Fluid Mechanics & its Applications
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Lecture 29A
Von Karman Momentum Integral Method

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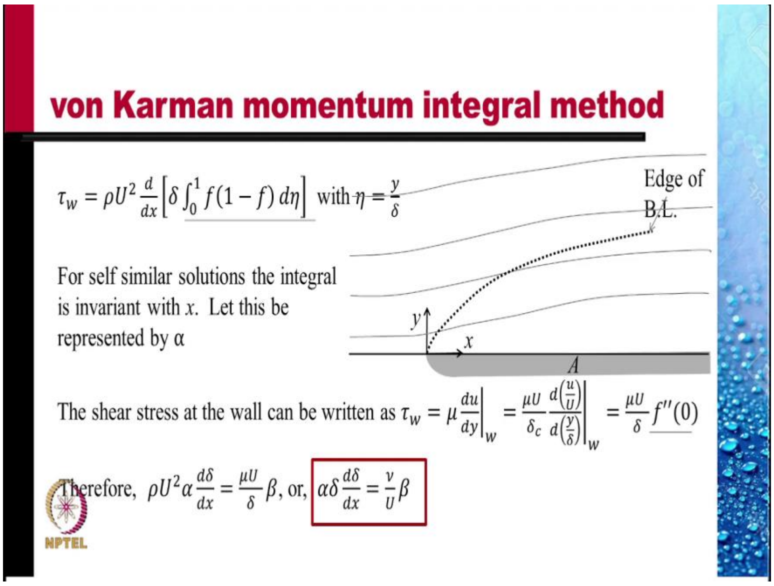

Von Karman momentum Integral equation.

We obtained τ_w in terms of the momentum thickness and that is the expression that we have with $\eta = \frac{y}{\delta}$. The moment of thickness is written like this, and so for self-similar solutions, this integral is invariant with this x. Let this integral be represented by α. Now, α is not a function of x for self-similar solutions, the kind that we can assume on a flat plate for sure. For many other geometries we get self-similar solutions, but that is beyond the level of this course.

The shear stress at the wall, the left hand side of the equation, can be written as $\tau_w = \mu \frac{du}{dy}\Big|_w$,

and which can be shown as equal to $\frac{\mu U}{\delta} f''(0)$. Therefore, $\rho U^2 \alpha \frac{d\delta}{dx} = \frac{\mu U}{\delta}\beta$, where β is $f''(0)$, that is $\frac{d\left(\frac{u}{U}\right)}{d\left(\frac{y}{\delta}\right)}\Bigg|_w$, and we call this we name this beta.

So, with this β, I get an equation $\alpha\delta \frac{d\delta}{dx} = \frac{\nu}{U}\beta$. Now, α and β are known once we know the shape of the velocity profile, and that is independent of x.

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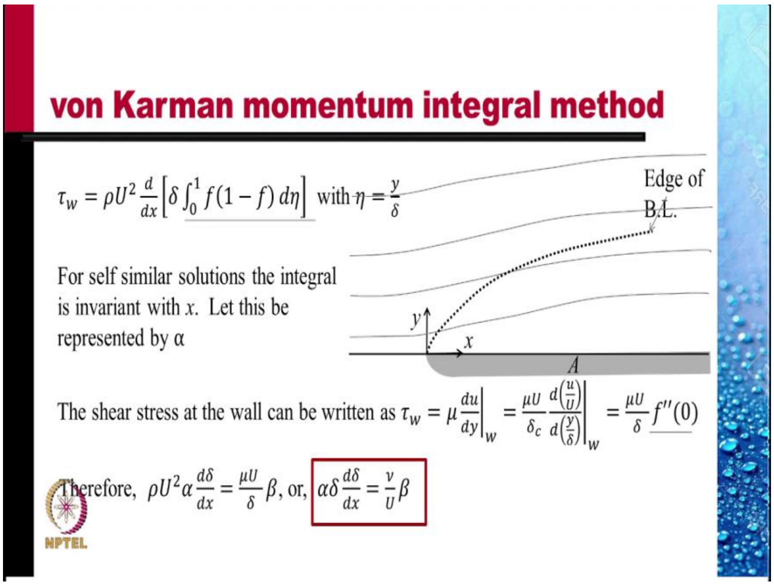

So, this equation can be easily integrated to give $\delta = \sqrt{\frac{2\nu}{U}\frac{\beta}{\alpha}x}$ for the condition $\delta = 0$ for $x = 0$,. The boundary layer thickness at the start of the plate is obviously 0. This is the essence of the Karman momentum integral method.

von Karman, one of the greatest name of fluid dynamics, almost of the same stature as Prandtl, proposed that the value of β and αa are quite insensitive to the actual shape of the velocity profile as long as we meet the basic requirements of the velocity profile. And what are the basic requirements of velocity profile? u at the wall must be 0, v at the wall must be 0, and at the edge of the boundary layer, u must tend to U asymptotical.

That is, the derivative of the velocity profile should also tend to 0 as y tends to infinity, or at the edge of the boundary layer. So, Von Karman proposed that we choose a velocity profile to meet these conditions. And if you do this, evaluate α and β, which are simple integration, and then the growth of δ can be obtained in this manner.

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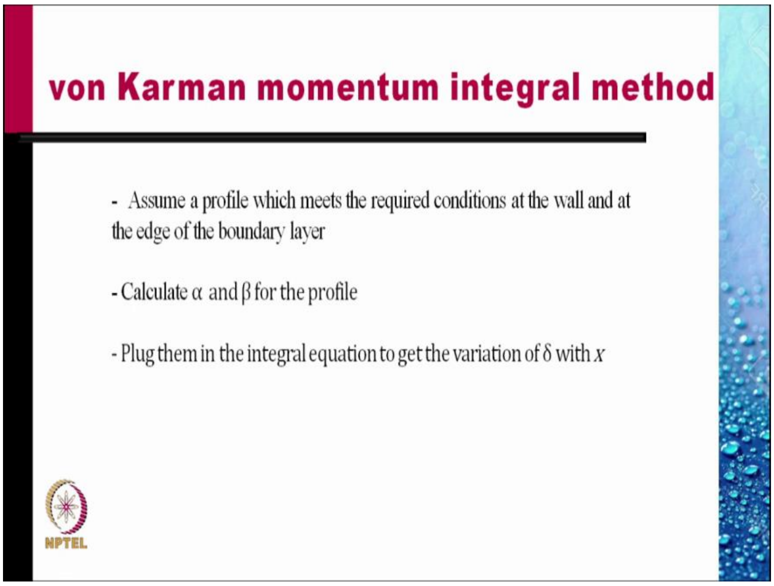

So, in method has the following steps.

Assume a profile which meets the required condition at the wall and at the edge of the boundary layer.

Calculate alpha and beta for the profile.

Plug them in the integral equation to get the variations of δ with x.

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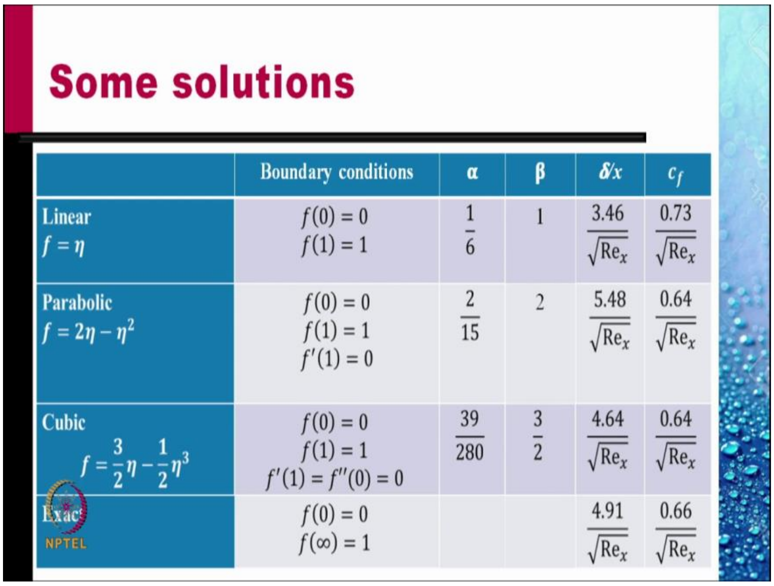
Some solutions

	Boundary conditions	α	β	δ/x	c_f
Linear $f=\eta$	$f(0)=0$, $f(1)=1$	$\frac{1}{6}$	1	$\frac{3.46}{\sqrt{Re_x}}$	$\frac{0.73}{\sqrt{Re_x}}$
Parabolic $f=2\eta-\eta^2$	$f(0)=0$, $f(1)=1$, $f'(1)=0$	$\frac{2}{15}$	2	$\frac{5.48}{\sqrt{Re_x}}$	$\frac{0.64}{\sqrt{Re_x}}$
Cubic $f=\frac{3}{2}\eta-\frac{1}{2}\eta^3$	$f(0)=0$, $f(1)=1$, $f'(1)=f''(0)=0$	$\frac{39}{280}$	$\frac{3}{2}$	$\frac{4.64}{\sqrt{Re_x}}$	$\frac{0.64}{\sqrt{Re_x}}$
Exact	$f(0)=0$, $f(\infty)=1$			$\frac{4.91}{\sqrt{Re_x}}$	$\frac{0.66}{\sqrt{Re_x}}$

In this table, we have tried 3 different profiles: linear, parabolic and cubic. And compared them with the exact profile, really not exact, but the solution obtained by Blasius. For the linear profile, we need only 2 conditions, $f(0) = 0$, the velocity at the wall is 0, and the velocity at the edge of boundary layer, $\frac{u}{U} = 1$. A linear velocity profile.

For this the value of α is readily obtained as 1/6, the value of β which depends on the slope of the velocity profile is obtained as 1. δ/x is then obtained as $\frac{3.46}{\sqrt{\mathrm{Re}_x}}$ and the skin friction coefficient C_f is evaluated as $\frac{0.73}{\sqrt{\mathrm{Re}_x}}$.

For a parabolic profile, we did three conditions, and we use the $f(0) = 0$, $f(1) = 1$, and $f'(1) = 0$. The slope of the velocity profile at the edge is also 0. $f = 2\eta - \eta^2$ meets these three conditions. And for this profile, the value of α is 2/15, the value of β is 2, and we can calculate δ/x and C_f.

For the cubic profile with these four conditions. so it is not just the velocity is to 1, but the first derivative of the velocity, and second derivative of the velocity are also 0 at the edge of the boundary. So, with this four condition, the cubic profile has this simple form. For this the values α is obtained is 39/280, β is 3/2, and these are the values of δ/x and C_f.

For the Blasius profile, we get the result δ/x as 4.91 and C_f as 0.66 divided by $\sqrt{\mathrm{Re}_x}$. The Blasius solution in his day took months of calculations on mechanical calculator, and von Karman results were obtained in minutes, and you can see that the results are quite accurate. That is the power of integral methods. Obtain very quick results which are good enough for most purposes.

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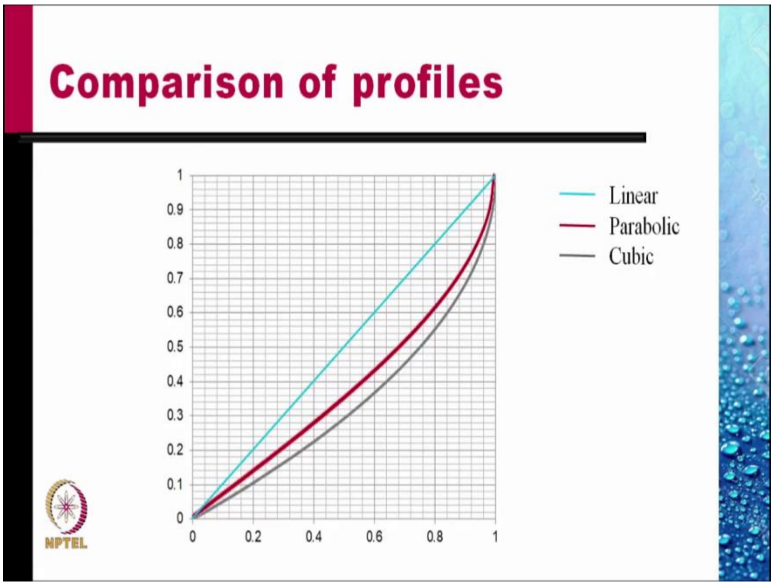

The comparison of the three profiles that were chosen the linear, parabolic and cubic profile across the boundary layer compared to the actual profile due to Blasius, which is shown in black.

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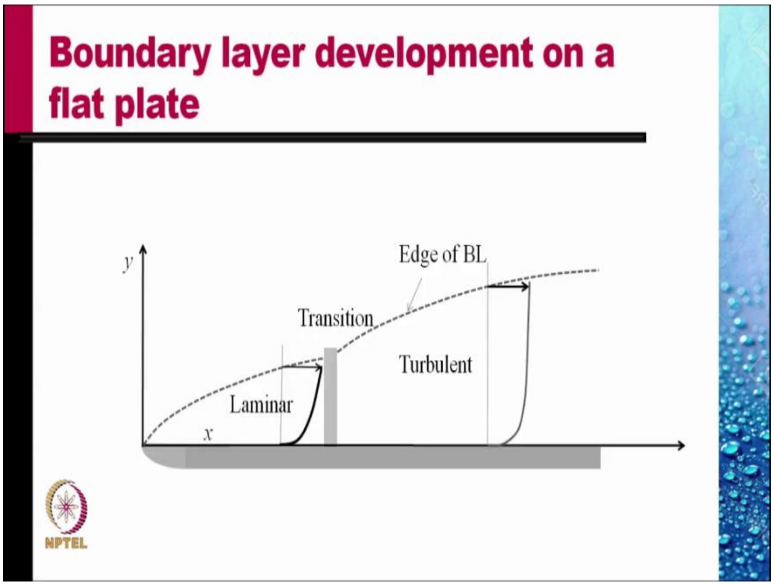

Now, all this development is for the laminar flows, because the nature of the velocity profile that we have taken. The Blasius profile is only for the laminar flows, because we have written shear stress to be $\mu\frac{\partial u}{\partial y}$.

In turbulent flows that will no longer be true. Though at the wall, the shear stress is still $\mu\frac{\partial u}{\partial y}$, but that is within the laminar sub-layer at the wall. As the distance along the plate increases, the value of x increases, the Reynolds number Re based on x increases. And at a value of about 5×10^5, the boundary layer because unstable, it becomes turbulent. And the growth pattern of the boundary layer changes.

There is not sufficient theory for the turbulent boundary layers. Various attempts have been made, and the structure of the turbulent boundary layer is too complicated to be discussed at this first course level. But today, using direct numerical simulation of the Navier Stokes equation, researchers can calculate the turbulent flow on flat plate or any other shaped body with great amount of accuracy.

Here we will discuss some very elementary ideas of what we can do with the turbulent boundary layers.

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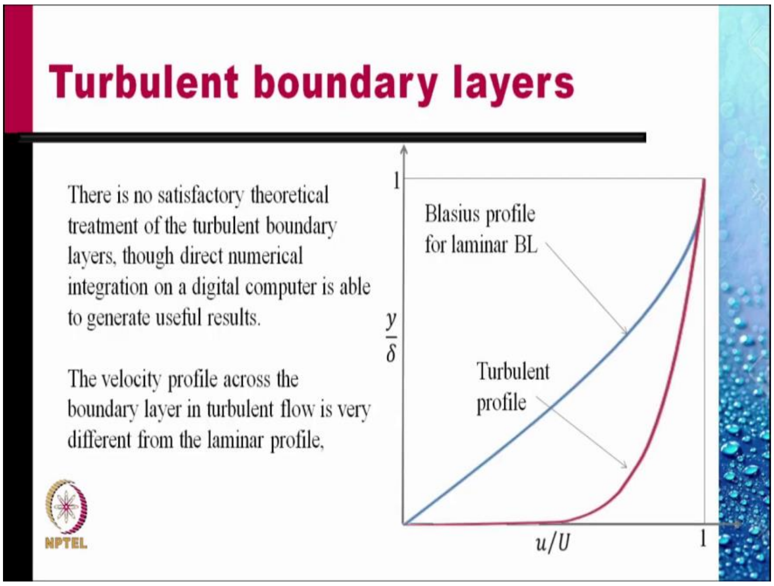

First of all, notice that the velocity profile across the boundary layer in turbulent flows is very different from the laminar flows. The Blasius profile is shown in blue, and the turbulent profile is shown in red. The slope du/dy at wall, that is, at $y = 0$, is very sharp, and then the growth is slow.

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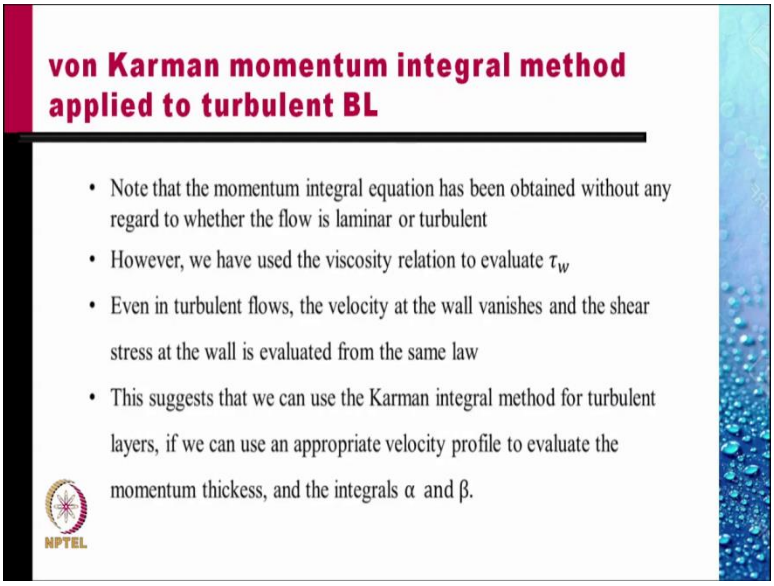

In fact, notice that we have been able to obtain the momentum integral equation without any regard to whether the flow is laminar or turbulent. However, we have used the viscosity relation to evaluate τ_w. Even in turbulent flows, the velocity of the wall vanishes, and the shear stress at the wall is evaluated from the same law. This suggests that we can use the Karman integral method for turbulent layers, if we can use any appropriate velocity profile to evaluate the momentum thickness, and the integrals α and β, used in the integral equation.

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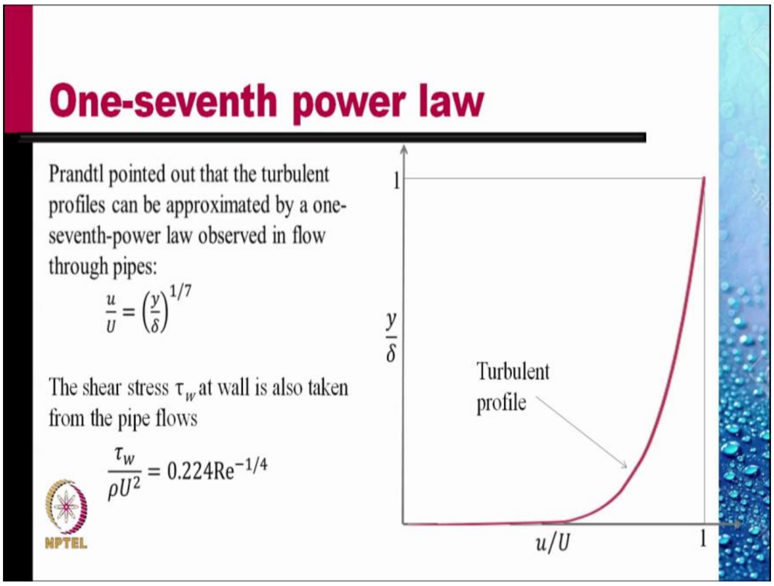

Prandtl pointed out that the turbulent profile can be approximated by a one-seventh power law, observed in flow through pipes. That is, $\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7}$. The red line here is what I have plotted, one-seventh power law. This is very close to the turbulent profile. This was first experimentally obtained in flows through pipes. And Prandtl also suggested that we could use the expression for the shear stress at the wall from the pipe flow itself, and for the pipe flow, he had obtained $\frac{\tau_w}{\rho U^2} = 0.224\text{Re}_\delta{}^{-1/4}$, for turbulent flow through pipes.

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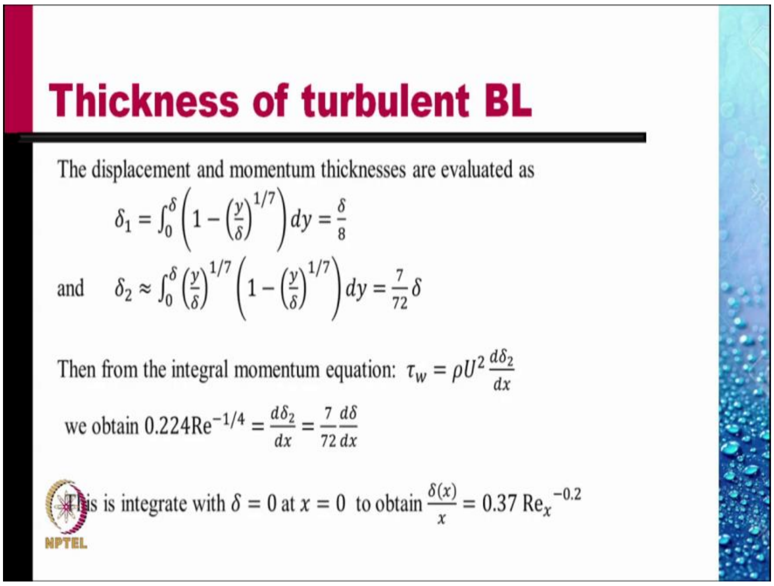

With these the value of δ_1 and δ_2 are obtained as $\delta/8$, and $7\delta/72$. And then from the momentum integral equation using the appropriate relation for τ_w, we obtain that for the turbulent flows the thickness of the boundary layer grows like this.

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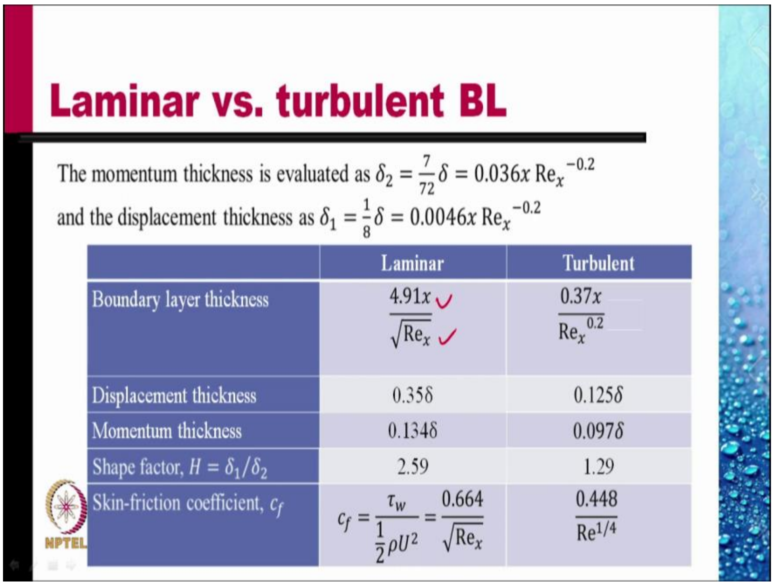

This table shows the difference between the laminar and the turbulent flows. You see the boundary layer thickness in the laminar flows, grows like $x^{0.5}$, like $\sqrt{x}$, because there is x in the numerator, and there is a $\sqrt{x}$ in the denominator. Reynolds number based on x contains x in the numerator.

So, when we take square root of that, we have x raised to power half, and so, the net variation of the boundary layer thickness of the laminar flows is like x raised to power half. And for the turbulent flows by the same argument this is like $x^{0.8}$. The displacement thickness in laminar flows is one-third, about one third, a little more than a one-third of the boundary layer thickness, but in the turbulent flow it is only one-eighth. The deficit of mass flow is much less.

Similarly, the momentum thickness from 0.134 it reduces to 0.097. The shape factor values reduce from 2.59 to 1.29. Turbulent flows are less likely to separate. Skin friction coefficient decreases as root x in laminar flow, but x raised to power one fourth in the turbulent flows.

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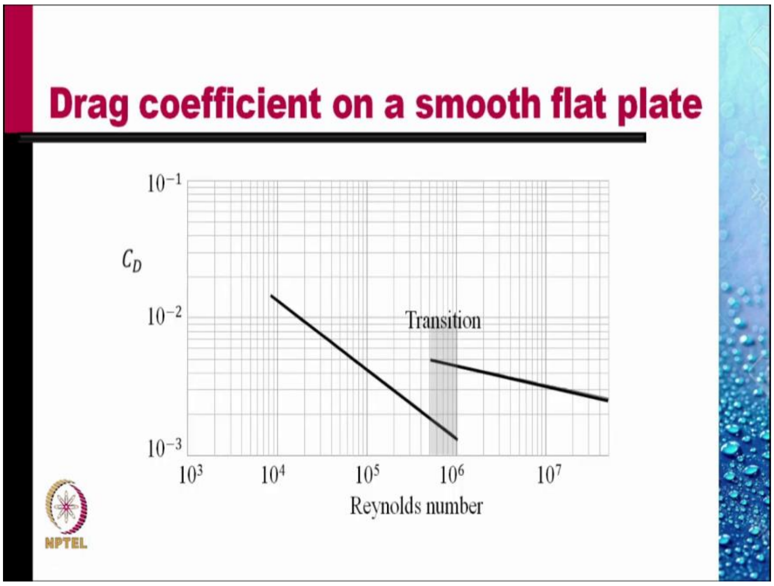

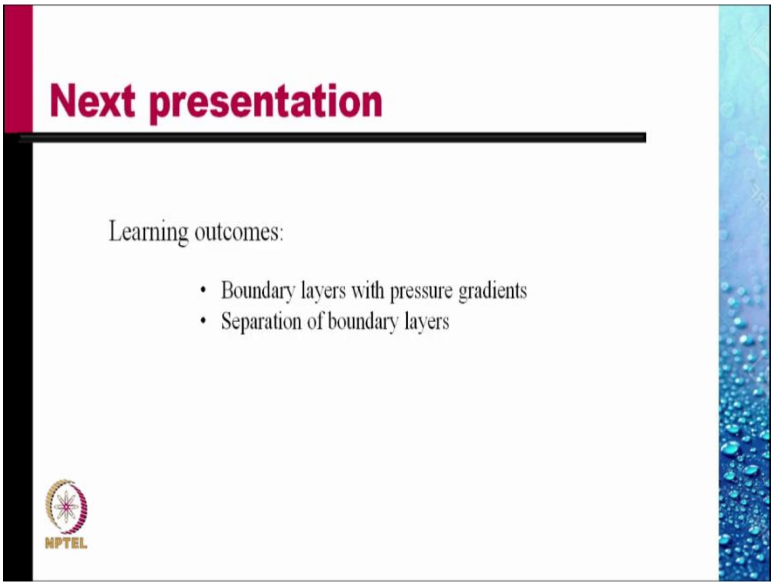

This graphs shows typically the variation of drag coefficient with the Reynolds number. The transition occurs at the value of Reynolds number based on the location distance x of about 5 into 10 raised to power 5. The drag coefficient reduces at a much lower rate in the turbulent flows than in the laminar boundary layer.

Thank you very much.