Advanced Mathematical Techniques in Chemical Engineering

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Lecture No. # 35

Solution of non-homogeneous Elliptic PDE

Very good morning, everyone. So, we are looking into the solution of linear, but non-homogeneous partial differential equation using Green's function method.

We have developed the theory in last few classes for the Green's function solution, and checked out whatever the procedures; let me repeat it, first, given a problem in non-homogeneous partial differential equation in the governing equation as well as in the boundary condition, we have to find out the adjoint operator. Once, we find out the adjoint operator, then you have to check whether the operator is self adjoint or not.

Next, you have to formulate the causal Green's function, and there are three rules, thumb rules for formulation of the causal Green's function. First of all, the non-homogeneous, the operator remains the same for the causal Green's function as in the original problem. The non-homogeneous term in the governing equation has to be replaced by a unit step function at a particular location in the domain, and then we have to force the boundary conditions to be homogeneous, once, boundary and initial condition to be homogeneous. So, once we do that, then we will be getting the solution of causal Green's function, after that, basically the idea is have to see the effect of the 1 unit step function, a non-homogeneity, it will be replaced by the 1 unit step function forcing others to be 0, so, that one can track, hardly, with the unit impulse or unit non-homogeneous term will be giving the response in the whole system, and based on that we will be mapping on and connect it with the actual problem.

So, step number one is construction of causal Green's function; step number two is the construction of evaluation of adjoint operator. If adjoint operator is same as the operator, then the whole problem has been simplified. From the expression of Green's function we will be writing the expression of adjoint Green's function by changing the subscript x 0 and x 1, and then we will be connecting the adjoint problem with the original problem, and we will be getting the complete solution.

If the operator is not self adjoint operator, then you have to find out the self adjoint operator, and then we will be getting the solution of adjoint Green's function by changing the subscript, and then we will be connecting the adjoint Green's function with the original problem, and we will be getting the complete solution.

Once, we do that, we demonstrated this method by solving an ordinary differential equation, and we have identified that, depending upon the number of non-homogeneities in your system whether in the governing equation or whether in the boundary conditions, there will be, the solution will be containing those many number of terms. Because of the presence of non-homogeneity in the governing equation, the solution will be containing a volume integral that will be corresponding to the non-homogeneity in the governing equation.

Because of the presence of non-homogeneity in the boundary conditions, the solution will be containing a surface integral over the boundaries, which will be corresponding to a non-homogeneous boundary conditions.

So, if there are three sources of non-homogeneity, one, we will be getting with, the solution will be composed of three terms- one volume integral term and two surface integral term; if there are four sources of non-homogeneity, one volume integral and three surface integrals will be present corresponding to four sources of non-homogeneity in the, in the actual definition of the problem.

So, we looked into the parabolic partial differential equation first, because the parabolic partial differential equations are quite common, and they represent an unsteady state process in any chemical engineering operation.

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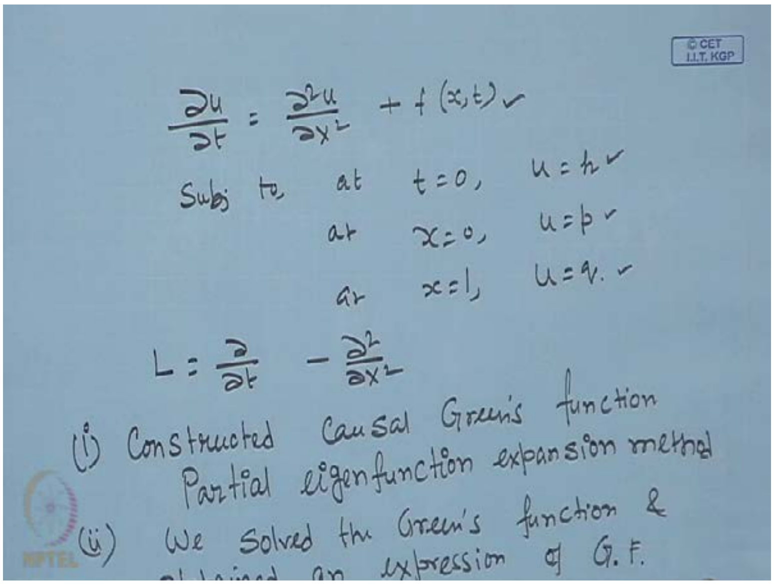

So, we are looking into the system del u del t is equal to del square u del x square plus f of x t subject to the boundary condition at t is equal to 0 it was u is equal to h, boundary conditions at x is equal to 0 we had u is equal to p, at x is equal to 1 we had u is equal to q.

So, we consider the dirichlet boundary conditions on u, and there are four sources of non-homogeneity in our problem. So, in the last class, we constructed the causal Green's function, we solved the causal Green's function using partial Eigenfunction expansion method, and then we have obtained the adjoint Green's function, and with the adjoint operator was not the same operator.

So, L is equal to del del t minus del square del x square. So, I am just summarizing whatever we have done in the last class in context of solution to this problem. So, what we did? We constructed causal Green's function; the method that we have adopted is partial Eigenfunction expansion method. Second, we completed the solution of Green's function, and we solved the Green's function and obtained an expression of Green's function.

Next, what we did? We evaluated the adjoint operator- that was the step number three.

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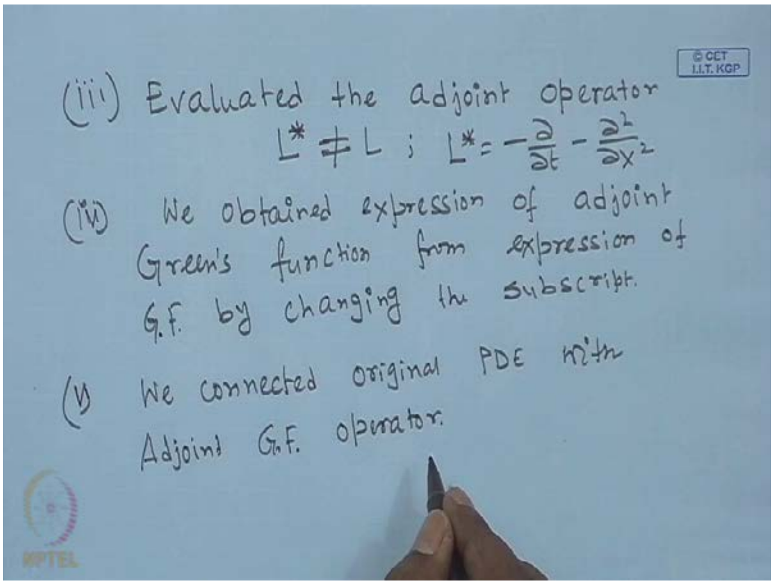

We evaluated the adjoint operator and found out that adjoint operator is not same as the original operator, the adjoint operator turned out to be in this case, minus del del t del square del x square.

Next, what we did? We obtained expression of adjoint Green’s function from expression of Green’s function by changing the subscript and using the relationship between g and g star; and after that, what we did? We connected original PDE with adjoint Green’s function operator. And what we did after that? We simplified, so, basically we have taken the inner product of original equation with the g star and the inner product of governing equation of g star with u, and then we subtracted, and that way these two are connected.

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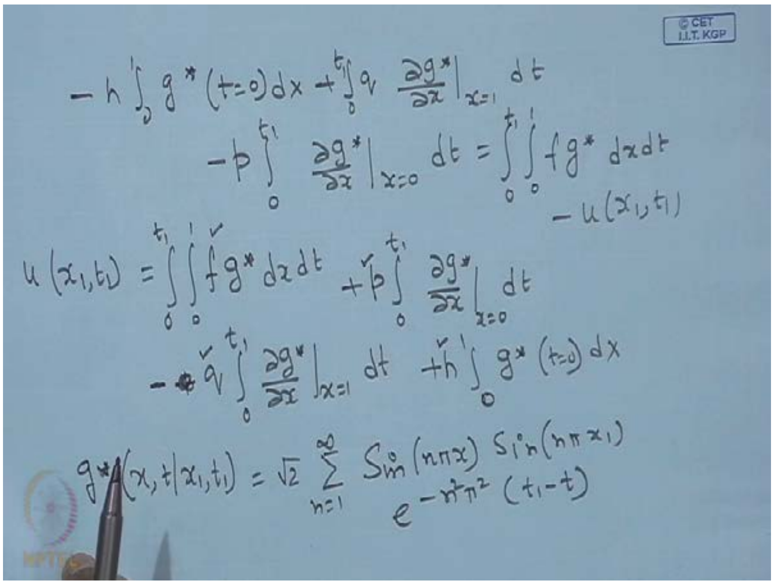

So, finally I am just writing the final- and we simplified the in between steps- and the final expression we obtained is that: on the left hand side, you had minus h, if h is constant, it will be 0 to 1 g star at t is equal to 0 d x plus 0 to t 1 q, q is constant, del g star del x evaluated at x is equal to 1 d t minus p integral 0 to t 1 del g star del x at x is equal to 0 d t is equal to right hand side, 0 to t 1 0 to 1 f g star d x d t minus u of x 1 and t 1. Now, what we did? We obtained up to this expression, then what we should do? We take this one on the other side and see what we get.

So, u x 1 t 1 is nothing but double integral f g star d x d t 1 over x 1 over t plus p 0 to t 1 del g star del x evaluated at x is equal to 0 d t plus q 0 to t 1 del g star del x evaluated at x is equal to 1 d t- this will be minus when you change the sign- and this will be plus h 0 to 1 g star at t is equal to 0 times d x. So, if you see on the right hand side, there are four terms corresponding to four non-homogeneities in the governing equation.

So, this volume integral, since it is a 2 dimensional problem, double integral is nothing but a volume integral, this volume integral corresponds to the non-homogeneous term in the governing equation; this surface integral over t corresponds to the non-homogeneous initial condition in the governing equation in initial condition of the original problem; and this non-homogeneous term corresponds to the non-homogeneity occurring at the boundary condition of the original problem; and this non-homogeneous term will be corresponding to the non-homogeneity present in the initial condition of the original problem; and this non-homogeneous term is corresponding to the non-homogeneous term present in the boundary condition of the original problem at the boundary x is equal to 0.

So, there will be four integrals, I 1, I 2, I 3 and I 4. So, what I will do, I will just evaluate one of these integrals one after another to demonstrate this problem. So, for demonstration, what you need? You require the expression of g star, and we have already developed and obtained the expression of g star, that is nothing but root over 2 summation n is equal to 1 to infinity sine n pi x sine n pi x 1 e to the power minus n square pi square t 1 minus t.

Now, in order to evaluate these three integrals, what we need? To evaluate, we need to evaluate g star at t is equal to 0, that is number one; second, we have to evaluate del g star del x at x is equal to 0; we have to evaluate del g star del x at x is equal to 1.

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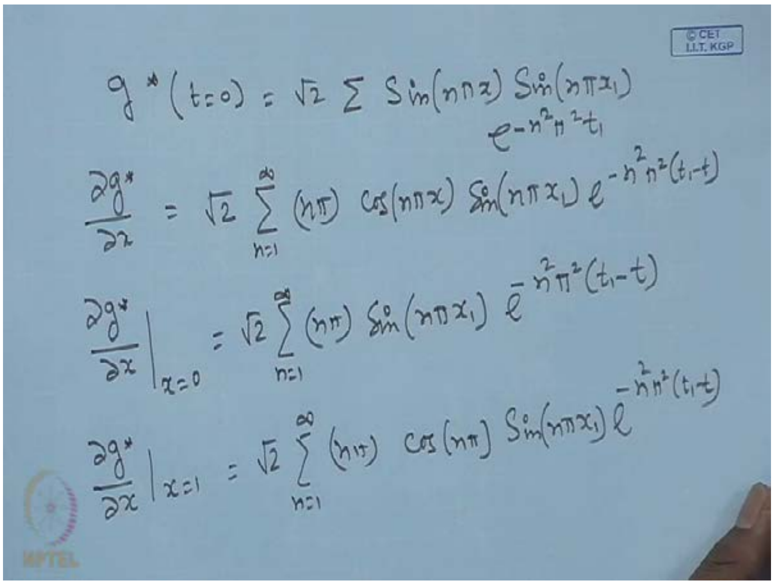

So, let us do that. So, g star evaluated at t is equal to 0 is nothing but root over 2 summation sine n pi x sine n pi x 1 e to the power minus n square pi square t 1 del g star del x will be nothing but, so, let us first differentiate this one, n is equal to 1 to infinity n pi cosine n pi x sine n pi x 1 e to the power minus n square pi square t 1 minus t, so, that is, del g star del x. So, del g star del x evaluated at x is equal to 0 is root over 2 n is equal to 1 to infinity n pi sine n pi x 1 e to the power minus n square pi square t 1 minus t, and del g star del x evaluated at x is equal to 1 is equal to root over 2 summation n is equal to

1 to infinity n pi cosine n pi sine n pi x 1 e to the power minus n square pi square t 1 minus t. So, we evaluate these four terms, and then will be in a position to evaluate all these integrals one after another.

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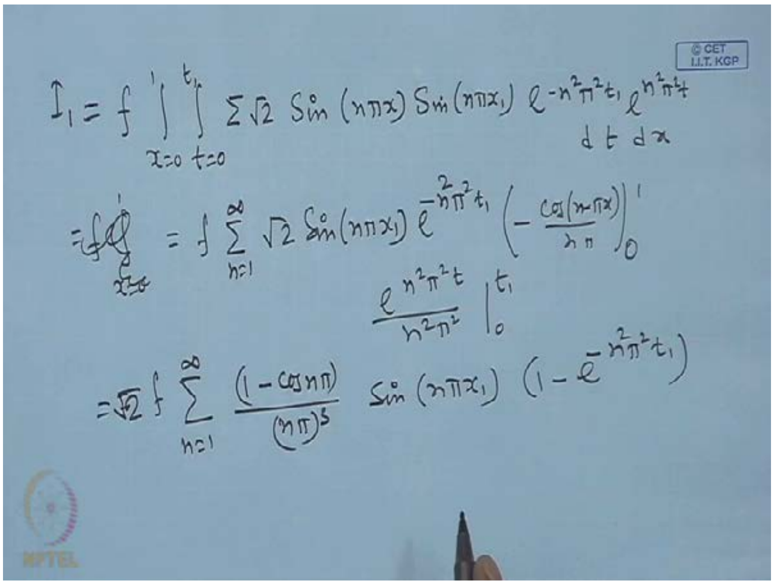

Let us talk about the volume integral is I 1, and consider, assume f is constant, so, f is brought out of the integral sign, so, integral from 0 to x is equal to 0 to 1 from t equal to 0 to t 1, so, this will be summation root over 2 sine n pi x sine n pi x 1 e power minus n square pi square t 1 e to the power n square pi square t d t d x.

So, this will be nothing but f x is equal to 0 to 1, so, we do the integration with respect to t first, then we will do the integration with respect to x, or we can do it simultaneously; so, since this is a product term we can do the integration over t varying part, whenever we will be doing the integration of t and x, the terms containing x 1 and t 1 will be treated as constant- so, I am just writing directly, probably omitting one step in between- n equal to 1 to infinity root over 2 sine n pi x 1 e to the power minus n square pi square t 1, and then the integration of sine n pi x d x, integration of e to the power n square pi square t d t.

So, this will be minus cosine n pi x divided by n pi, so, this from 0 to 1, and e to the power n square pi square t divided by n square pi square, so, this will be 0 to t 1.

So, this will be f root over 2 f summation n is equal to 1 to infinity 1 minus cosine n pi divided by n pi to the power 3 sine n pi x 1 1 minus e to the power minus n square pi square t 1- so that is I 1.

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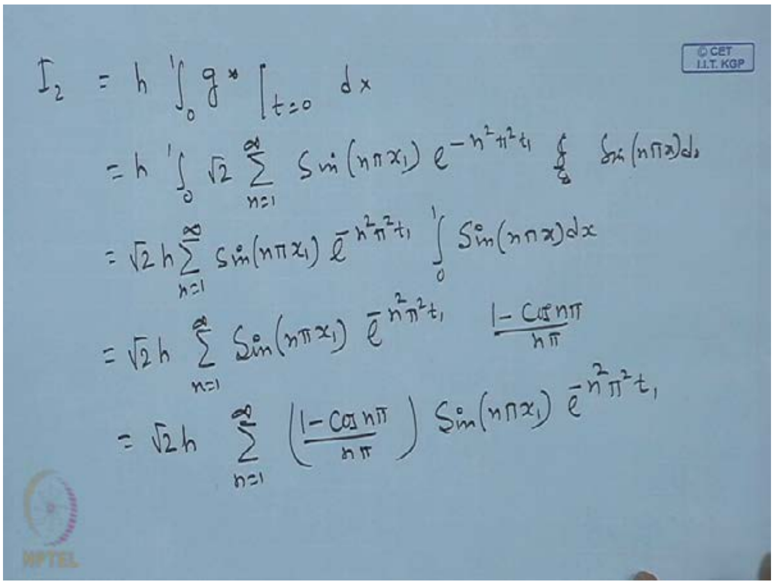

Now, let us evaluate I 2, I 3 and I 4. I 2, I am just evaluating in detail. I 2 will be h 0 to 1 g star at t is equal to 0 times d x; so, h and g star at t is equal to 0, we just put 0 to 1 root over 2 summation n is equal to 1 to infinity sine n pi x 1 e to the power minus n square pi square t 1 0, integration is there, so, it will be sine n pi x d x; so, this will be nothing but root over 2 h n is equal to 1 to infinity sine n pi x 1 e to the e power of minus n square pi square t 1 integral 0 to 1 sine n pi x d x; root over 2 h n is equal to 1 to infinity sine n pi x 1 e to the power minus n square pi square t 1 1 minus cosine n pi divided by n pi; so, second term becomes root over 2 h n is equal to 1 to infinity 1 minus cosine n pi divided by n pi sine n pi x 1 e to the power minus n square pi square t 1.

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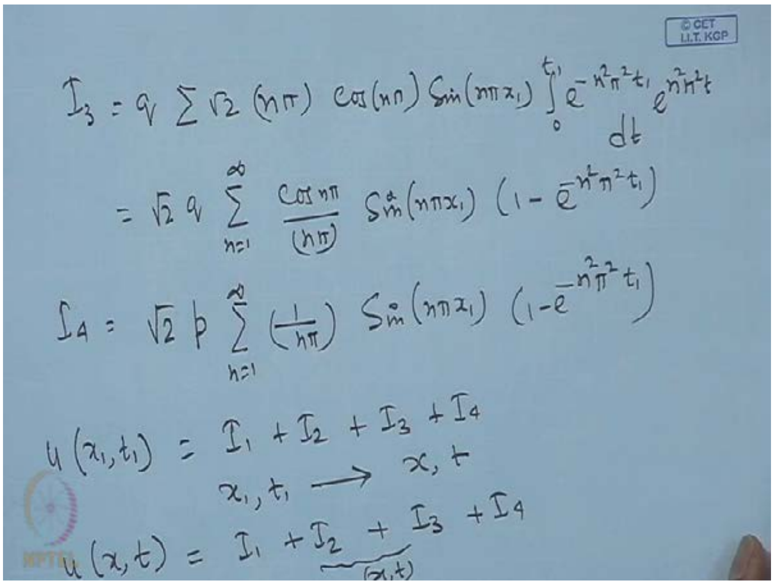

So, we evaluate the third integral, so, this goes, the second integral, then we evaluate the third integral. Third integral goes like this, I 3 is equal to q summation root over 2 n pi cosine n pi sine n pi x 1 0 to t 1 e to the power minus n square pi square t 1 e to the power n square pi square t d t, this term, t 1 continuing term will be treated as a constant; so, this will be root over 2 q summation n is equal to 1 to infinity cosine n pi over n pi- so, there will be n square pi square there, so, 1 n pi will be cancelled out- sine n pi x 1 1 minus e to the power minus n square pi square t 1.

Then, I evaluate the term number four, I 4. And I 4, I am just writing the complete solution directly, omitting a step in between, n is equal to 1 to infinity 1 over n pi sine n pi x 1 1 minus e to the power minus n square pi square t 1.

So, if you see that each of the term I 1 to I 4, they are containing x 1 and t 1, so, we will be getting x 1 t 1 is equal to I 1 plus I 2 plus I 3 plus I 4, each of these terms containing their function of x 1 and t 1; then we change the running variable x 1 t 1 into x and t on both the sides and we will be getting the complete solution of u x t at I 1 plus I 2 plus I 3 plus I 4, which are functions of x and t. And you will be getting an analytical solution for this particular problem using Green’s function method, and these methods demonstrates that, how one will be getting an analytical solution using Green’s function method, and combining the separation of variable problem with the Green’s function method to obtain the complete solution.

Now, so, we finish the parabolic partial differential equation, non-homogeneous, which has to be solved using Green's function method, and I demonstrated how a parabolic partial differential equation can be obtained by using partial Eigenfunction expansion method.

Now, next, what I will be taking up? I will be taking up the parabolic partial differential equation, but with the various other boundary conditions, and see how the Green's functions will be look like and how the solution will be composed of the eigenvalues and eigenfunctions.

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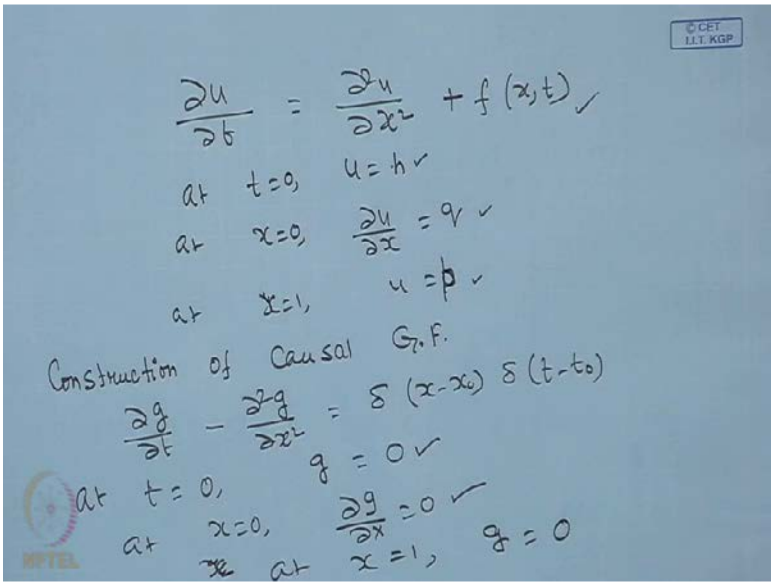

Now, let us look into this problem. I will not be completely solving this problem because earlier problem is completely solved, so, if anyone is interested, you can solve these problems completely, I will just proceed up to some crucial points, and then hand it over to you.

So, this is the governing equation, parabolic partial differential equation we are talking about at t is equal to 0 u is equal to h at x is equal to 0 we have del u del x is equal to q, and at x is equal to 1, we have u is equal to 0, u is equal to p. Suppose these are the solutions, this is the governing equation, there are three sources of non-homogeneity, all the initial and boundary conditions are non-homogeneous, and the governing equation containing this non-homogeneity f as a function of x and time.

And, so, what will be the construction of causal Green's function? The governing equation of causal Green's function looks something like this, it will be del g del t minus del square g del x square is equal to delta x minus x naught delta t minus t naught; so, that is the construction of causal Green's function. We have replaced the non-homogeneous term in the governing equation by a Dirac delta function of 2 dimension, and then we write down the initial and boundary conditions at t is equal to 0 g is equal to 0 at x is equal to 0, and x is equal to 0, we should have del g del x is equal to 0, at x is equal to 1 we have g is equal to 0.

Now, if you look the initial condition was, u equal to h, so, we write g is equal to 0, we homogenize the initial condition, then the boundary condition was, x is equal to 0, del u del x was equal to q, so, del g del x will be equal to 0; there we just wrote the same form of the boundary conditions, only the non-homogeneous term is forced to be homogenous, and at x is equal to 1 the dirichlet boundary condition prevailed, u is equal to p; so, therefore, this will be, g is equal to 0.

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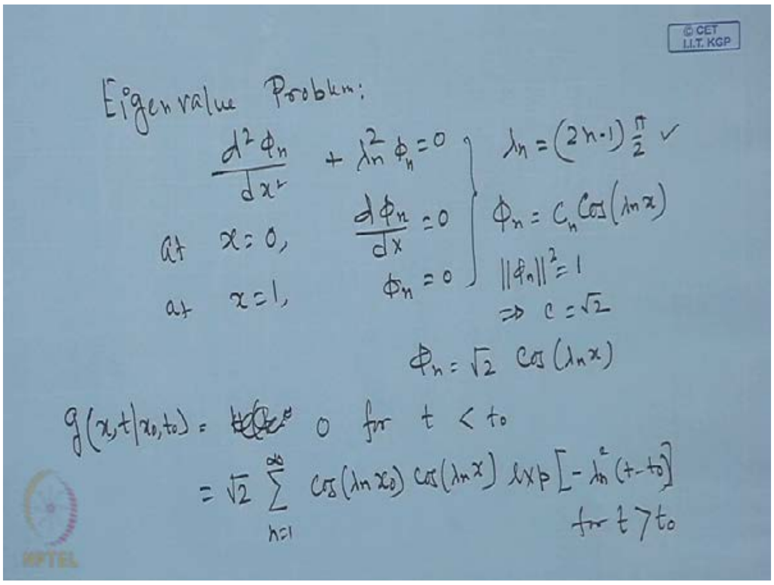

Now, if you look into the partial eigenfunction method, the corresponding eigenvalue problem, eigenvalue problem will be, d square phi n d x square plus lambda n square phi is equal to 0; and for this partial eigen, this eigenvalue problem the mother problem is the Green's function, so, it must satisfy the boundary conditions of the Green's function, and both the boundaries are homogeneous. So, we write the boundary condition on this eigenfunction. So, at x is equal to 0, we had del g del x is equal to 0, so, therefore, this will be d phi n d x is equal to 0, and at x is equal to 1, we had g is equal to 0, therefore, phi n should be equal to 0.

Now, we have already seen this problem, this problem is a standard eigenvalue problem, we have solved this problem and we have found out the eigenvalues will be, 2 n minus 1 pi by 2, and eigenfunctions will be nothing but some constant c cosine lambda n x; and if you make the eigenfunction orthonormal, this becomes 1, and c n becomes root 2,so, the orthonormal eigenfunction becomes root over 2 cosine lambda n x. So, therefore, if you remember, compared to the earlier problem, in the earlier problem we had dirichlet boundary condition, so, therefore, in this problem, we have a Neumann boundary condition, and if you, the eigenvalues will be, 2 n minus 1 pi by 2 instead of n pi, and eigenfunction will be root over 2 cosine 2 n minus 1 pi by 2 x or cosine lambda n x instead of sine n pi x or sine lambda n x.

Now if you look into the corresponding Green's function solution, the expression of Green's function becomes x t x naught, t naught will be nothing but H t minus, it will be equal to 0 for t less than t naught, and this will be is equal to root over 2 summation n is equal to 1 to infinity cosine lambda n x naught cosine lambda n x exponential minus lambda n square t minus t naught for t greater than t naught.

So, therefore, this is the Green's function solution, will be obtained; and if you look into the problem that, in the earlier problem, we had the sine functions as the eigenfunctions, and in this case we will be having the cosine functions as the eigenfunctions, and the eigenvalues will be 2 n minus 1 pi by 2 instead of n pi.

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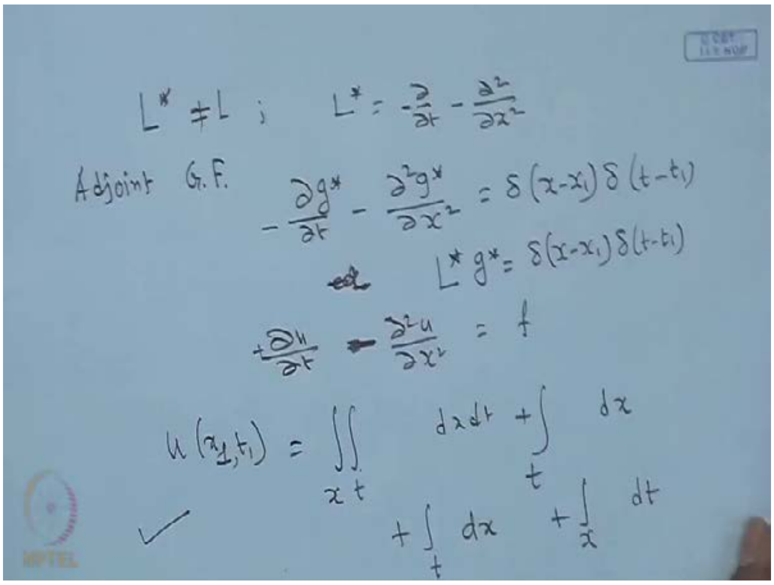

Now, if you, again in this problem, the operator remains is not a self adjoint operator, because L star remains the same minus del del t minus del square del x square.

So, again, we will be constructing the causal Green's function, the adjoint Green's function, the governing equation of adjoint Green's function will be nothing but del g star del t minus minus del g star del t minus del square g star del x square is equal to delta x minus x 1 delta t minus t 1.

This is in compact form, it can be written as l star g star is equal to delta x minus x 1 delta t minus t 1.

So, we now connect this equation with the original problem, del u del t is equal to- this is plus there is no minus- del u del t is equal to del square minus del square u del x square is equal to f, which may be a constant, which may be a function of x and t, a prescribe function.

So, then we connect u and this equation and this equation by taking the inner product of this equation with u, taking the inner product of this equation with g star, and subtract, and we will be getting the corresponding solution. And I am not going to solve this problem for you; in the last problem we have solved this problem completely.

So, we will be getting an expression of x 1 t 1, and there will be again four terms on the right hand side; first term will be corresponding to a volume integral one is over x and t d x d t, so, this term corresponds to the non-homogeneity term in the governing equation, then another term over non-homogeneity on the surface, so, it will be the non-homogeneous term in the in the governing equation at x is equal to 0, then one surface term will be a non-homogeneous term because of the non-homogeneity at x is equal to 1, then one more equation we will be getting over space, that is, the non-homogeneous term present in the initial condition. So, there will be, four such terms will be appearing on the right hand side corresponding to the four non-homogeneities in the governing equation and the boundary conditions and initial condition.

Please note, since the problem is a 2 dimensional problem, the volume integral is a double integral and the surface integrals will be basically the single integral or 1 integral. In case of a 3 dimensional problem, the volume integral would have been a triple integral, the surface integrals would have been a double integral. For a single variable problem, 1 dimensional problem like in ordinary differential equation, we have already seen earlier that the volume integral will be a single integral and the surface conditions would be, that is, it will be the algebraic equation corresponding to the bi-linear concomitant term type of thing.

So, we have got the complete solution for this problem as well. Please complete this problem- that will give you a fair practice of solving such problems.

Then, I will be considering one more problem which will be a robin mixed boundary condition, and let us see how things look like.

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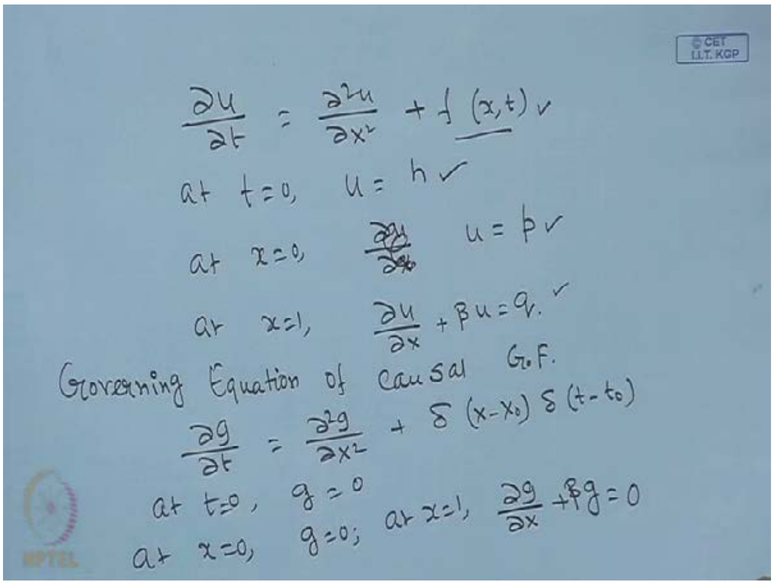

So, in this case, del u del t is equal to del square u del x square plus f of x t, in general that is the governing equation. At t is equal to 0 u is equal to h, at x is equal to 0 we have u is equal to p, at x is equal to 1 we had a robin mixed boundary condition, that is, del u del x plus beta u is equal to q. So, there are again four sources of non-homogeneity, one in the governing equation, another is initial condition, rest two are on the boundary conditions present in the boundary at x is equal to 0 and x is equal to 1.

Now, please note that again, the corresponding, let us look into the form of governing equation of causal Green's function. This will be del g del t is equal to del square g del x square plus delta x minus x naught delta t minus t naught, so, at t is equal to 0 we had u equal to h, so, therefore g is equal to 0; at x is equal to 0 we had u is equal to p, so, therefore, g is equal to 0; at x is equal to 1 we had del u del x plus beta u is equal to q, so, this is del g del x plus beta g is equal to 0.

So, if you just see, we have kept intact the forms of the boundary conditions, but made them homogeneous, there is the only thing, and the non-homogeneous term in the governing equation is replaced by the Dirac delta term.

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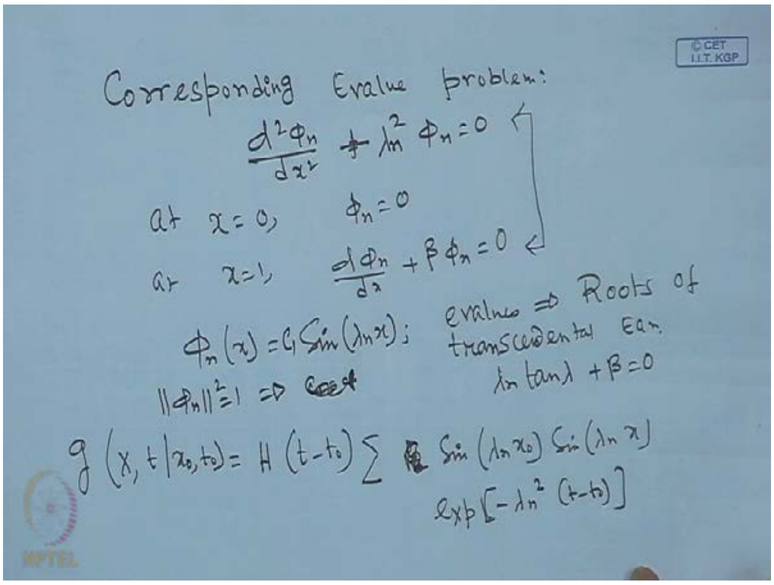

Now, if we see the corresponding eigenvalue problem by partial eigenfunction expansion method, the corresponding eigenvalue problem will be d square phi n d x square is equal to plus lambda n square phi n is equal to 0.

So, boundary conditions of this problem must be satisfying the boundary conditions of the Green's function, at x is equal to 0 your phi n was equal to 0, at x is equal to 1 d phi n d x plus beta phi n must be equal to 0.

So, we have already solved this problem. This is a standard eigenvalue problem with homogeneous boundary conditions; we have already solved this problem earlier.

If you remember the eigenfunctions are sine functions in this case as well sine lambda n x, and eigenvalues are lambda n's are basically roots of the transcendental equation, that is, lambda n tan lambda n plus beta is equal to 0.

So, this was the form of the transcendental equation, and the eigenvalues are the roots of this equation. So, therefore, in this case, if you look into the expression of Green's function, the expression of Green's function becomes x t x 0 t 0, is nothing but H heavy side function t minus t 0 summation root over 2 sine lambda n x naught sine lambda n x exponential minus lambda n square t minus t naught.

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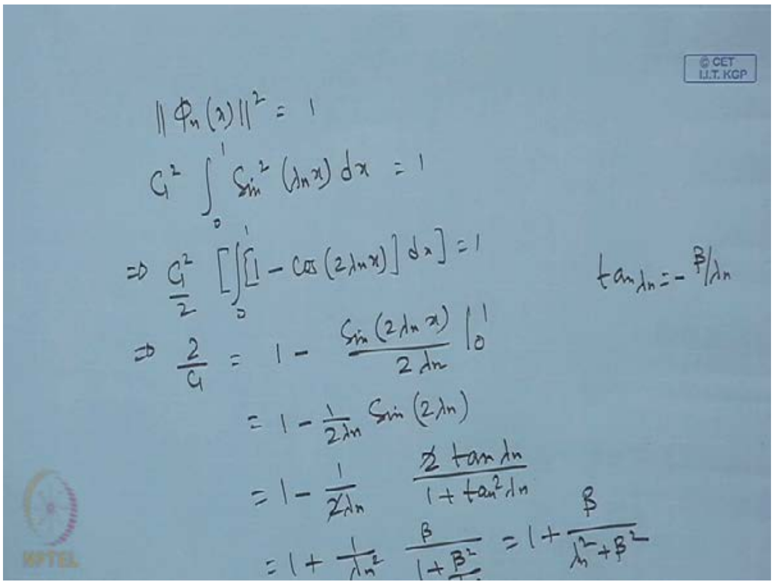

So, again, we can make the eigenfunction orthonormal to some constant; so, eigenfunction orthonormal means norm of phi n square is equal to 1, that will give C 1 is equal to some value, but in this case, let us look what C 1 will be obtained, so, it may not be root over 2; we will be solving this problem. Let us say we make norm of phi n square is equal to 1, so, this becomes C 1 square integral sine square lambda n x d x from 0 to 1 is equal to 1.

So, this will be C 1 square by 2 sine square lambda n x, so, it will be 1 minus cosine 2 lambda n x d x is equal to 1, so this will be from 0 to 1- you carry out the integral- so, this becomes 2 C 1 square is equal to 1 minus cosine 2 lambda n x divided by integral cosine 2 lambda n x, so, this becomes sine 2 lambda n x divided by 2 lambda n from 0 to 1, so, it will be 1 minus 1 over 2 lambda n and sine 2 lambda n, because the, if you put the lower limit this becomes 0, so, sine 0 is 0

So, we put it like this form, 1 by 2 lambda n, sine 2 lambda n, we write it in tan 2 lambda n, so, this becomes 2 tan lambda n divided by 1 plus tan square lambda n- and we know tan lambda n is nothing but minus beta by lambda n, we have already proved that, these two will be cancelled out- so, 1 minus 1 over lambda n it will be minus minus plus beta lambda n square divided by 1 plus beta square divided by lambda n square, so, this becomes 1 plus beta divided by lambda n square plus beta square.

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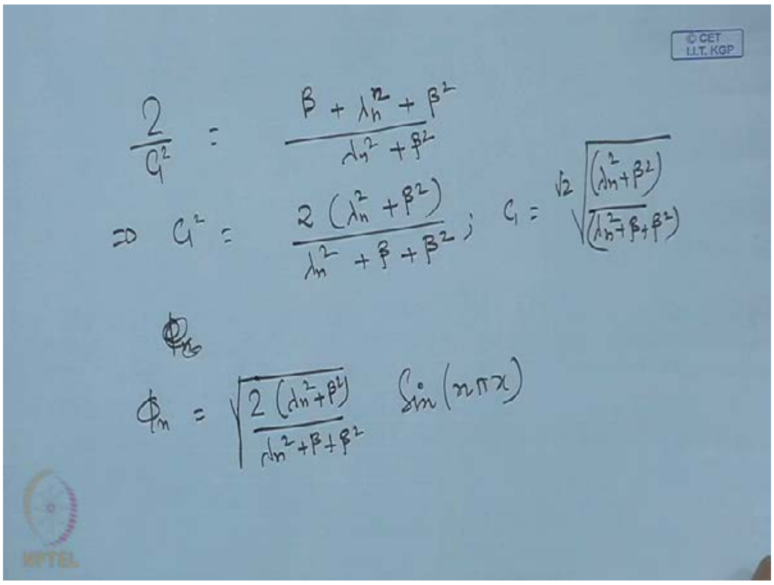

So, we get an expression of C 1 from here. So, 2 by C 1 will be nothing but beta plus lambda n square plus beta square divided by lambda n square plus beta square, so, therefore, this becomes C 1 equal to 2 lambda n square plus beta square divided by lambda n square plus beta plus beta square. So, that will be the C.

Therefore, the eigenfunction becomes quite straightforward; the eigenfunctions, since it was C 1 square by 2, so, this will be 2 by C 1 square, so, there is a square there, so C 1 square will be there, so, you will be getting C 1 as root over 2 lambda n square plus beta square under root divided by lambda n square plus beta plus beta square, so, that is the constant; so, eigenfunction becomes root over 2 lambda n square plus beta square divided by lambda n square plus b plus beta square sine n pi x in this particular problem. So, these are the eigenvalues. So, one can get the expression of eigenfunction, so these are the eigenfunction, so one can get the expression of Green's function.

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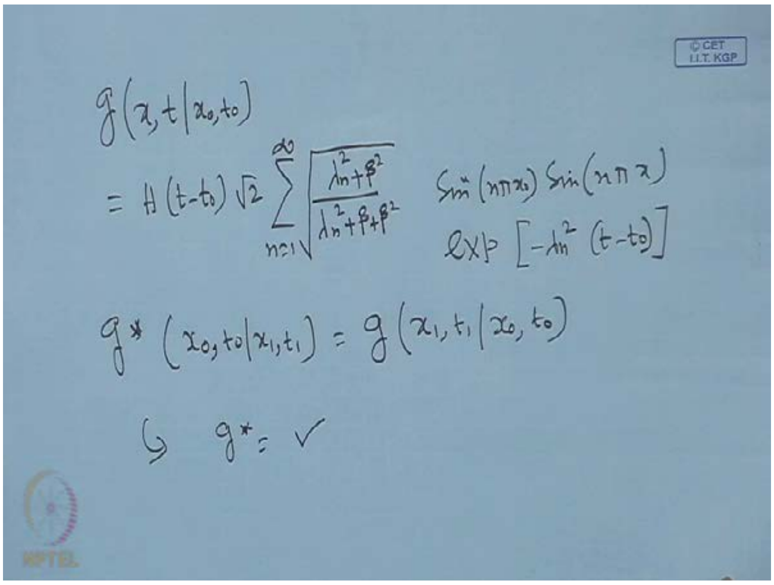

In this particular case, so, Green's function becomes g x t x 0 t 0 is nothing but H t minus t naught, then this constant will be appearing root over 2 n is equal to 1 to infinity, then this becomes lambda n square plus beta square divided by lambda n square plus b plus beta square root over, and then sine n pi x naught sine n pi x exponential minus lambda n square t minus t naught; so that is the expression of Green's function.

Once we get the expression of Green's function one can get the expression of adjoint Green's function by using the formula g x 0 t 0 x 1 t 1 is equal to g star x 0 t 0 x 1 t 1 is nothing but g x 1 t 1 x 0 t 0, and by changing the subscript we can get the expression of g star- the analytical expression we can obtain as we have demonstrated earlier.

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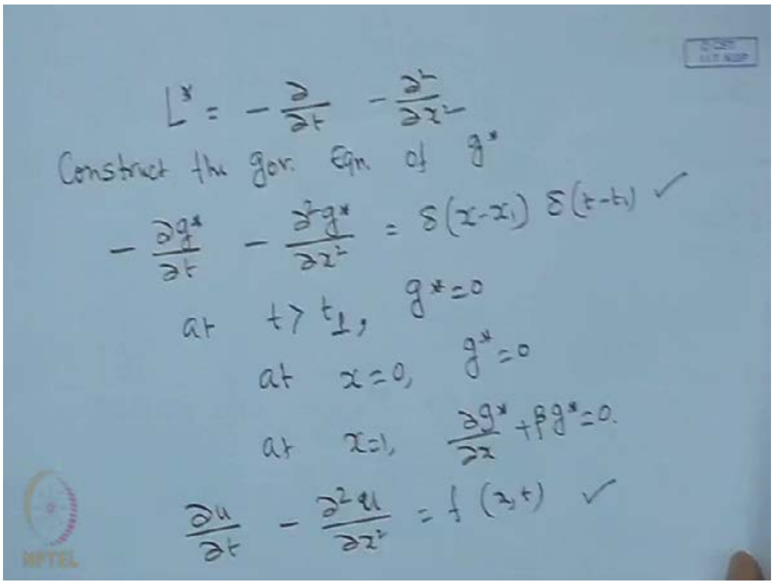

So, once we do that, then what we do? We formulate, again, the operator, the operator remains the same because it becomes del del t minus del square del x square; so, you formulate, construct the governing equation of g star that will be, minus del g star del t minus del square g star del x square is equal to delta x minus x 1 delta t minus t 1, and we know the boundary conditions on g star, the way we have done it earlier, at t greater than t 1 g star is equal to 0, t greater than t 1 g star is equal to 0, at x is equal to 0 your g star is equal to 0, at x is equal to 1 del g star del x plus beta g star is equal to 0.

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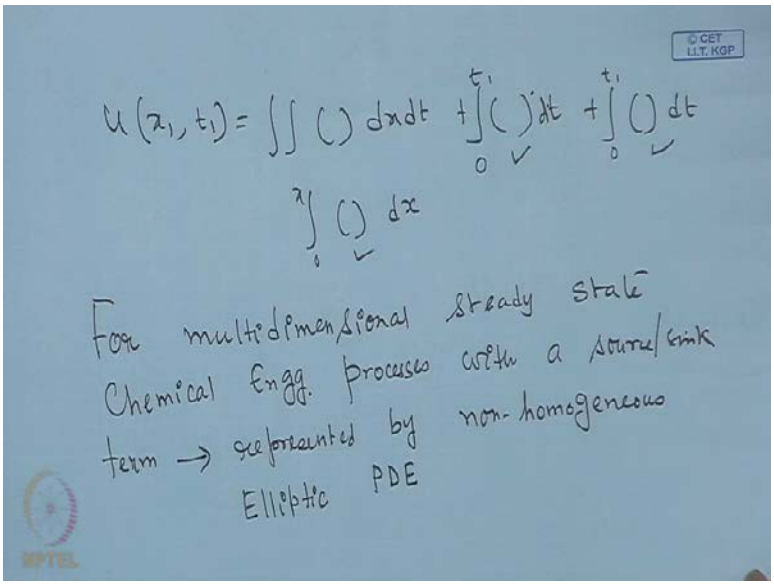

So, this is the governing equation and the boundary conditions on adjoint Green's function. Then, what we do? We connect the actual problem with this adjoint Green's function.

The actual problem was del u del t minus del square u del x square is equal to f of x and t- it may be constant, it may be a function of x, it may be function of time, may be function of both- now, we connect we take the inner product of this equation with respect to u, we take the inner product of this equation with respect to g star, and we subtract and we will be getting the solution of u x t x 1 t 1. So, there will be four terms again on the right hand side, these four terms will be corresponding to the volume integral or the non-homogeneous term in the governing equation; there will be one term over t, so, that will be from 0 to t 1, and another term over t, so, 0 to t 1 d t, these two terms corresponds to two non-homogeneous term present in the boundary condition at x is equal to 0 and boundary condition at x is equal to 1; and then we will be having one term over x, that will be something multiplied by d x, and this term corresponds to the non-homogeneous initial condition of the original problem.

So, that is how, we have seen that, how the parabolic partial differential equation will be tackled for different boundary conditions and initial conditions. And we have seen that, we have looked into all the three boundaries, that is, the dirichlet boundary condition, Neumann boundary condition and robin mixed boundary condition. Only one thing we changed, the eigenvalue, eigenfunctions and eigenvalues will change, and the corresponding expression of g star, corresponding expression of Green's function will be different because all these three problems will be having three different eigenvalues and three different eigenfunctions.

So, once we do that then we will be able to construct the causal Green's functions by the expression of the adjoint Green's function by changing the subscript simply. Then we will formulate the adjoint Green's function and connect it with the original problem, and we will be getting the complete solution for the three different boundary conditions of the parabolic partial differential equation.

So, that completes the solution of non-homogeneous parabolic partial differential equation. Next, we will be looking into another popular partial differential equations non-homogeneous, that is, the elliptical partial differential equation with a non-homogeneity in the governing equation as well as in the boundary conditions.

Now, this type of problems, the elliptic problems becomes very common and rampant for the steady state chemical engineering processes, which is multidimensional.

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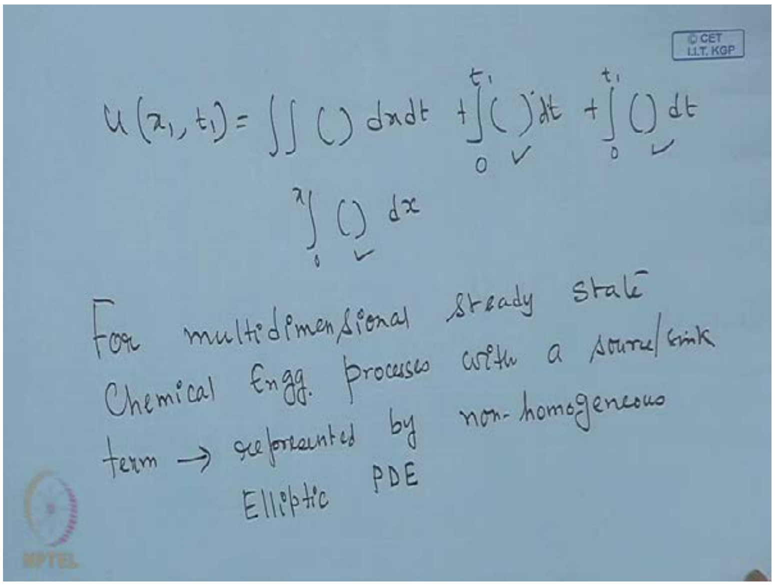

So, for multidimensional steady state chemical engineering processes with a source or sink term is represented by non-homogeneous elliptic partial differential equation.

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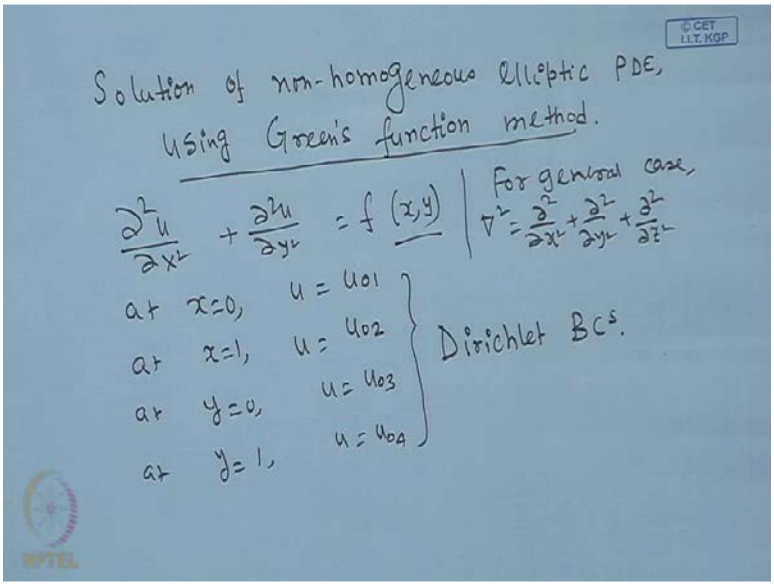

So, I will just take up an example of non-homogeneous elliptic PDE using Green's function method.

Now, again I will take up the problem in rectangular coordinate, we can convert all this problem in cylindrical polar coordinate or spherical polar coordinate by selecting the appropriate operator.

So, I am just demonstrating these problems in case of in cartesian coordinates or in rectangular coordinate because the solutions, the eigenfunctions, etcetera, becomes very simplified.

Now, let us look into the elliptical problem: del square u del x square plus del square u del y square is equal to some function of x and y, it may be a constant, it may be a sole function of x, it may be a sole function y, it may be a function of x and y both, but we are talking about a 2 dimensional problem, but in a general problem for general case, the Laplacian is represented by del square del x square plus del square del y square plus del square del z square.

So, at x equal to 0 there are four boundaries; at x equal to 0 u is equal to u 0 1, at x is equal to 1 we had u is equal to u 0 2, at y is equal to 0 we had u is equal to u 0 3, and at y is equal to 1 you have u is equal to u 0 4.

So, there are four sources of non-homogeneity, and we have considered dirichlet boundary conditions on all these four boundaries. So, dirichlet boundary condition on all the four boundaries and this four are non-homogeneities, and one boundary, the initial condition was non-homogeneous.

Now, if you examine this problem, unlike the earlier problem, the parabolic problem, if you looked into a parabolic problem, the parabolic problem was having, the operator was del u del t minus del square u del x square, and the corresponding eigenvalue problem will be d square phi n d x square minus plus lambda n square phi n.

How the eigenvalue problem will be cropping up? The eigenvalue problem will be cropping up by the formulation of construction of causal Green's function. So, if you remember, I am just talking about the earlier problem, in the earlier problem the causal

Green’s function was formulated by forcing all non-homogeneities to be 0 except the non-homogeneity in the governing equation is replaced by a Dirac delta function.

Now, since the boundary conditions are homogeneous in the earlier problem we formulated the corresponding eigenvalue problem in the x directions simply because the mother problem, which was the Green’s function problem, was having homogeneous boundary conditions; so, in this case, if we look into the corresponding causal Green’s function...

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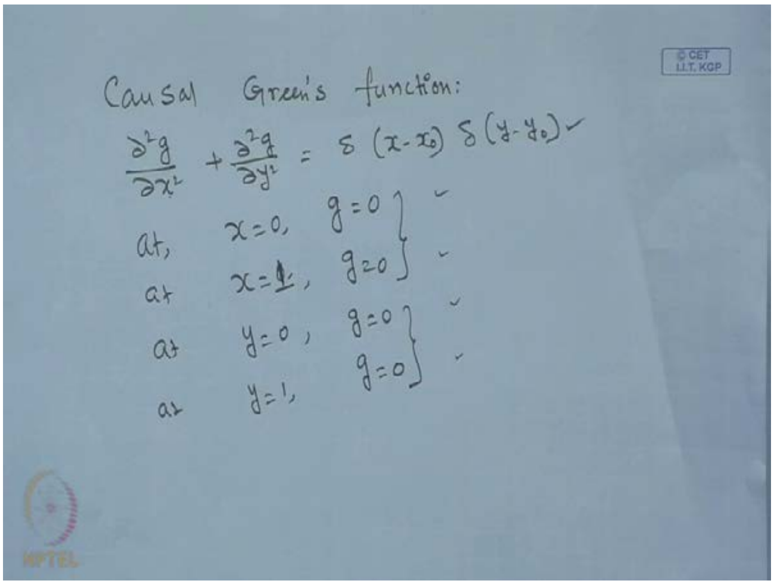

So, causal Green’s function will be del square g del x square plus del square g del y square is equal to delta x minus x naught delta y minus y naught; so, what we did? We replaced the non-homogeneity in the governing equation by Dirac delta function located at x naught and y naught, then we make all the boundary conditions to be homogeneous at x is equal to 0 g is equal to 0, at x is equal to 1 g is equal to 0, at y is equal to 0 g is equal to 0, at y is equal to 1 g is equal to 0.

So, if you look into the construction of causal Green’s function, we replaced the non-homogeneous term in the governing equation by a Dirac delta function located at x naught and y naught, and all the non-homogeneous boundary conditions are forced to be homogeneous.

Now, if you look into the boundary conditions, both the boundary conditions in x directions are homogeneous; both the boundary conditions in y direction are homogeneous. So, therefore we can formulate the standard eigenvalue problem both in x direction and y direction independently; and there are the dimension, this is a 2 dimensional problem, the independent variables are x and y, in both x direction we have an independent eigenvalue problem, in the y direction as well we have an independent eigenvalue problem.

So, this problem can be broken down into the eigen, the Green's function can be expressed as a function of the eigenfunctions in the x direction as well as eigenfunction in the. So, we are looking into the causal Green's function of elliptical partial differential equation, and we have seen how to formulate the Green's function in such case, and if you remember that the, even the original problem the boundary conditions are not homogeneous, we have to force the boundary conditions to be homogeneous in case of formulation of Green's function.

So, the non-homogeneity in the governing equation is replaced by the Dirac delta function in the others, unit step function in the governing equation of Green's function, but all the boundary conditions forced to be homogeneous, in other words, we are trying to find out what is the effect of the non-homogeneous, source term, in the governing equation keeping all the other non-homogeneities to be vanish so that we can ideally identify what is the non-homogeneity in the governing equation of the original problem will reflect in the final solution. Then, we will look up this problem with the actual problem governing equation of u, and then coupling with that we will be getting the complete solution.

So, in this class we have looked into the complete solution of parabolic partial differential equation non-homogeneous and how to solve that equation by using Green's function method. So, in the next class, I will take up the complete solution of elliptical partial differential equations, which will be quite common in steady state chemical engineering processes.

So, in case of partial differential equation, in case of the causal Green's function formulation in elliptical problem, you must have understood that all the boundary conditions are homogeneous, that means we can have a standard eigenvalue problem in the x direction and we can have a standard eigenvalue problem in the y direction as well. Therefore, we will be utilizing the full eigenfunction expansion method in order to obtain the Green's solution of Green's function, then we will be looking it up with the actual problem denote to get the complete solution.

So, I will stop here, in this class. I will take up this problem in the next class and completely solve the Green's function solution for the elliptical problem, and then I will again just couple the solution of Green's function with the original problem and see how the complete solution evolves out of it. Thank you very much.