Advanced Mathematical Techniques in Chemical Engineering

Prof. S. De

Department of Chemical Engineering

Indian Institute of Technology, Kharagpur

Lecture No. # 14

Stability Analysis (Contd.)

So, welcome to the next session of this next class, of this session. So, we are discussing about the stability of the steady state. So, in the last lecture, we finished - we have studied - the classification of the steady state and we have discussed about the two steady states - one is the unstable node and another is the stable node.

Now, in this class, we will finish the classification of the steady state and further we develop our theory for the stability analysis.

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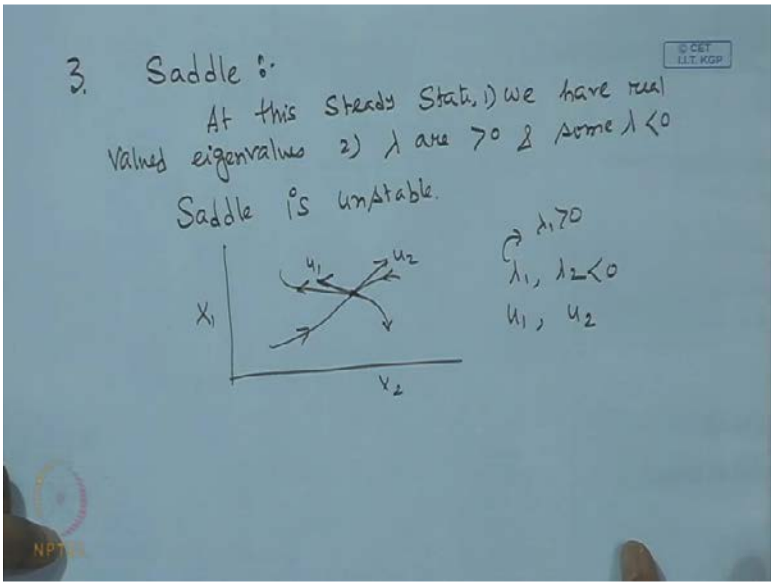

So, next steady state that we will talk about is the saddle steady state. So, let us see what a saddle is - at this steady state, some Eigenvalues are positive and some Eigenvalues are negative. And we have real first condition is for the saddle, we have real valued Eigenvalues. Secondly, some lambdas are positive and some lambdas are negative; so the saddle is always unstable.

So, if you look into the phase plane plot for the saddle, it will look something like this; so, in one steady state branch, you will be - let us say - this is u 2 and this is u 1 (Refer Slide Time 00:55). So, lambda 1, lambda 2 are two Eigenvalues, corresponding eigenvectors are u 1 and u 2. Now, in this case, any disturbance will decay, so lambda 2 is negative and along with this eigenvector, and along with this eigenvector u 1, any disturbance will grow in time; so, therefore, lambda 1 is ever positive.

So, this case of saddle, saddle is basically a steady state, where at least some of the Eigenvalues are positive and some of the Eigenvalues are negative.

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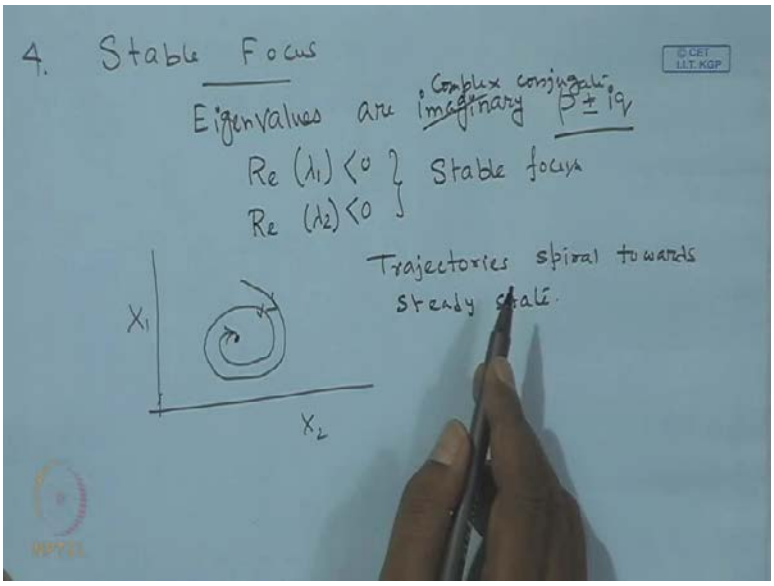

Next steady state will be - talking about is - called as stable focus. Now, in this case, the Eigenvalues are imaginary and they are in the form of p plus minus i q. So, in case of real values of lambda 1 is less than 0 and real value of lambda 2 is also less than 0, then we call this focus as stable focus.

So, basically if you look into the phase plane plot, it will be the steady state here; so whenever you will start from here, it will spiral and then we come into this steady state; so, trajectories spiral towards - steady state - steady state and from why it will be spiraling? That will be depending on the initial conditions.

So, it will be spiraling in that particular path and as you have seen that, it contains an imaginary part, it will be the solution then the deviation variable or part of variable will be in the form of sine and cosine function; So, it will be having a spiral nature towards the steady state or away from the steady state.

So, in this case of real values of lambda 1, lambda 2 are negative, then it will be landing with a stable focus and focus will happen only in case of a complex. So it is not imaginary, it is basically complex and complex conjugate; so Eigenvalues are complex conjugate (Refer Slide Time 02:42).

So, then we will be having a focus. So, the condition for having a focus is that, complex conjugate Eigenvalues must occur, that is, number 1. And secondly it will be a stable focus; if the real values of the Eigenvalues are negative then you will be having a stable focus.

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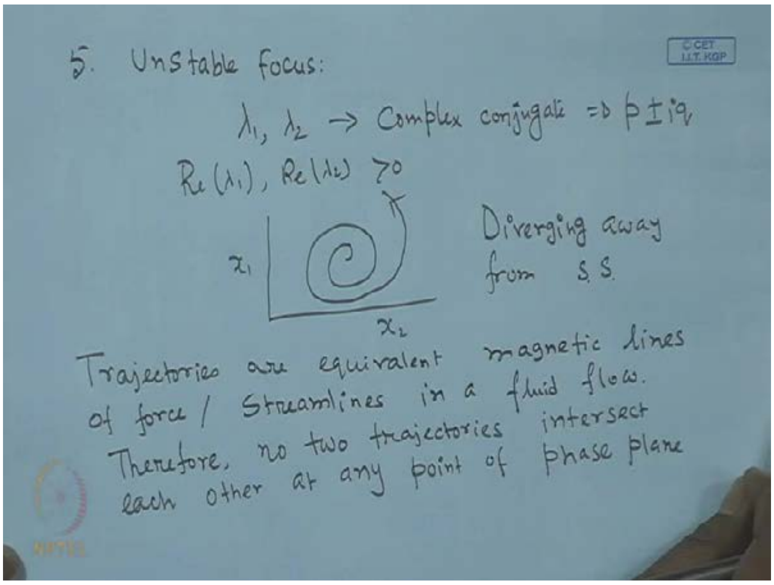

In case of unstable focus, the lambda 1, lambda 2’s are complex conjugate and they are in the form of p plus minus i q and real values of lambda 1 and lambda 2 are always greater than 0.

Now, if you look into the phase plane plot x 1 and x 2, then in this case the steady state and the focus will be unstable and it will be diverging away from the steady state; the deviation or perturbation will be diverging away from the steady state.

Therefore now in case of the trajectories whatever we are talking about in the phase plane plot this trajectories are equivalent to magnetic lines of force or the streamlines in a fluid flow. Therefore, no two trajectories will intersect each other at any point of phase plane. So, not that is a typical property of the trajectories in the phase plane plot.

And next we look into the bifurcation theory and the actual theory for the stability analysis in any chemical engineering system

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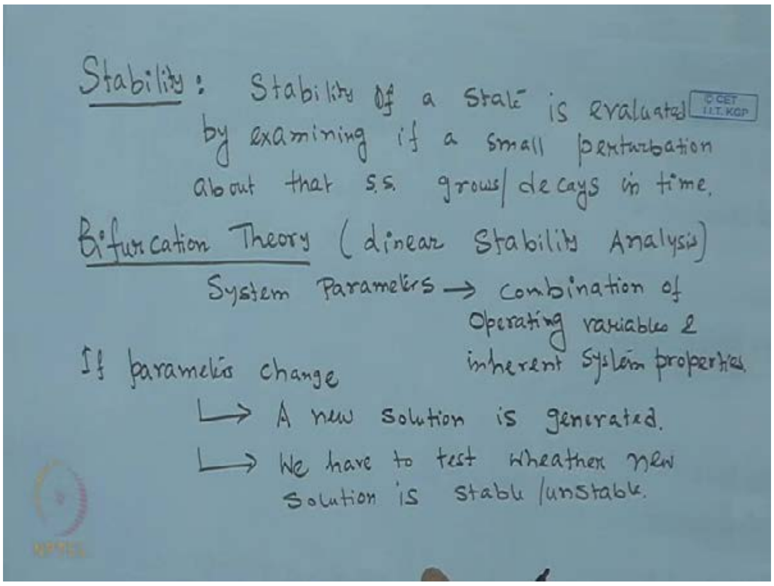

So, next we define what stability is, with this background we are able to take up the stability analysis using the bifurcation theory, which is also known as the linear stability analysis.

So, stability of a state – can be determined - is evaluated by examining if a small perturbation about that steady state grows or decays in time that is the definition of the stability of a particular state.

Next, we talk about, if basically for a steady state you just give a perturbation, if that perturbation grows in time, it becomes an unstable steady state; if the perturbation decays in time, it is a stable steady state.

Next, we look into the bifurcation theory. This is also known as the linear stability analysis. So, basically if the bifurcation theory says that we have several system parameters.

System parameters are nothing but combination of operating variables and inherent system properties; so, that is the system parameters and it says that, if parameters change a new solution is generated.

Now, we have to test whether the new solution is stable or unstable. In other words one can get the bound on the parameters for where the solution or the steady state is always stable or always unstable or you can get the combination of the parameters or the conditions in the parameters where the solution becomes moves from stable to unstable region. So if you want to operate under the stable steady state, those conditions are have to be avoided.

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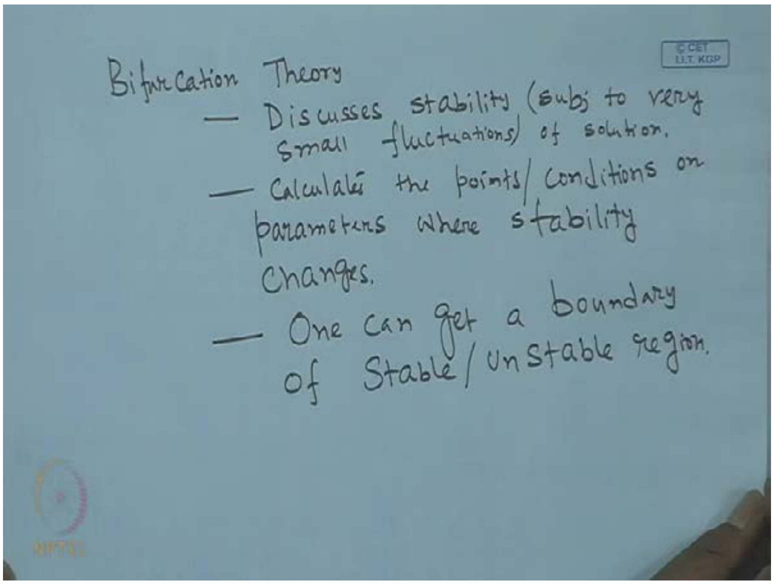

So, what a linear stability analysis or bifurcation theory does? Bifurcation theory has 2 functions. First one, it discusses stability the 2 stability subject to very small fluctuations - discusses stability - of solution. It calculates the points or the parameters conditions on parameters where stability changes. One can get a boundary of stable or unstable region.

Now, we just take up an example and the example I preferred is that the example whatever we have done with the earlier in context of whatever we discussed about the contraction mapping - the CSTR problem.

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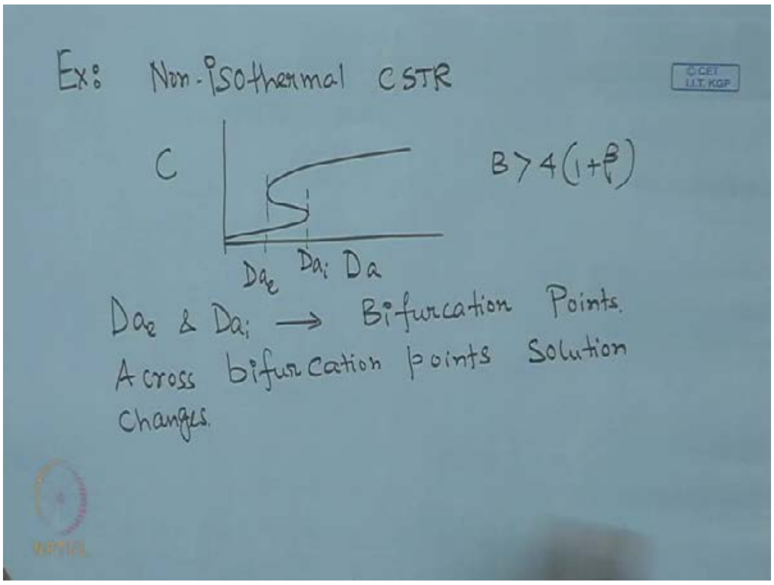

So, let us take up the example of CSTR problem - non isothermal CSTR. So if you looked into the unstable solution, the variation is something like this (Refer Slide Time 14:17), if B is greater than 4 into 1 plus beta, then this is D a i ignition, this is D a e Damkohler number extension; so, extension D a e and D a i are called the bifurcation points.

If we slightly shift the Damkohler number from D a e or D a I, it will be getting a new steady state either in the upper half branch or in the lower half branch.

So, across the bifurcation point the solution changes; so, if we write down the governing equation of this system thing becomes clearer and we can go further with the analysis.

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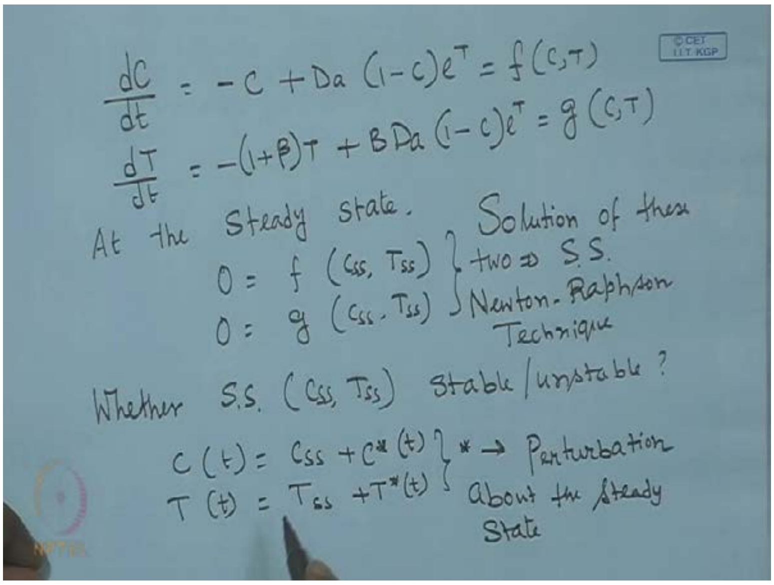

So, for mass balance will give you dC dt is equal to minus C plus D a 1 minus C e to the power T and this is function of C and T - concentration and temperature, this is non-dimensional concentration is basically conversion and temperature. So, dT dt is nothing but minus 1 plus beta T plus BD a 1 minus C e to the power T, this will be some function g of concentration and time.

So, at the steady state, we have 0 is equal to function of C s s and T s s and 0 is equal to some function the g of C s s and T s s. Now, the steady state will be obtained by solution of these equations. Solution of these two equations will give you the steady state values of C s s and T s s, but the point is in this particular form; these two equations cannot be solved analytically. So, one has to take request to the non-linear, because these two equations are non-linearly connected and therefore one has to take the iterative technique, may be a Newton-Raphson technique will be suitable.

So, some kind of iterative technique can be adopted to solve these two equations because subject to given a set of the parameters, because they are not linear in nature and one can take recourse to the Newton-Raphson technique numerically to solve these set of equation in order to get C s s and T s s - the steady states.

Now, the question is, whether the steady state C s s and T s s they are stable or unstable?

So, what we do? We write C as a function of T as C s s plus C star t and T as a function of time as T s s plus T star t, so this star indicates the perturbation about the steady state.

Now, we write the governing equation in terms of the part of variable C star and T star and see what we get.

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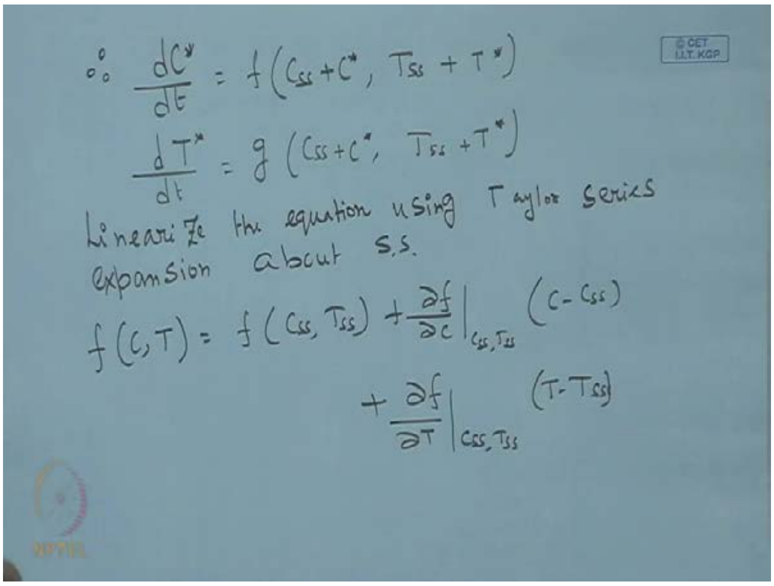

So, if we really do that we will be getting dC star dt is equal to function of C s s plus C star and T s s plus T star and dT star dt will be function of C s s plus C star and T s s plus T star.

Now, again we use to the tell up a linearize the problem, linearize the equations using Taylor series expansion and retaining the first order term and neglecting higher order term assuming the deviations are quite close to the steady state. So, we have a linearization f of C T will be in about the steady state, this has to be done about the steady state.

So, f of C T will be nothing but f of C s s and T s s plus del f del C about the steady state C s s T s s C minus C s s plus del f del T about the steady state T minus T s s and T minus T s s is nothing but T star C minus C s s is nothing but C star.

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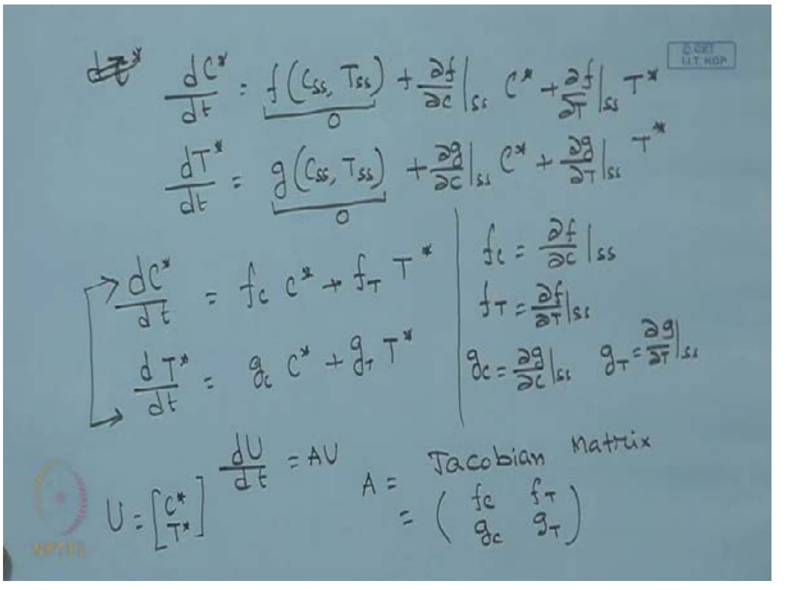

So, therefore, we will be getting the, so similarly we can expand the Taylor series expansion with respect to f of g of C and T about the steady state and what will be getting is that, dC star dt is nothing but f of C s s T s s plus del f del C about the steady state times C star plus del f del T about the steady state times T star. And dT star dt will be is equal to g of C s s and T s s plus del g del C steady state C star plus del g del T steady state T star.

Now at the steady state, this will be equal to be 0, this will be equal to 0 (Refer Slide Time 21:29). So, you have dC star dt is equal to f of f c - we just call this as f c - C star plus f subscript T T star and dT star dt is equal to g c C star plus g of T T star.

So, what is f c? f c is nothing but del f del C at the steady state; f T is del f del T evaluated at steady state; g C is del g del C evaluated at steady state; g T is del g del T evaluated at the steady state.

So, therefore, in a compact notation - matrix notation - these two equations can be written as dU dt is equal to AU, where U being a vector comprising of C star and T star and A is the Jacobian matrix, that is f of c f T g c g T which are nothing but these derivative with respect to steady state evaluated at the steady state and that forms the Jacobian matrix.

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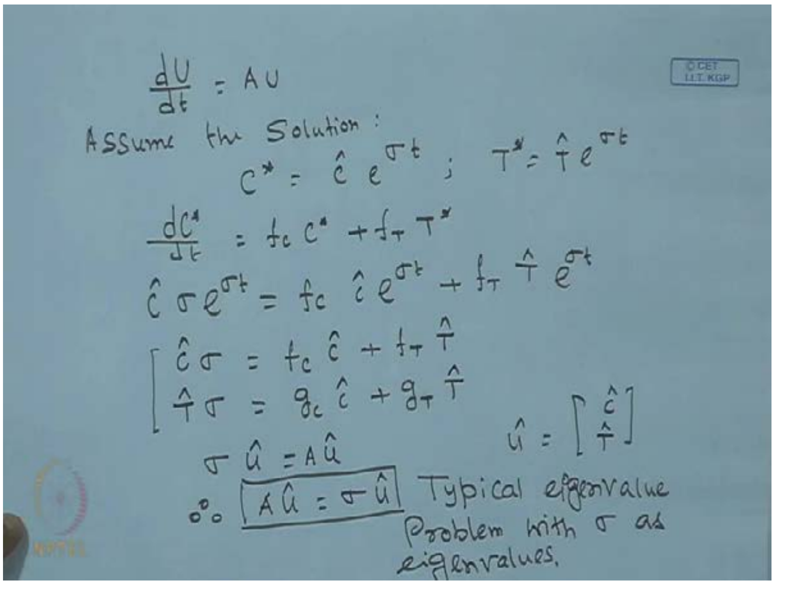

So, we are basically having an equation in the form of dU dt is equal to A U and assume the solution in the form of, so this variable U - the vector U - is nothing but it is made of elements which are in the form of part of variables.

So, assume the solution in the form of C star is C hat e to the power sigma t and T star is T hat e to the power sigma t. So, if we put all these in this equation, so this becomes C hat we put dC star dt is equal to f c C star plus f T T star. So, dC star dt will be C hat sigma e to the power sigma t is equal to f c C hat e to the power sigma t plus f T T hat e to the power sigma t and you will be having this C star C hat sigma that e to the power sigma t will be cancelled out, C hat sigma has f c C hat plus f T T hat and from the other equation dT star dt is equal to f c C star that plus f T T star, we will be getting T hat sigma is equal to g c C hat plus g T T hat. And in the compact notation, you can write this equation as sigma u hat is equal to A u hat and in that u hat is nothing but the vector comprising of the element as C hat and T hat.

So, basically you will be getting A U hat is equal to nothing but sigma u hat and this is a typical Eigenvalue problem with the sigma as the Eigenvalues; so, this a typical Eigenvalue problem with sigma as Eigenvalues. So, we can come to some kind of analysis. Now we will be looking into the values of Eigenvalues and their signs in order to establish the stability. So, how the sigmas are, so how to obtain the Eigenvalues?

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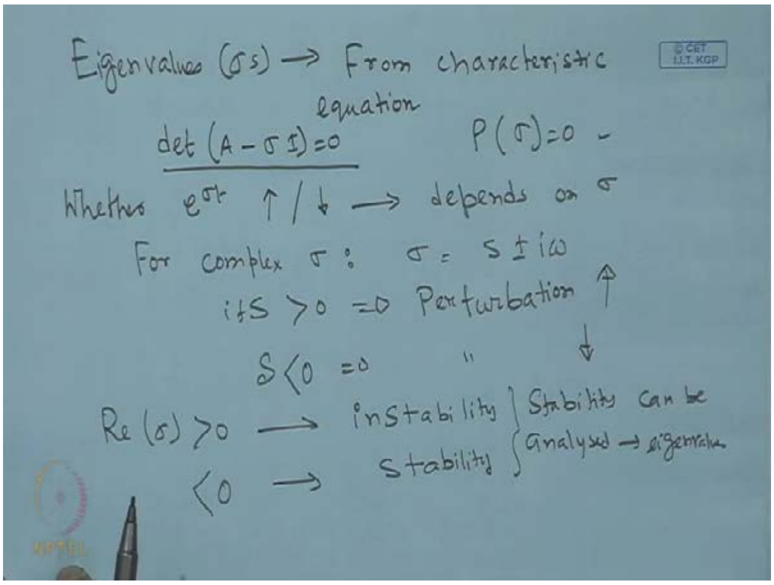

Sigmas Eigenvalues are obtained from the characteristic equation and this characteristic equation will be obtained by solving the determinant of A minus sigma I is equal to 0, but we solve this determinant is equal to 0 and we will be getting the characteristic equation, P of sigma is equal to 0.

If it is 2 into 2 matrixes, you will be getting two Eigenvalues, it is a quadratic root quadratic characteristic equation will be landing up with the two roots; if it is 5 into 5 matrixes, then you will be getting the five roots, fifth order of polynomial of the characteristic equation and you will be getting the five roots; so, whether sigma e to the power sigma t increases or decreases it entirely depends on sigma.

For complex conjugate roots, for complex sigma, the sigma will be occurring in the form of complex and conjugate and if S is positive, then perturbation grows; if S is negative then perturbation decays. So, the point is, real part of sigma should be greater than 0 for instability and real part of sigma is less than 0 for the stability.

So, stability can be analyzed by looking into the - by evaluating the - Eigenvalues and let us see how the Eigenvalue and the nature of the Eigenvalues will dictate the perturbation from the steady state, it will grow in time or decay down in time or not. Now, let us see how the Eigenvalues are evaluated in this particular case.

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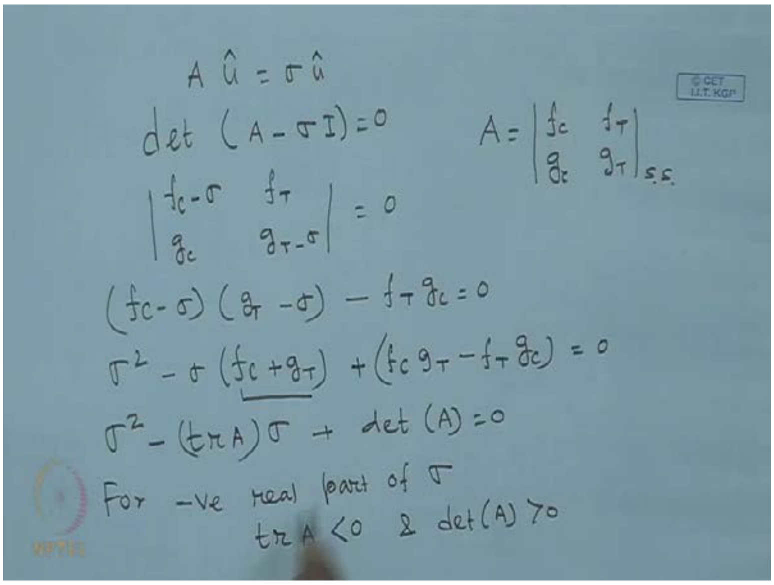

So, evaluation of Eigenvalues, if you look into the matrix A u hat is equal to sigma u hat and Eigenvalues are obtained by evaluating this determinant of A minus sigma I is equal to 0. So, if you look into the matrix A, matrix A is nothing but the Jacobian matrix containing f c f T g c and g T and this partial derivatives with respect to concentration and with respect to temperature are evaluated at the steady state; so these are evaluated at the steady state.

So, determinant of A minus sigma I become f c minus sigma f T g c g f T minus sigma is equal to 0. So, f c minus sigma times g T minus sigma minus f T g c will be is equal to 0. So, if you just multiply this thing, this will becomes sigma square minus sigma f c plus g T is equal to plus f c g T minus f T g c is equal to 0. So, if you look into these matrix what is f c and f c plus g T? It is nothing but the trace of the matrix A and f c g T minus f T g c is nothing but the determinant of the matrix A.

So, sigma square we can write, we can replace this by trace of A multiplied by sigma plus determinant of A is equal to 0. So, therefore, the coefficient of sigma is nothing but the trace of the Jacobian matrix and with the third term in this equation is nothing but the determinant of the Jacobian matrix.

Now, this equation will be having real root for negative real part of sigma, we can have the condition trace of A should be negative and determinant of A should be positive, then only you will be having the real negative part of the root sigma.

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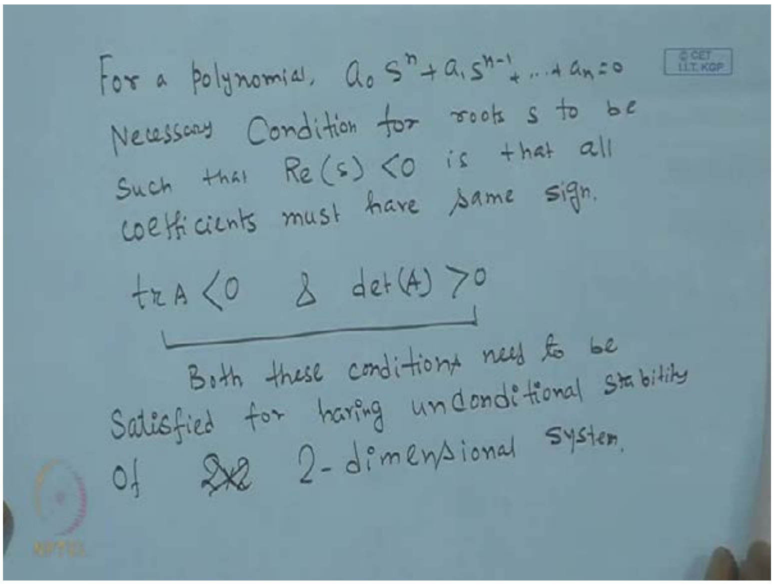

So, if we just invoke how this thing will come? How this condition will come? If you remember for a polynomial a 0 S to the power n plus a 1 S to the power n minus 1 up to a n is equal to 0. The necessary condition for roots S to be such that real part of S will be less than 0 is that all coefficients must have same sign.

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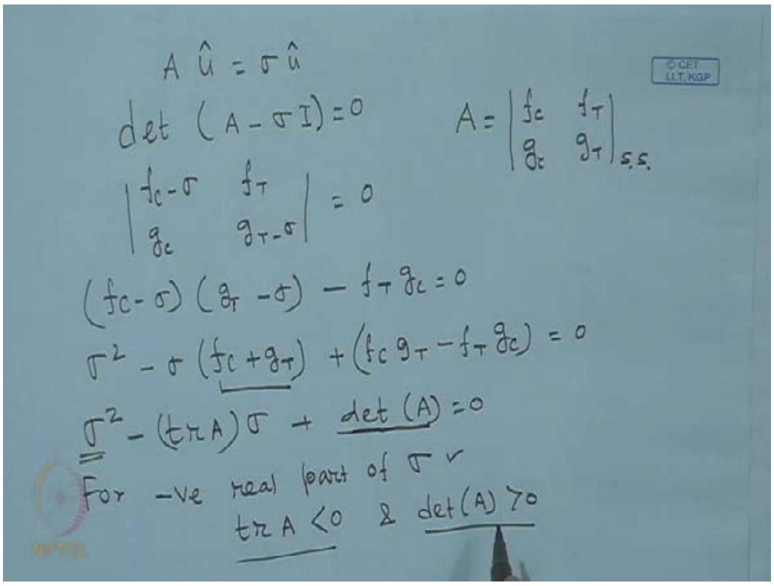

So, therefore, this is positive determinant of A is positive, when the coefficient of sigma square is positive in order to have a positive sign coefficient of sigma trace of A has to be negative, in order to have a positive value of determinant of A has to be positive.

So, therefore, these are the other conditions of stability for a two-dimensional system. So, trace of A Jacobian matrix has to be negative and determinant of A is positive; if both these conditions need to be satisfied for having unconditional stability of a two-dimensional system, we can write it as two-dimensional system; for higher dimensional system what will be the condition, that we will discuss shortly.

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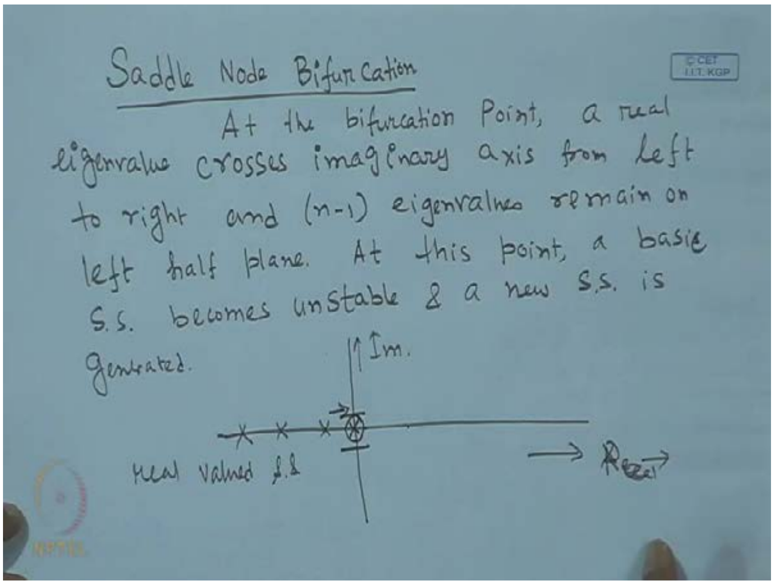

Now, let us talk about two kinds of bifurcation, one is the saddle node bifurcation. Now, in this definition, at the bifurcation point, a real Eigenvalue crosses the imaginary axis from the left to right and n minus 1 Eigenvalues remain on left half plane. Then, if this is the case what happens? At this point, a basic steady state becomes unstable and a new steady state is generated.

So, the situation is something like this, we have the real axis, it is a real axis and this is the imaginary axis and you will be having - let us say - n number of real valued steady state for a set of combination of the parameters.

Now, for a particular parameter combinations, this steady state starts moving, if we change the value of the system parameters - let us say - this steady states starts moving in the right direction and for a particular value of steady state this crosses over the imaginary axis and comes to the positive real obtain or the right obtain. So, in that case that is a saddle point and at this point the whole solution one of the steady state becomes positive and the solution becomes unstable from the stable.

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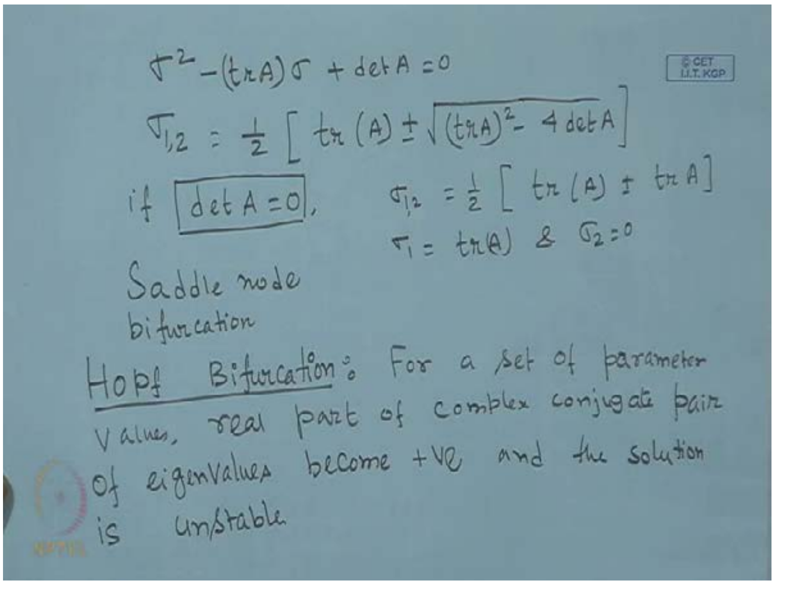

So, one can identify the saddle point by looking into this (Refer Slide Time 35:54). So, what is the saddle point? If you look into the equation for the roots of the Eigenvalues of a two-dimensional process, sigma square minus trace of A times sigma plus determinant of A should be equal to 0.

If you look into the roots of this quadratic equation, this will be half, sigma 1 2 will be half, times trace of A minus b plus minus under root b square trace of A square minus 4 times A c, so 4 times determinant of A. Now, if determinant of A is equal to 0, let us see what we get? We will be getting sigma 1 2 is nothing but half trace of A plus minus trace of A. So, therefore, sigma 1 becomes trace of A and sigma 2 becomes 0.

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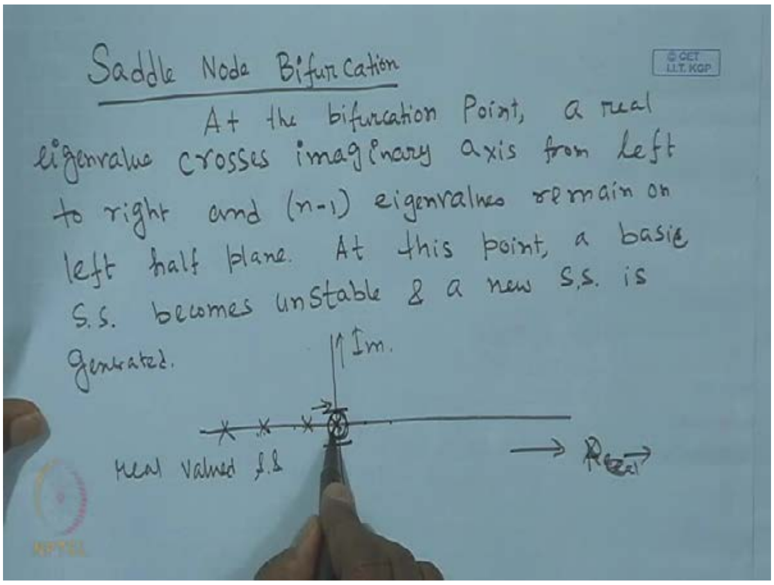

So, this is the condition where one of the Eigenvalues of the steady state, one of the Eigenvalues becomes 0 and steady state becomes from the unstable from the stable steady state, it becomes unstable steady state.

So, one of the Eigenvalue becomes 0, what is the condition for one of the Eigenvalue to be 0? It is the determinant of A has to be equal to 0 in order to have a saddle node bifurcation, so this is the point of saddle node bifurcation. So, the condition is determinant of A should be equal to 0 and by setting this condition determinant of A is equal to 0, one can get the appropriate condition of the system parameters, so that one can get a saddle node bifurcation point and can identify so that is the saddle node bifurcation.

Next, we talk about the hopf bifurcation. Hopf bifurcation point for A this occurs, for a set of parameter values real part of complex conjugate pair of Eigenvalues become positive and hence the solution is unstable. Now, if you again plot it on the real and imaginary axis, then you can understand graphically.

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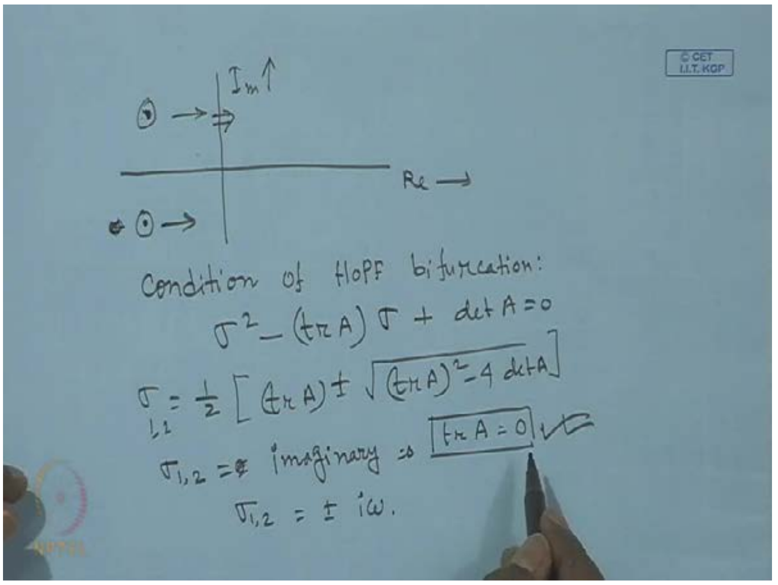

Suppose, this is the real axis and this is the imaginary axis and let us say it is a two-dimensional system, so we have two Eigenvalues which are complex conjugate. Let us say this is one Eigenvalue, this is a matching Eigenvalue (Refer Slide Time 42:38), which have both of them are having the negative real part. So, both the Eigenvalues are having negative real part, so they will be lying on the left half plane.

Now, by changing one parameter, so if you change the parameters one of this will be moving towards left and another will be moving towards right, so there may come that for a particular value of parameters, this may cross over the imaginary axis and moving into the right half plane. The point where it moves over the crosses over the imaginary axis and moves to the right half plane the Eigenvalues becomes positive and the steady state is no longer a stable steady state, it becomes an unstable steady state.

So, therefore, if you look into the condition of hopf bifurcation, if you look into the equation sigma of the characterization of the Jacobian matrix, sigma square minus trace of A times sigma plus determinant of A is equal to 0 and the roots are become 1 by 2 trace of A plus minus under root trace of A square minus 4 times determinant of A, so sigma 1 2 one of them becomes 0 - a purely imaginary.

Now, at this point when trace becomes 0, sigma 1 2 becomes plus minus i omega, so in fact they move in the same direction. So, at this point where trace of A becomes 0, then both of these complex conjugate Eigenvalues will move from the left half plane and goes to the right half plane and they becomes and they will crosses over the imaginary axis and the real part turns out to be positive when the real part turns out to be positive, then they becomes unstable. So, the bifurcation point will be the condition, when it moves over the left half plane to the right half plane and the condition is trace of A. So, by setting the condition, trace of A will be getting a condition on the parameters, so that hopf bifurcation point can be realized.

So, one can get an idea about the bifurcation points the hopf bifurcation point and the saddle node bifurcation point and based on that one can get an idea of the combination of the parameters so that one can land up into the critical points or the bifurcation point. So, in order to avoid the instability in the steady state, one can get an idea what will be the bifurcation point and what will be the values of the parameters to be selected. So, one can select the values of the parameter set by the bifurcation points and can ensure that the operator can operate or the plant can be operated at the stable steady state.

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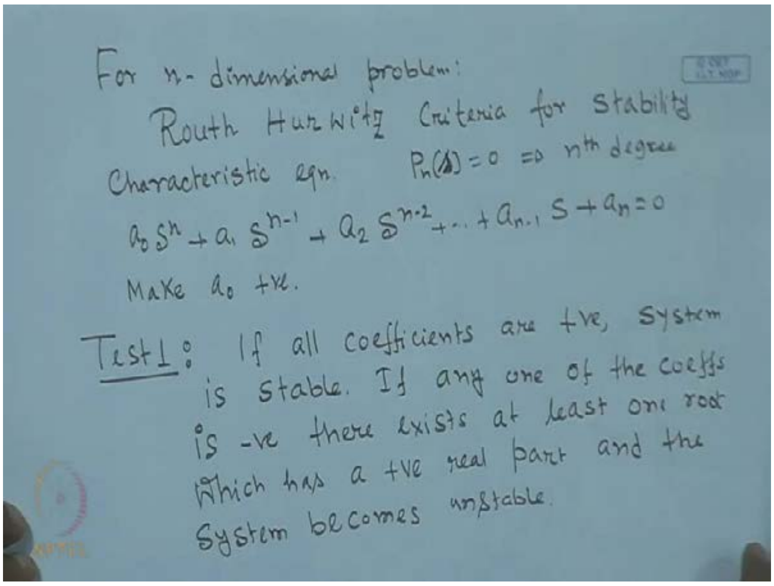

Next, we will look into a multi-dimensional problem; for a multi-dimensional problem we have just looked into the stability of a two-dimensional problem. For n dimensional problem one can use the Routh Hurwitz criteria for stability - the characteristic equation in this particular case - for an n dimensional matrix equation becomes a polynomial may be s, is the root polynomial of nth degree polynomial.

So, this becomes a 0 S to the power n plus a 1 S to the power n minus 1 plus a 2 S to the power n minus 2 up to a n minus 1 S plus a n is equal to 0. This is the nth degree of polynomial which becomes a characteristic equation of this particular problem.

Now, you make a 0 positive, even if a 0 is not positive then if it is negative, then multiply both side by minus 1 and make a 0 as positive, then the first test goes like this (Refer Slide Time 47:08). So, this is the test 1, test 1 says that, if all coefficients are positive, system is unconditionally stable - system is stable. If any one of the coefficients is negative there exists at least one root which has a positive real part and the system becomes unstable, that is the first test.

If we will make all the coefficients in the characteristic equation to be positive, if it has been observed that all coefficients becomes positive, then the system is stable. If anyone of the coefficients is negative, now there exists at least one root which will be having a positive real part and the system becomes unstable.

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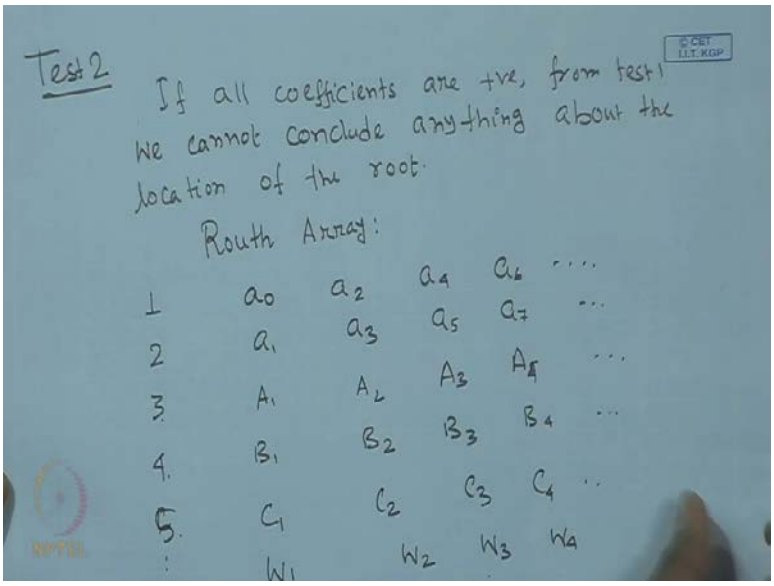

So, that is the first test and then we go to a more detailed test that is test number 2, this test 2 is that if all coefficients are positive, from the first test - from test 1 - we cannot conclude anything about the location of the root, so we can say that the roots are positive but all coefficients are positive from test number 1, we cannot conclude anything about the location of the root.

So, for that, to locate the root one has to form the Routh array, the Routh array is this, we write down the row 1 will be a 0 a 2 a 4 a 6, all the even coefficient row 2 a 1 a 3 a 5 a 7, odd coefficients row 3 capital A 1 capital A 2 capital A 3 A 5 like that, 4 will be B 1 B 2 B 3 B 4 like that, fifth row will be C 1 C 2 C 3 C 4; likewise, then we will be having n plus one throw will be W 1 W 2 W 3 W 4. Now, these are the coefficients of the characteristic equation, then A 1 A 2 and B 1 etcetera, are expressed in terms of this coefficients.

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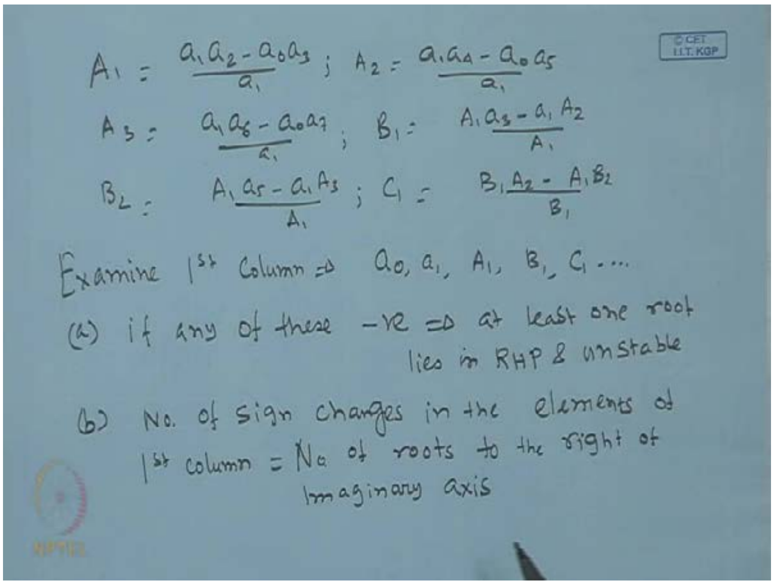

So, we write A 1 as a 1 a 2 minus a 0 a 2 divided by a 1; write A 2 as a 1 a 4 minus a 0 a 5 divided by a 1; A 3 as a 1 a 6 minus a 0 a 7 divided by a 1; We write b 1 as A 1 a 3 minus a 1 A 2 divided by A 1; B 2 as A 1 a 5 minus a 1 A 3 divided by A 1; C 1 as B 1 A 2 minus A 1 B 2 divided by B 1 like that.

Now, examine the first column of the Routh array, this will constitute of a 0, a 1, capital A 1, capital B 1, capital C 1 like that. Now, conclusion is, if anyone of these is negative, then at least one root lies in right half plane and system becomes unstable. Number b is, second interpretation of this Routh array is that, number of sign changes in the elements of the first column is equal to number of roots to the right of imaginary axis.

So, the condition is that, all the elements of the first column of Routh array should be positive for having a stable condition, as well as all the elements in the characteristic equation has up to be positive for stable conditions.

If we look into the first column, if any one of these is negative, then at least one root lies in the right half plane and the system becomes unstable and number of sign changes in the elements of the first column is equal to number of roots to the right half plane of the imaginary right half or the right half imaginary axis to have the unstable condition, this completes the theorem of stability for an n dimensional system.

So, by using the Eigenvalues of the characteristic equation and the next we will be taking about some of the example to demonstrate the chemical engineering system, how to utilize this method to identify the steady state and how to test the stability of this steady states and the conditions for the saddle node bifurcations and hopf bifurcations.

Thank you very much.