The,

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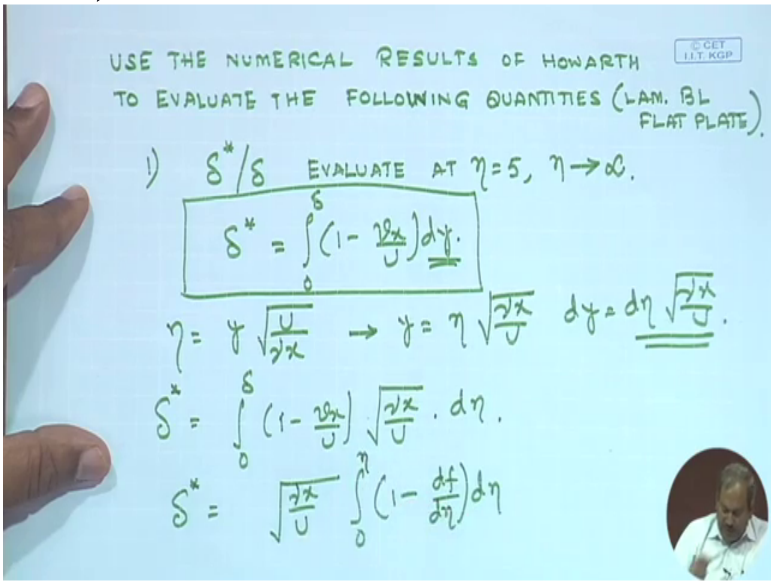

the limit of integration changes from delta to eta now since everything is in terms of eta, the independent variable is eta, so a simple substitution of v x by u from our, from our definition would essentially give me the expression for the displacement thickness as, as a function of this. We also understand that we have obtained a relation between, relation between this, the eta

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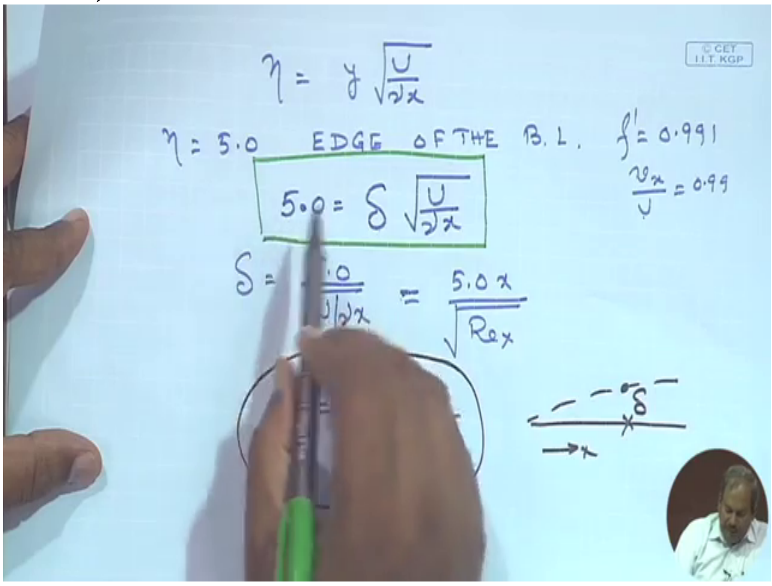

for a value of eta to be equal to 5, y would become equal to delta and this would be there. So from here, one can write that this part is simply going to be phi by delta. So if I see in here, this is simply going to be delta by 5. So my delta star would simply be equals delta by 5, zero to some value of eta, one minus d f d eta times d eta.

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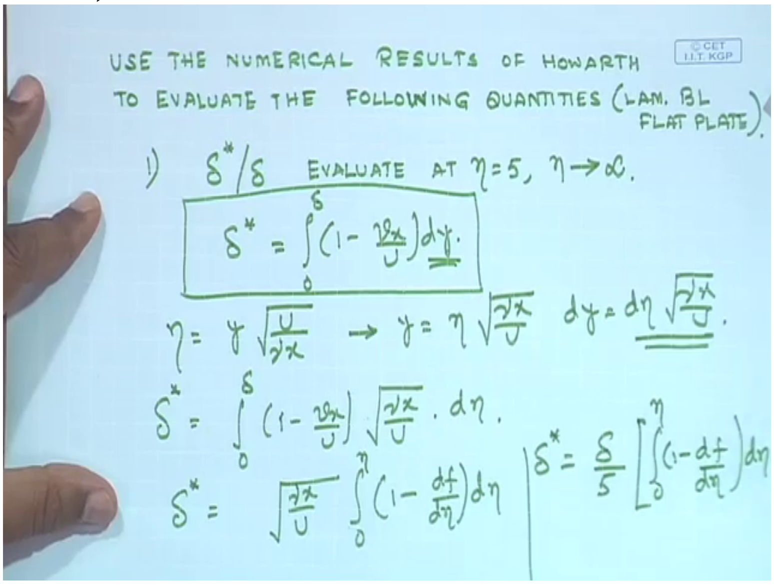

I will say it once again. This part is something that I am going to, I am substituting

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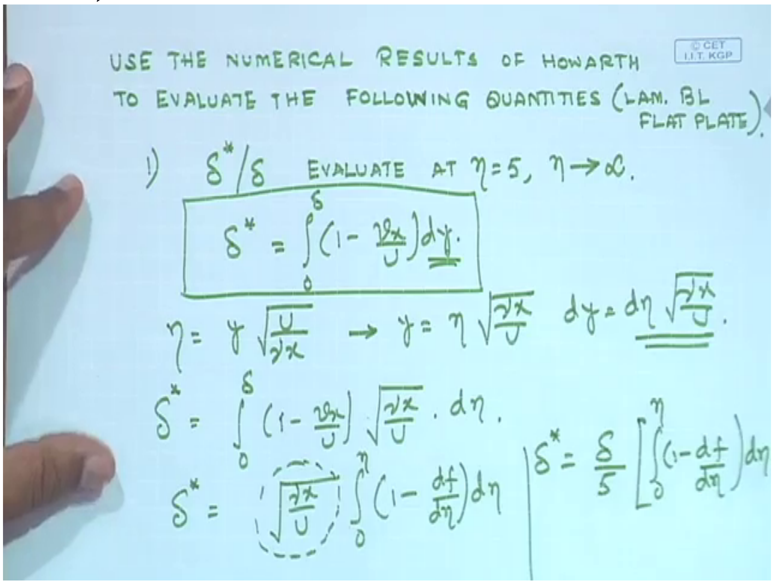

from here. So nu x by u is simply going to be equal to 5

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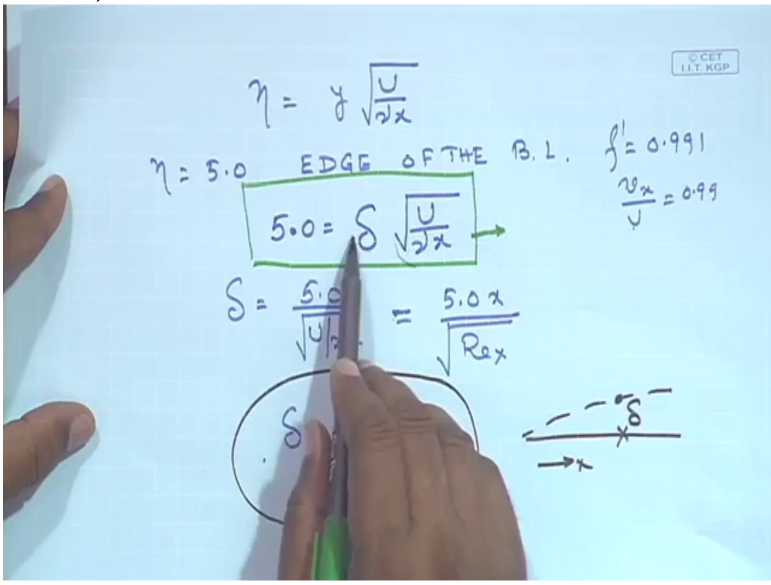

by delta, sorry delta by 5. If I bring it to this side nu x, root over nu x by u is simply going to be equal to delta by 5 from whatever we have done previously. Since eta equals 5, we reach the edge of the boundary layer. So at the edge of the boundary layer, y must be equal to delta. So through this, I will bring simply this as my expression. So I take it one step further. So

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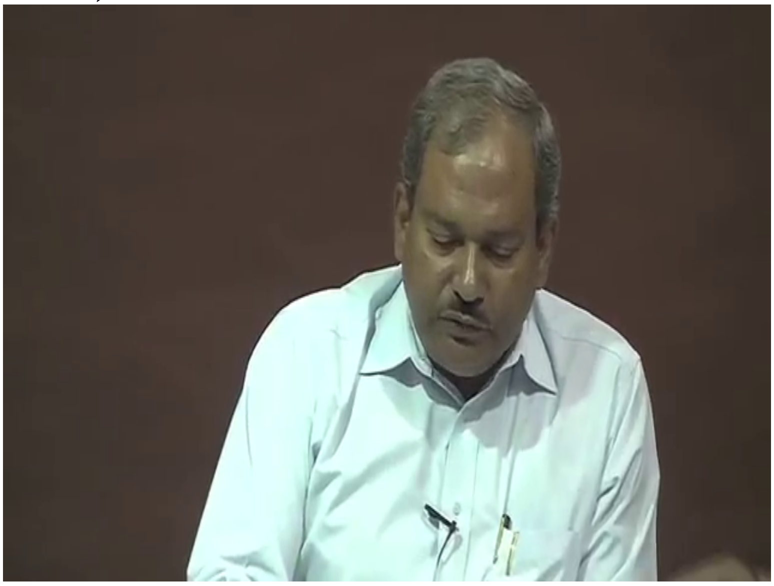

from this where we have written as this is equal to delta by 5, zero to some value of eta, 1 minus d f d eta times d eta I perform the integration now. So my

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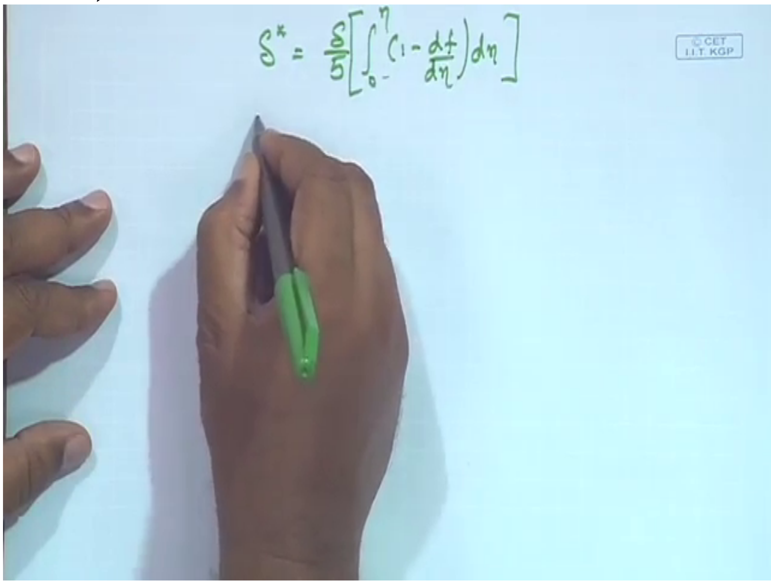

delta star by delta would simply be equals 1 by 5, if I do the integration, this becomes eta minus f and from zero to the value of eta.

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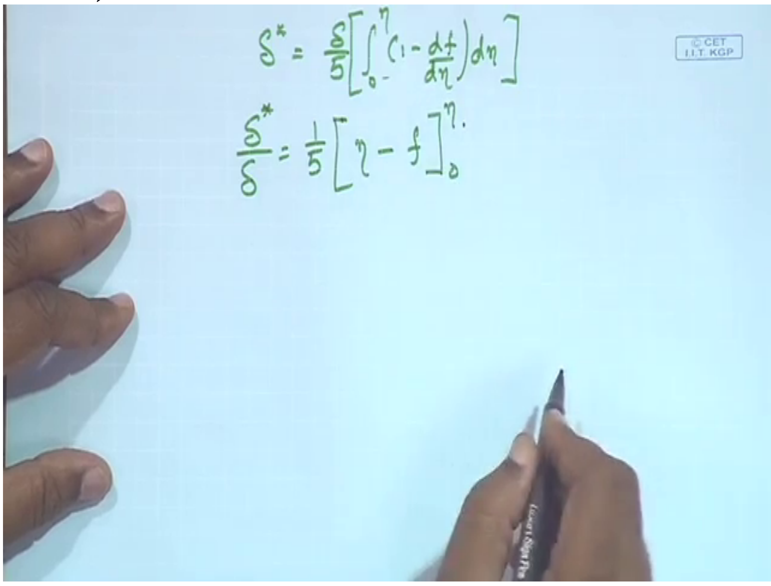

Now this is the complete expression that I am going to do, that I am going to use to obtain the, to obtain the solution to the problem where the value of delta star by delta was to be evaluated for eta equals 5 and as eta tends to infinity. So what

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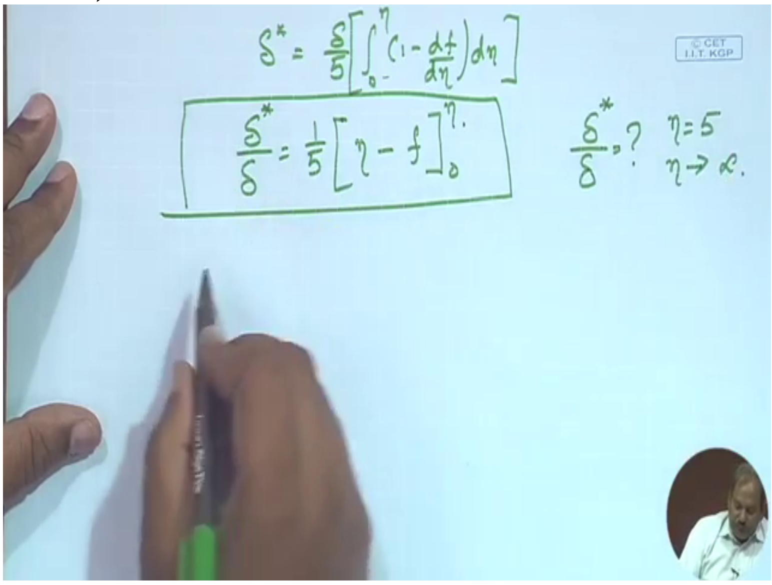

I do then is delta star by delta would simply be one by 5 and the value of eta is simply going to be 5 and the corresponding value of, corresponding value of f is to be provided and again I go back to

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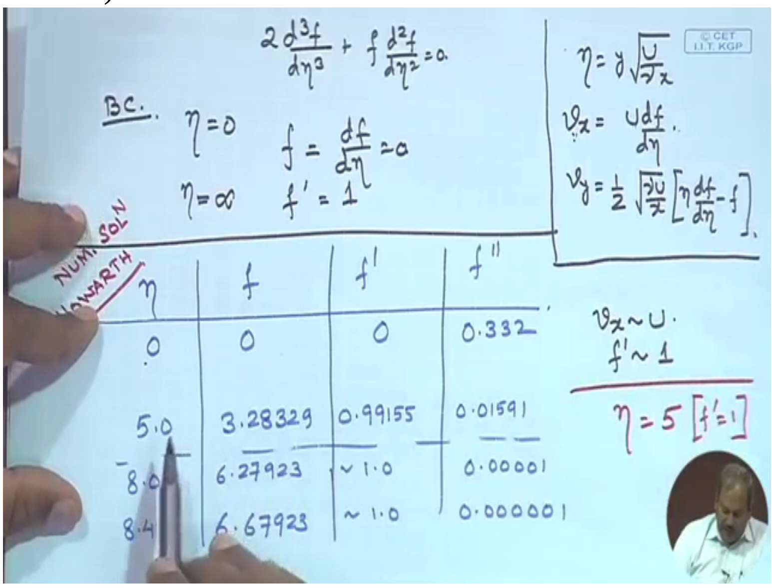

this table, for a value of eta equals 5, the value of f is 3 point 2 8 3 2 9. So this is going to be 3 point 2 8 3 2 9. So I calculate this and this is going to be equals point 3 4 3 3 4.

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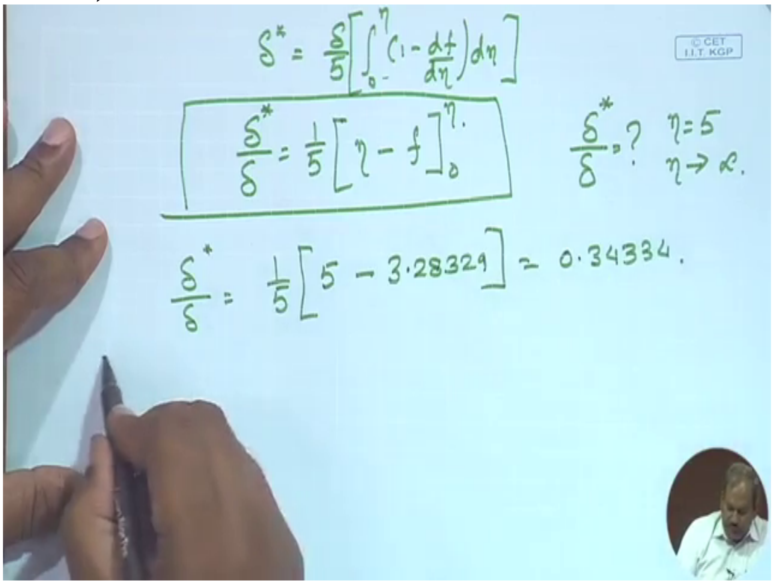

But the problem comes, how I am going to get the value of eta, how I am going to get the value of f when eta tends to infinity, Ok. Because if you look at the table and even if you look at the table,

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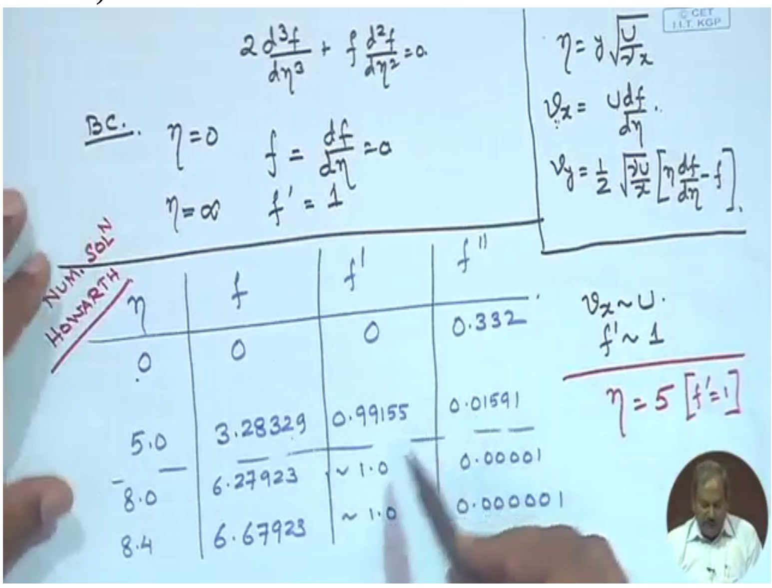

the complete table in your textbook the values of eta are provided, the values of corresponding f are provided but eta is never infinity. The value of f at eta at infinity is not provided. So how do I find out the value of this quantity eta minus f,

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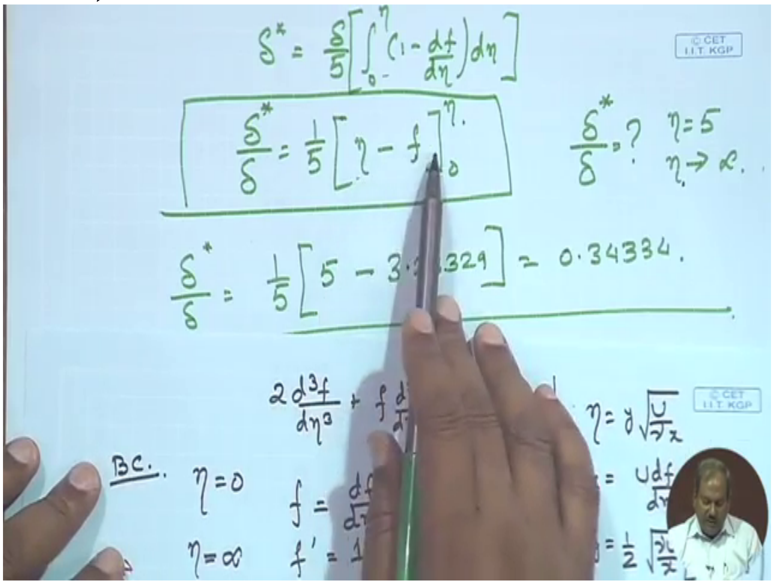

eta minus f when eta tends to infinity? You look at this table carefully.

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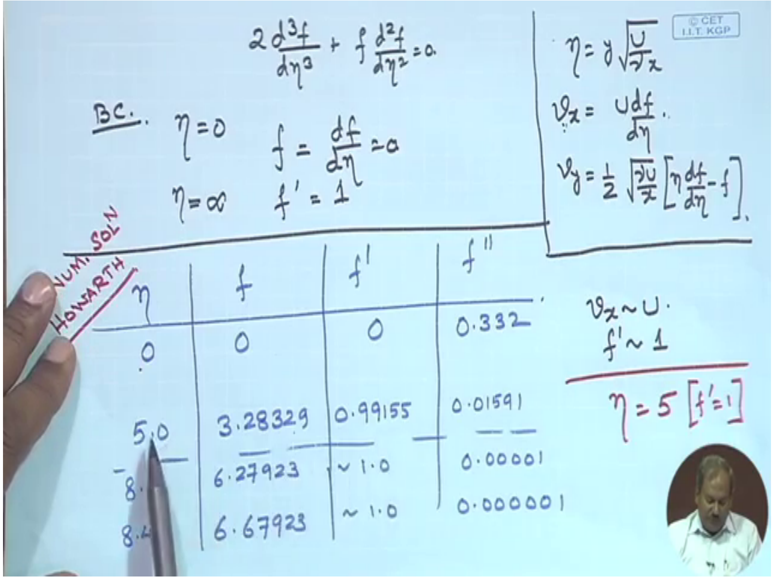

eta minus f, it is 5 minus 3 point 2 8, eta minus f and eta minus f, so if you look carefully that eta minus f and eta minus f, if you just take these two rows, the value of eta minus f,

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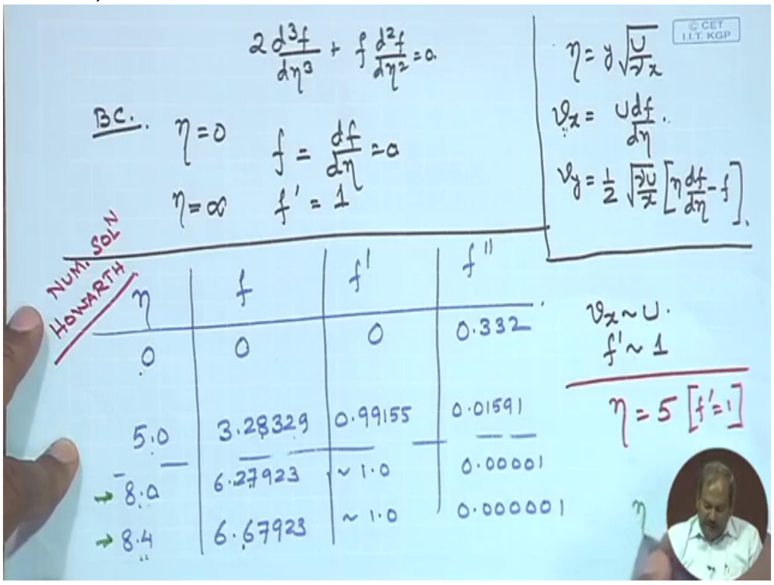

it becomes independent of the value of

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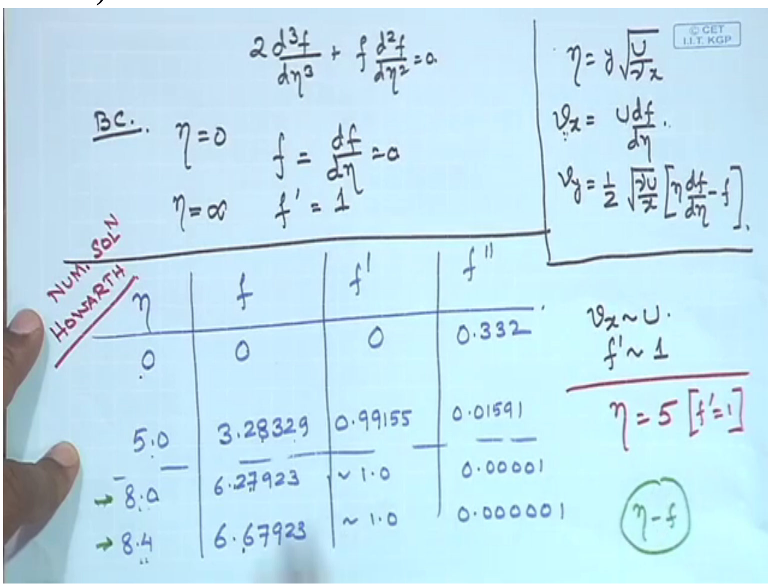

eta beyond eta equals 8, because 8 minus 6 point 2 7 and 8 point 4 minus 6 point 6 7, so the difference is independent of eta, so that is something which you need to, need to see, need to observe by looking at the data presented in the table. So really you do not know the value, do not need to know the value of f at eta equals infinity. Because it’s not the value of f that you would like to know. You would like to know the value of

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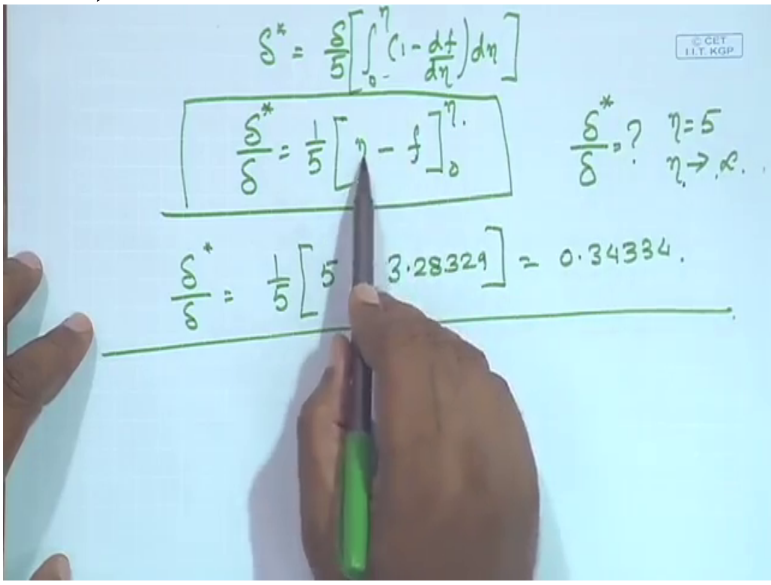

eta minus f when eta tends to infinity. And from the table it is apparent

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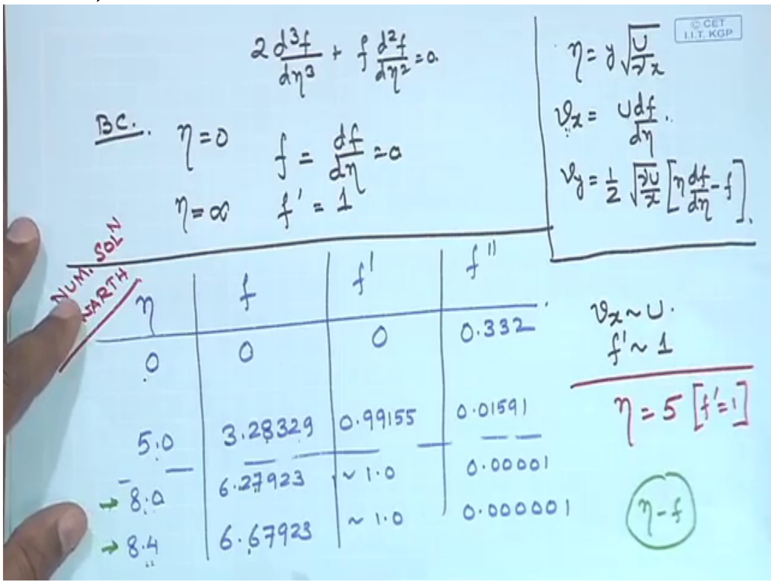

that eta minus f becomes independent or becomes a constant once you cross the value of eta, once you cross the value of eta to be equal to 8. So it really does not matter if you know what is the value of eta minus f at 8, at 4, at 8 point 4, at 100, at 1000 or at 10 to the power 5.

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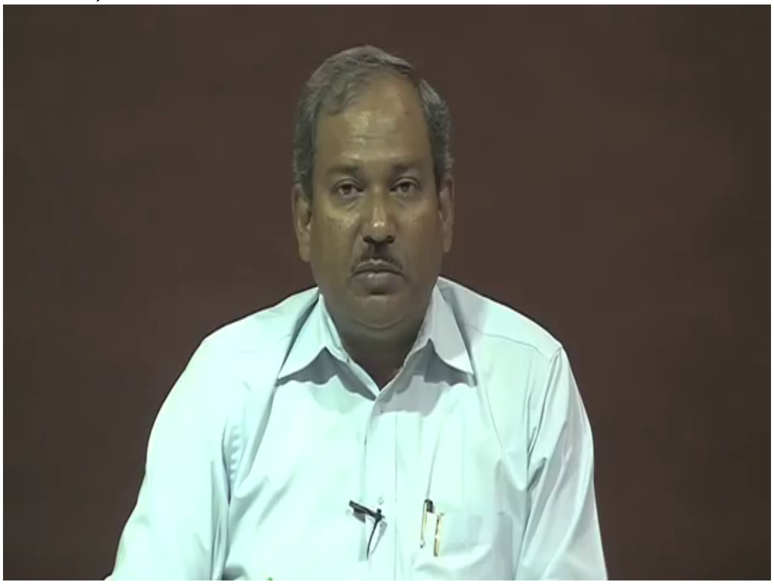

The difference will always remain the same. That is something which you have to identify, which you have to identify from the table numerical value, table that is provided in (()). So we use some value, any value since eta minus f eta becomes independent of eta for eta greater than 8, then

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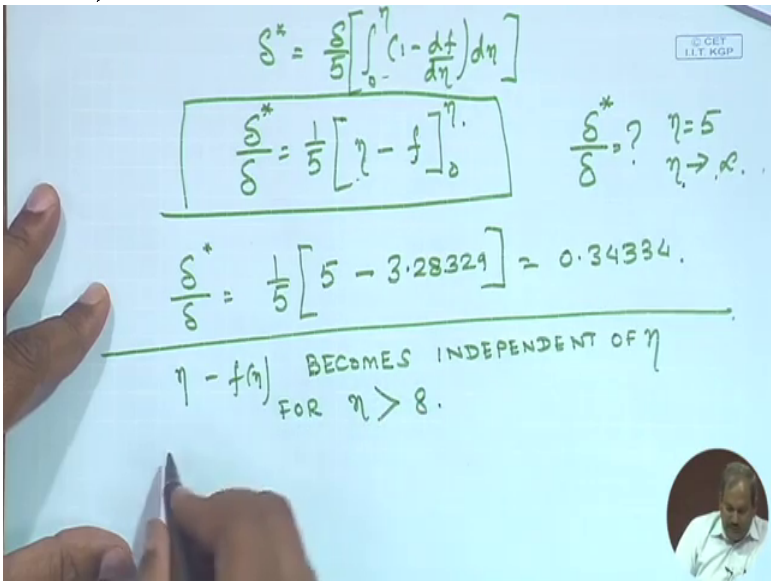

we would simply use del star by delta at eta equals 8 which is same as eta tends to infinity would be 1 by 5 8 minus 6 point 2 7 9 2 3 which would be equals 3 4 4 1 5. So you could compare these two values.

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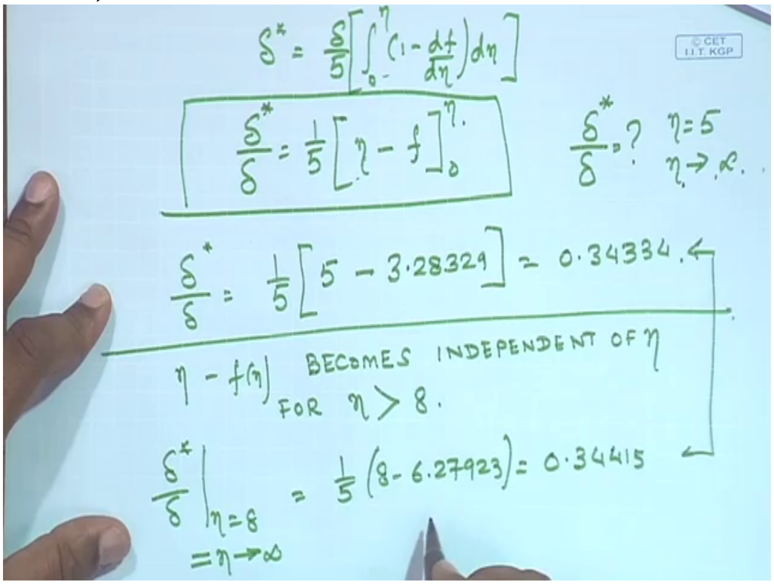

delta star by delta at eta equals 5 is almost equal to delta star by delta as eta tends to infinity. So this is, this is what you would see.

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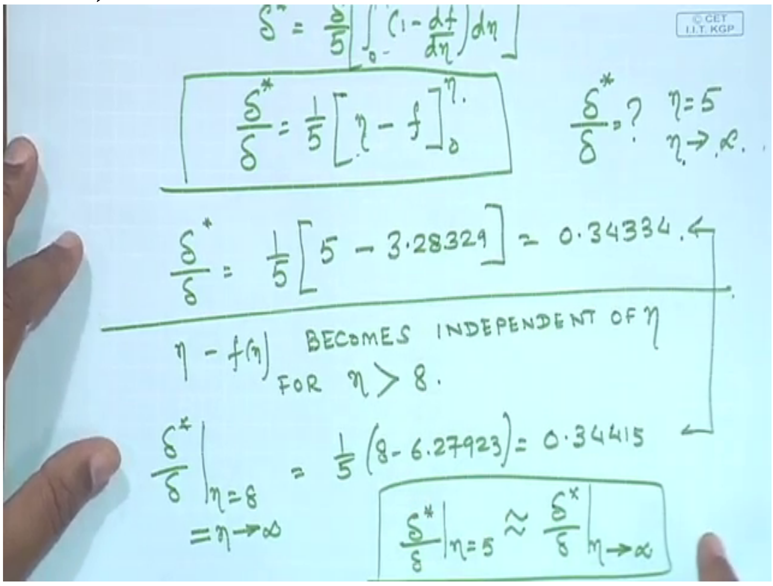

So just the use of the table would, would give you some idea of

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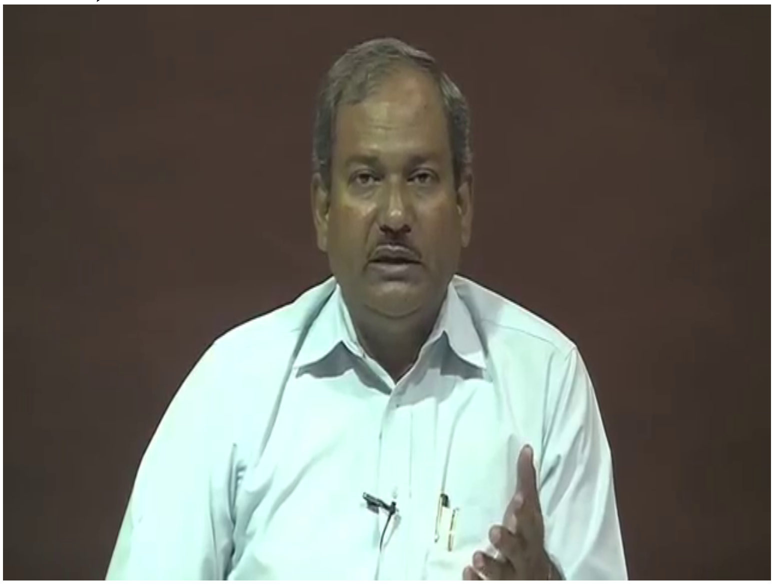

how does displacement thickness varies for different values of eta, for different values of y and so on.

The next question that I pose to you, I would not solve it is, in the, in this problem it was also asked that you calculate the value of v y by u at the, calculate v y by u at the edge of the boundary layer. So

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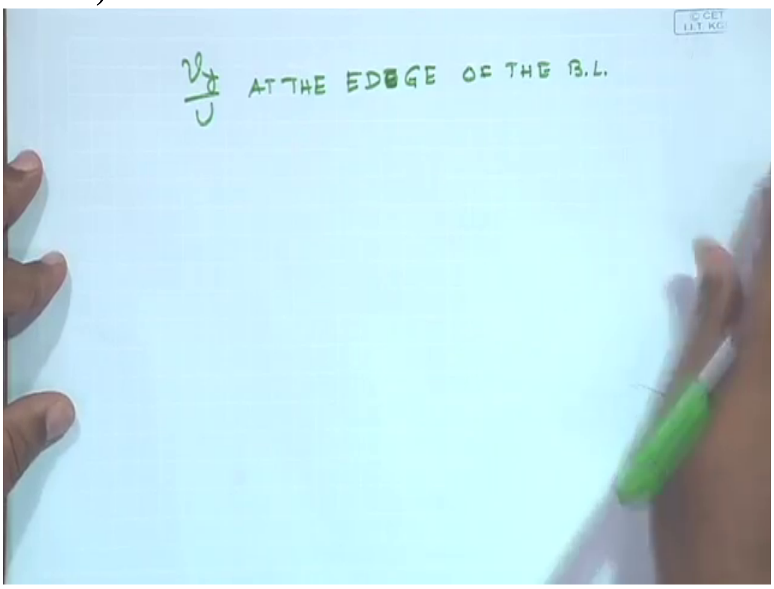

obviously I write the expression for v y which we have derived, which we have derived before. This is the expression for v y. So

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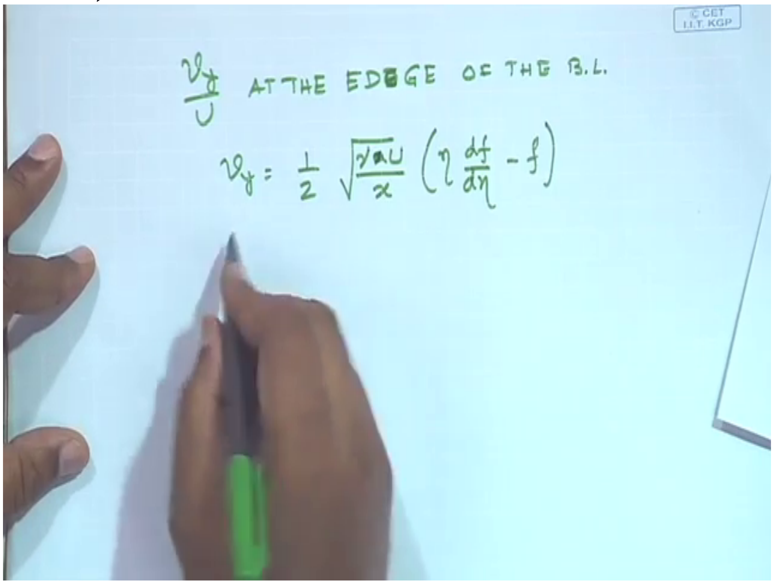

v y by u would simply be equals half nu, the u would come over this side and this can be, this can be expressed in terms of 1 by 2 root over R e x eta f prime minus f. So at

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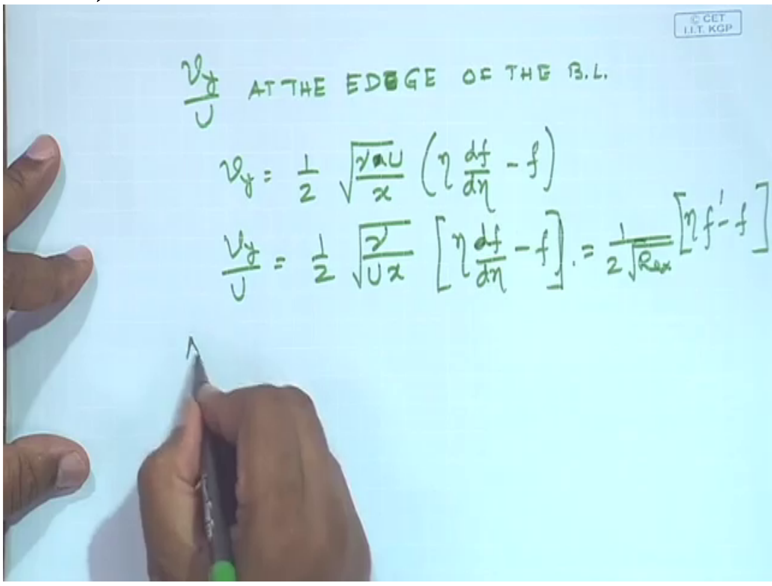

at the boundary layer, at the edge of the boundary layer, that means eta means 5, because we know that, we say that at eta equals 5, we reach the edge of the boundary

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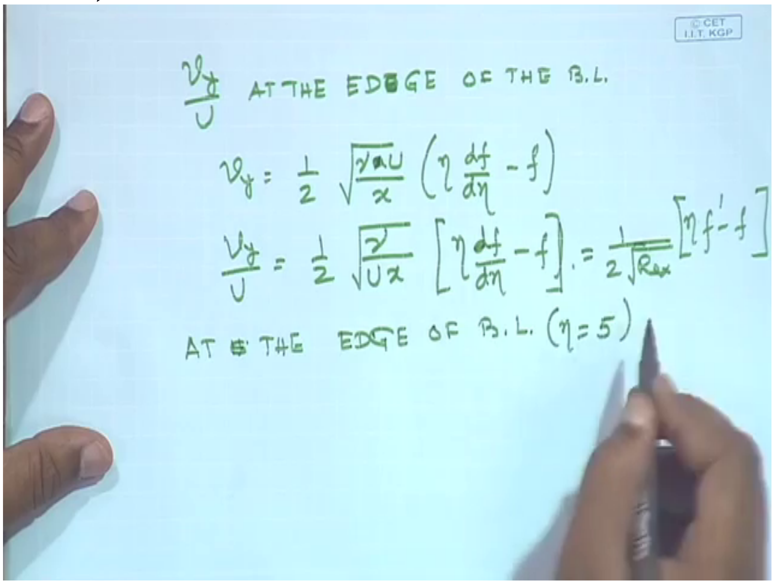

layer, v y by u would be 1 by 2 root over R e x. This is 5 into 0 point eta 5. f prime is point 9 9 1 5 5 an f is minus 3 point 2 8 3 2 9 or something, this you get from the table where v, therefore

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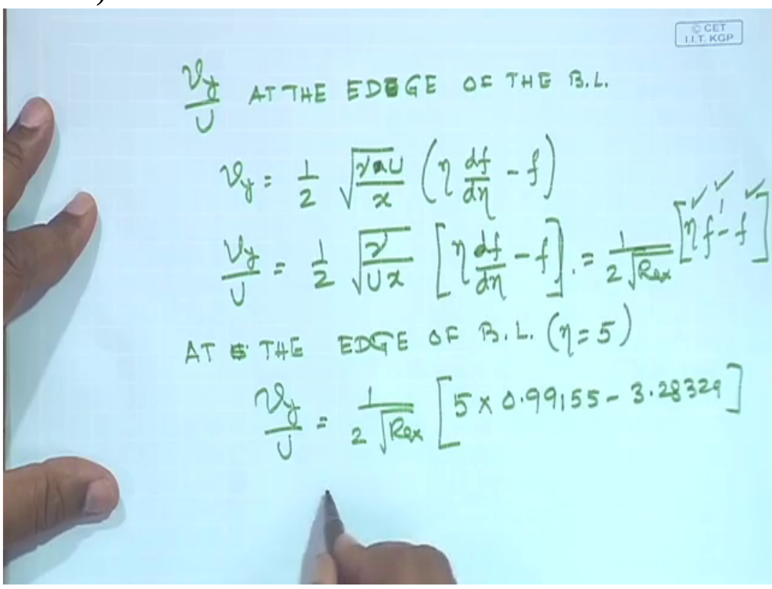

v y by u is equal to point 8 4 by root over R e x. This gives me

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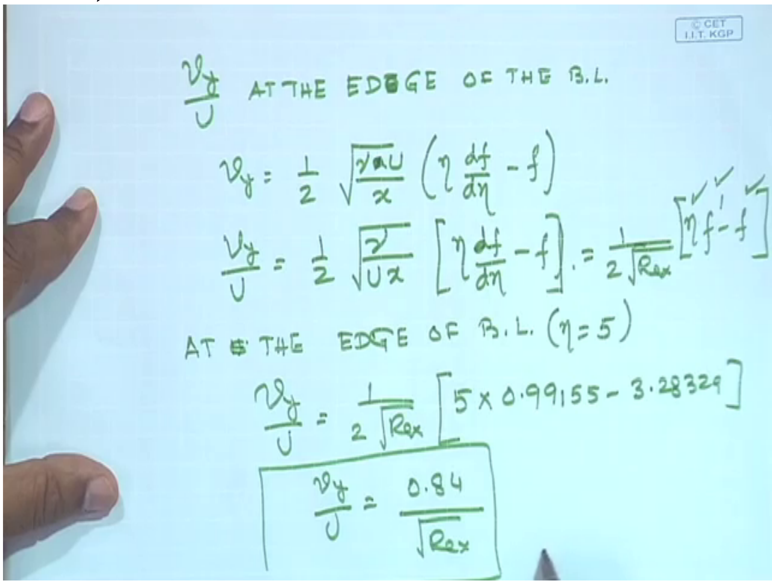

another insight into this boundary layer. So this is your boundary layer. Here you have v x and v y to be both zero. You have a v x in here and outside this, so you have v x, outside this this v x is going to be equal to u but what happens to v y in here? Do we have, we have a v x in here, do we have a v y? In other words

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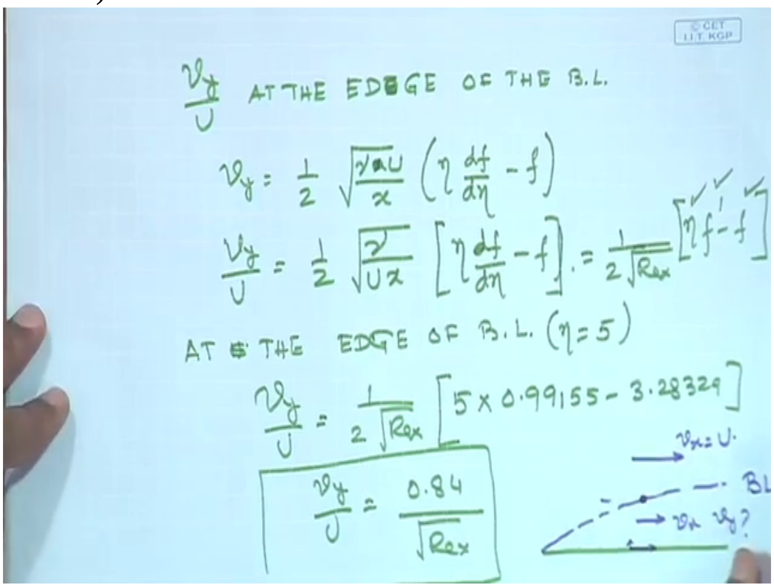

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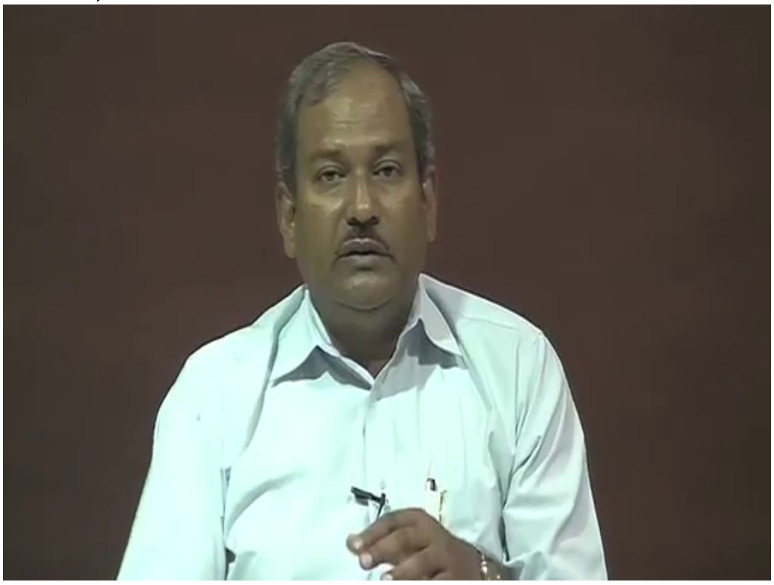

if someone poses this question to you is, can you call the edge of the boundary layer as a streamline? Is it a streamline? Now in order for it to be streamline, it has to satisfy the basic properties of the streamline. So what are the basic properties of a streamline? The basic properties of the streamline is that the particle which is on a streamline will remain on that streamline and nothing, no particle, no fluid can cross a streamline. The moment you have flow across a line, then that line cannot be a streamline. So you have yourself found out in the previous problem that at the edge of the boundary layer, v y is not zero. v y is a function of Reynolds number which you have obtained to be point 8 4 by root over R e,

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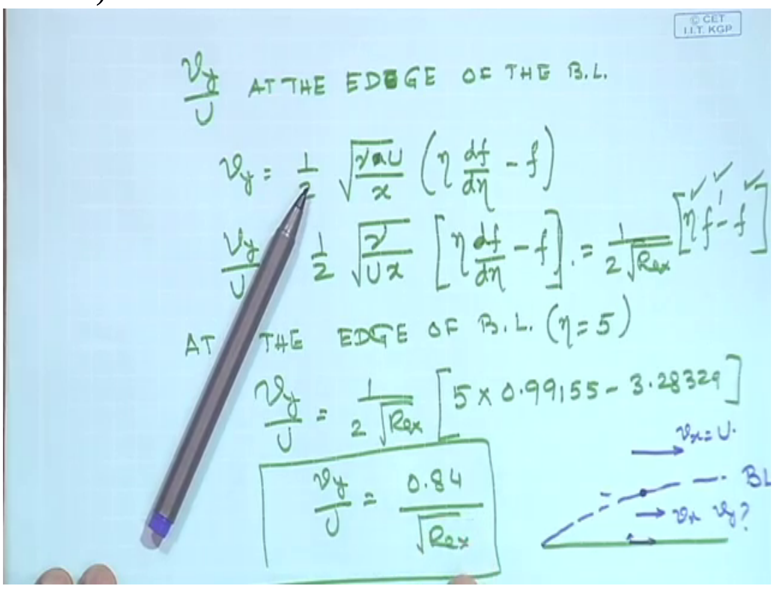

v y by u to be root over R e x but v y is not zero at the edge of the boundary layer. So if v y is not zero at the edge of the boundary layer

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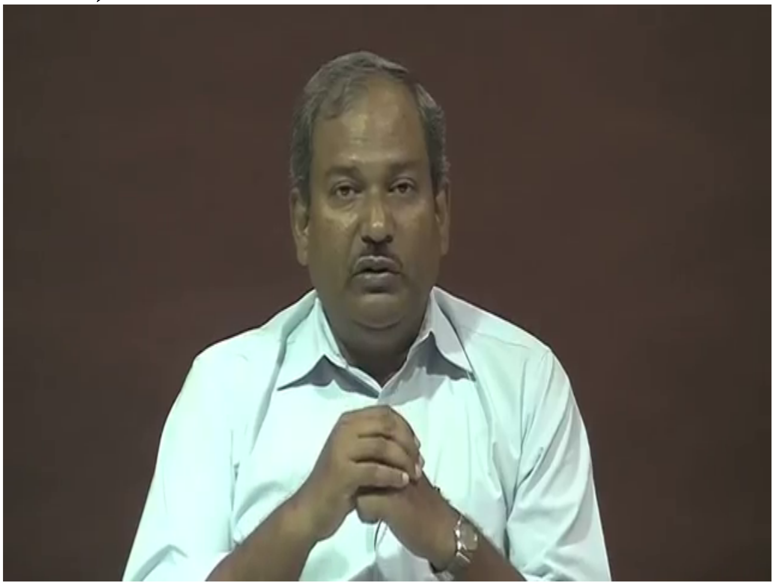

then the edge of the boundary layer cannot be a streamline, Ok. Since you have flow through the edge of the boundary layer, it is not a streamline. And of course since the boundary layer keeps on growing, keeps on, its thickness keeps on increasing; in order for, in order for, in order to sustain, in order to sustain the growth of the boundary layer, the progressive growth of the boundary layer along with flow you must have flow through the edge of the boundary layer. So the edge of the boundary layer is definitely not a streamline. So through these two examples I have tried to provide you with some insights into the concept of displacement thickness, the utility of numerical solution of Howarth and how we can use it to find the thickness of the boundary layer, the growth or the change in the pressure, the growth of the boundary layer, the concept that the edge of the boundary layer is not a streamline and so on.

However the limitations, I would stress on the limitations of this approach once again. It is valid for laminar flow. It is valid for the simplest possible geometry when you do not have, when you, simplest possible geometry of flow over a flat plate only is, it is therefore too restrictive, it is therefore represents, it therefore represents a situation which is an ideal situation. Because ideally you are probably going to get a turbulent flow and you are going to get a flow which is not over a flat surface and therefore the pressure gradient is not going to be equal to zero. So how do you handle such a problem?

In order to handle this problem, the simplest problem I had to solve numerically, first convert an o d e, first convert a p d e to an o d e and then numerically solve it. So this cannot be a convenient method to solve for complicated systems, Ok. So in the next class, in the next classes I would give you, I would show you a method which is known as the integral method or momentum integral equation which can be conveniently used to tackle problems of those types where the flow could be turbulent and where the flow could be on a surface which is not flat. So those are, those, those, that approach would be much more convenient than this approach. However as the name suggests, it's an integral method.

Whatever we have done so far is a differential approach where the profile, the information about the profile can be obtained at each and every point in the flow domain; that is differential approach. But when you go to the integral approach we are not interested in what happens at every point on the, in the flow field. I would like to know what is the, what is the condition, what happens at only those places which are places of interest for me, for example, what happens on the surface and what happens on the edge of the boundary layer?

What happens in between, I do not need to know exactly, a rough idea would do, but I would precisely like to know what happens at the interface, liquid-solid interface because that is essentially what is going to give me the expression for drag force; the idea of the drag force, the force exerted by the moving fluid on the solid plate. So that is where I would like to know what is happening precisely. But in-between I am not interested that much. So I am ready to accept little bit of approximation and errors; if I can quickly get to the solution of finding the drag force at the solid-liquid interface and not knowing everything in-between. So momentum integral approach is a convenient approach but it is also an approximate approach; so we would try to, we would try to do that in our subsequent classes.