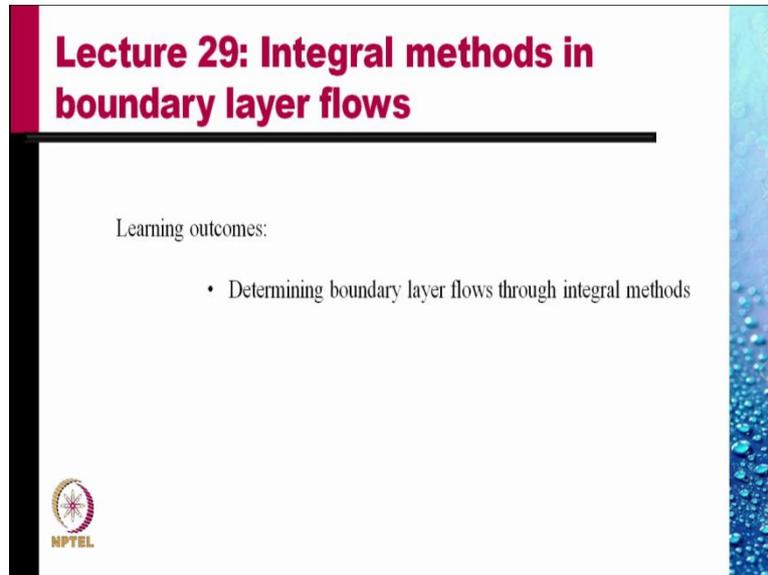


Fluid Mechanics & its Applications
Professor Vijay Gupta
Indian Institute of Technology, Delhi
Lecture 29
Integral methods in boundary layer flows

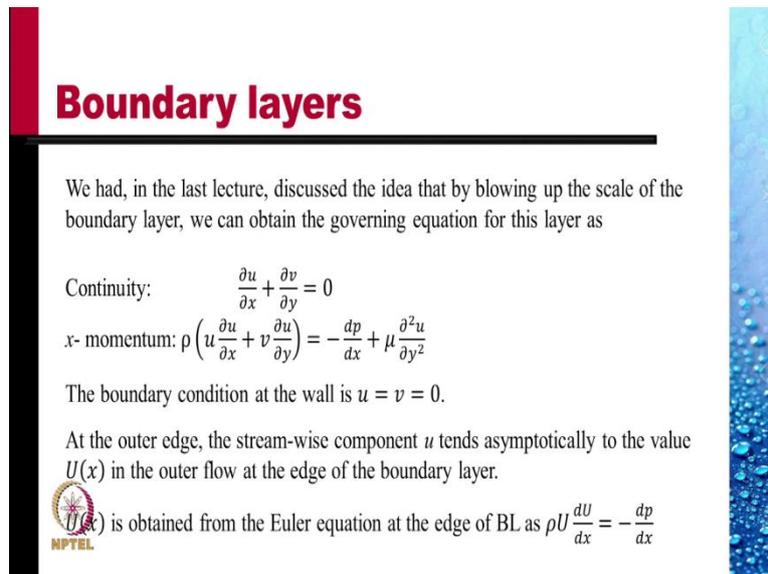
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Lecture 29: Integral methods in boundary layer flows

Learning outcomes:

- Determining boundary layer flows through integral methods



Boundary layers

We had, in the last lecture, discussed the idea that by blowing up the scale of the boundary layer, we can obtain the governing equation for this layer as

Continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

x- momentum: $\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{dp}{dx} + \mu \frac{\partial^2 u}{\partial y^2}$

The boundary condition at the wall is $u = v = 0$.

At the outer edge, the stream-wise component u tends asymptotically to the value $U(x)$ in the outer flow at the edge of the boundary layer.

$U(x)$ is obtained from the Euler equation at the edge of BL as $\rho U \frac{dU}{dx} = -\frac{dp}{dx}$

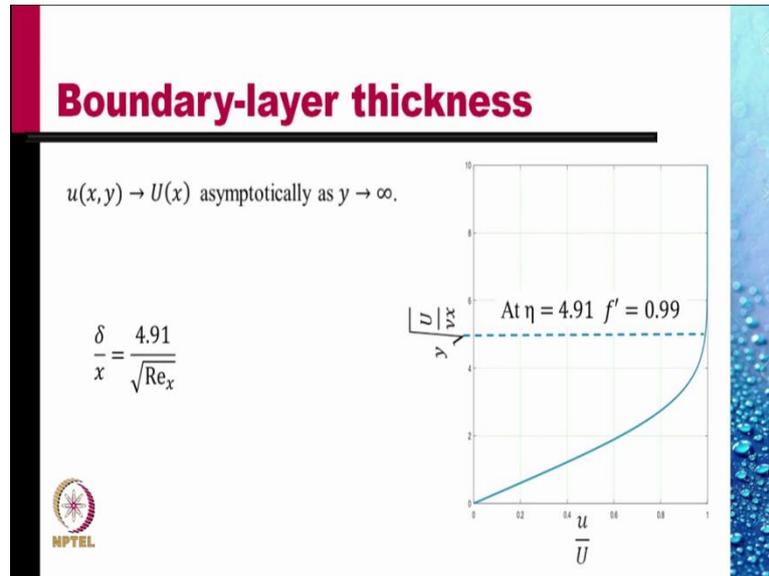


Welcome back.

We had, in the last lecture, discussed the idea that by blowing up the scale of the boundary layer, we can obtain the governing equation for that layer as $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ for continuity equation, and for momentum equation $\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{dp}{dx} + \mu \frac{\partial^2 u}{\partial y^2}$. Notice that we have been able to retain the viscous term for $\mu \frac{\partial^2 u}{\partial y^2}$, but the viscous term $\mu \frac{\partial^2 u}{\partial x^2}$ is still negligible.

The boundary conditions at the wall would be no slip, that is, $u = v = 0$. At the outer edge, the stream-wise component u tends asymptotically to a value $U(x)$ that is the velocity of the outer flow at the edge of the boundary layer, $U(x)$ that is obtained from the Euler equation at the edge of boundary layer as $\rho U \frac{dU}{dx} = -\frac{dp}{dx}$.

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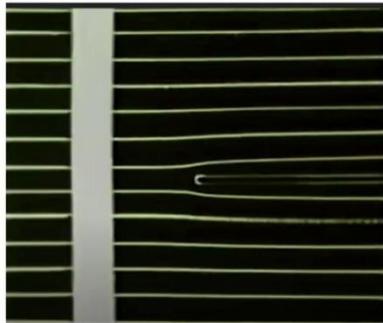
You notice here that the stream-wise velocity component $u(x, y) \rightarrow U(x)$, the velocity at the edge of boundary layer asymptotically as $y \rightarrow \infty$. We had obtained this solution, the Blasius solution, in terms of u/U as a function of η . The self-similar variable $y \sqrt{\frac{U}{\nu x}}$, where ν is the kinematic viscosity of the fluid.

The velocity u becomes $U(x)$ only very far away at $y \rightarrow \infty$. So, how do we determine or define the boundary layer thickness? The boundary layer thickness in this case is arbitrarily defined as the distance by which the velocity in the boundary layer reaches 99 percent of the value in the inviscid flow. Since, the velocity u/U is given by f' .

So, we determine the value of η at which the value of f' is 0.99, and that value η is 4.91. So, we can define $\frac{\delta}{x} = \frac{4.91}{\sqrt{Re_x}}$. In many places, this value is given as $\frac{5}{\sqrt{Re_x}}$. The difference is small and is largely because of approximations.

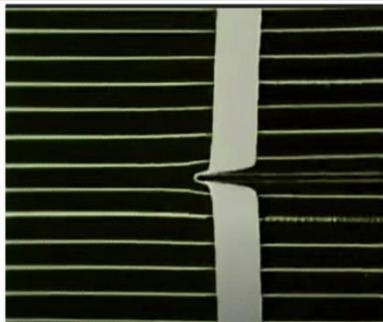
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Growth of boundary layer on a flat plate



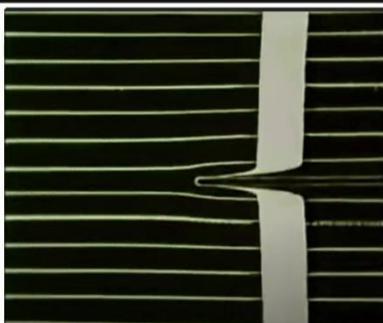
 Abernathy, F., *Fundamentals of Boundary Layers*, A 16-mm film produced for NCFMF, USA. Can be viewed at YouTube

Growth of boundary layer on a flat plate

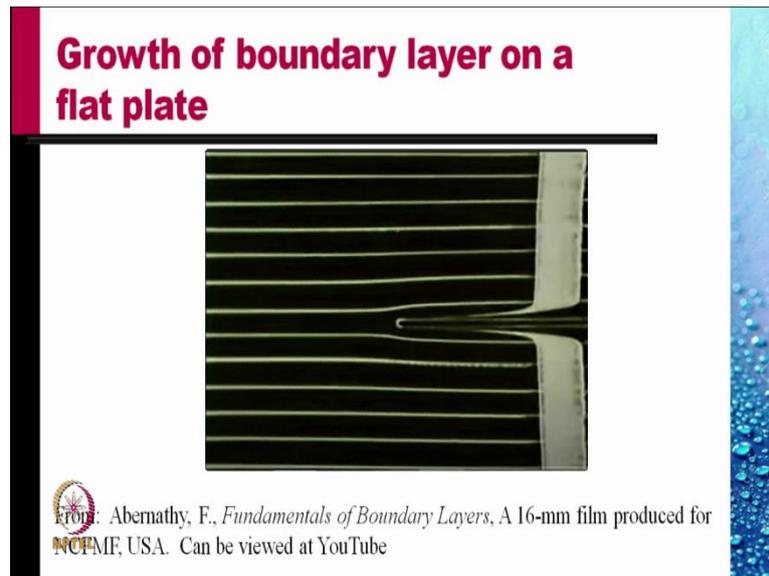


 Abernathy, F., *Fundamentals of Boundary Layers*, A 16-mm film produced for NCFMF, USA. Can be viewed at YouTube

Growth of boundary layer on a flat plate



 Abernathy, F., *Fundamentals of Boundary Layers*, A 16-mm film produced for NCFMF, USA. Can be viewed at YouTube



This series of shots from the movie *Fundamentals are Boundary Layers* by Abernathy that was produced in 1962 for the National Council of Fluid Mechanics Films that can be viewed at YouTube. It shows the development of boundary layer on a flat plate. The flow visualization is done in a water tunnel where hydrogen bubbles are produced by passing current through an electrode held perpendicular to the flow.

Current is passed through a small interval δt , and this white band of hydrogen bubbles are formed. These are washed down with the flow past the flat plate, and after a little while this is a kind of profile that you develop. This clearly shows the development of the boundary layer on a flat plate. The velocity at the plate surface is 0, and it tends to the velocity capital U far away.

At a little later time this band looks like this. This band is known as a timeline in fluid mechanics. This is at a given time. What does the flow velocity look like? At still later time, this pattern is this. This shows clearly the growth of the thickness of the boundary layer on the flat plate when that is aligned along the free stream.

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Displacement thickness

A physically meaningful measure of BL thickness is the displacement thickness, which is defined as the distance by which the external potential flow is displaced outwards as a consequence of the decrease in velocity within the boundary layer

$$\int_0^{\infty} (U - u) dy = U \delta_1$$

$$\delta_1 = \int_0^{\infty} \left(1 - \frac{u}{U}\right) dy$$

NPTEL

But, as we said, the boundary layer thickness defined in the manner earlier that is where the velocity of the flow reaches 99 percent of the inviscid velocity is arbitrary. A physically more meaningful measure of boundary layer thickness is the displacement thickness, which is defined as the distance by which the external potential flow is displaced outwards as a consequence of the decrease in velocity within the boundary layer.

This is the boundary layer velocity profile. And if we calculate this area, then this is the volume per unit depth of the fluid that flows past the flat plate aligned at $y = 0$. This same flow is obtained if the flow velocity does not decrease near the plate, but everywhere the flow velocity is U . Then the same flow is obtained in this distance. That means to say that because of the development of boundary layer, the flow past the flat plate decreases, and decreases by the amount which is equivalent to this thickness. And this is termed as the displacement thickness δ_1 .

So, if we shift the boundary by this is δ_1 , and we let the flow proceed with the velocity U everywhere, then the flow rate would be the same as the actual flow rate on the actual plate with the development of boundary layer. The flow deficit that develops because of the presence of the boundary layer, is this area, which is $\int_0^{\infty} (U - u) dy$. And we equate it to capital U times δ_1 , the displacement thickness.

This gives $\delta_1 = \int_0^{\infty} \left(1 - \frac{u}{U}\right) dy$. And if we use the self-similarity profile for the boundary layer discussed in the last lecture, $\frac{u}{U}$ is the function $f(\eta)$, where η is the self-similar variable

$\frac{y}{\sqrt{vx}}$. Then δ_1 is evaluated as this, where subscript 1 at the end denotes that the value of $[\eta - f(\eta)]$ should be evaluated far away from the boundary.

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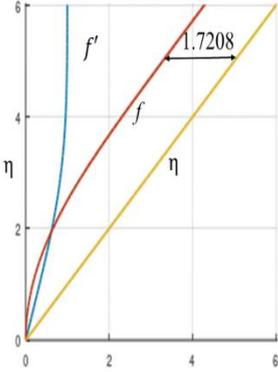
Displacement thickness

$$\delta_1 = \sqrt{\frac{vx}{U}} \int_0^{\infty} (1 - f') d\eta = \sqrt{\frac{vx}{U}} [\eta - f(\eta)]_1$$

$$\frac{\delta_1}{x} = \frac{1.7208}{\sqrt{Re_x}}$$

It is about one-third of the boundary layer thickness δ defined as the location where u is about $0.99U$





If you plot f' , which is u/U , and we plot f , then we plot η . Then $\eta - f$ tends to a constant value of 1.7208, as η tends to infinity. So, the value of this is 1.7208, and the displacement thickness is now given by this expression: $\frac{\delta_1}{x} = \frac{1.7208}{\sqrt{Re_x}}$. δ_1 varies like $x^{1/2}$. This thickness is about one-third of the boundary layer thickness delta defined earlier as the location where u is 0.99 of capital U .

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Displacement thickness

Another way to interpret displacement thickness is to regard it as the distance by which the streamlines of the external flow are displaced because of the slowing down of fluid within the boundary layer.

Edge of B.L.

Displacement thickness



Another way to interpret displacement thickness is to regard it as the distance by which the streamlines of the external flow are displaced because of the slowing down of the fluid within the boundary layer. Thus, if the flow proceeded straight it goes like this, but now, the streamlines are shifted upwards, and at this location A, this is the distance by which a streamline is shifted. This is the displacement thickness.

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Transverse velocity in the boundary layer on flat plate

With $\eta = y \sqrt{\frac{U}{\nu x}}$ and $u = Uf'(\eta)$, the stream function is given by $\psi = \sqrt{\nu x U} f(\eta)$.

From this we obtain the transverse velocity

$$v = -\frac{\partial \psi}{\partial x} = \frac{1}{2} \sqrt{\frac{\nu U}{x}} (\eta f' - f)$$

The vertical velocity at the edge of boundary layer is

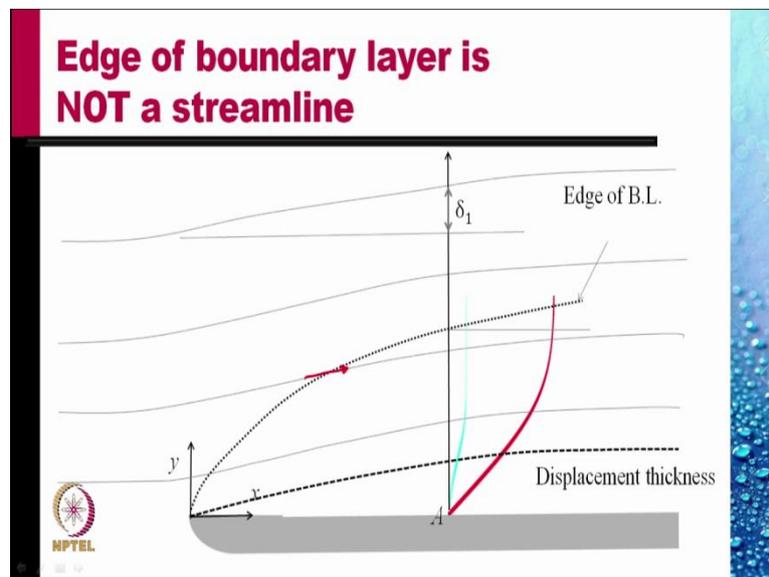
$$\frac{v}{U} = \frac{0.86}{\sqrt{Re_x}} \text{ positive}$$


To calculate the transverse velocity of the fluid in the boundary layer on the flat plate, that is the velocity in the y direction, we determine first the stream function with $\eta = y \sqrt{\frac{U}{\nu x}}$, and the stream-wise velocity $u = Uf'(\eta)$.

The stream function is given by integration of U as $\psi = \sqrt{vxU}f(\eta)$. And once we know ψ , we can find out the transverse velocity by taking its gradient along the x direction. $v = -\frac{\partial\psi}{\partial x}$, and this is obtained as $\frac{1}{2}\sqrt{\frac{vU}{x}}(\eta f' - f)$.

If you plot this, this is the stream-wise velocity we get as a function of η . It reaches a maximum value of 0.86 in terms of the non dimensionalized variable. So, that we can write $\frac{v}{U} = \frac{0.86}{\sqrt{Re_x}}$. Notice that this is positive. So, the transverse velocity is upwards in the boundary layer above the plate. You also notice, the transverse velocity is like $\frac{1}{\sqrt{Re_x}}$, that is, it is very small compared to U.

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This picture shows the transverse velocity and the stream-wise velocity, stream-wise in red transverse velocity in blue. Both the displacement thickness, the boundary layer thickness and the transverse velocity are shown exaggerated. But this shows that this is small compared to the value in the stream-wise direction.

Notice here that the edge of boundary layer is not a streamline. The streamlines of the flow enter the boundary layer as we go down the plate in the x direction, like shown here. This does not indicate that the v is negative, v is still positive. This streamline is sloped upwards, confirming that v is indeed positive, but the slope of the edge of the boundary layer is more upwards. That is why the flow is entering the edge of the boundary layer. There is entrainment of fluid within the boundary layer.

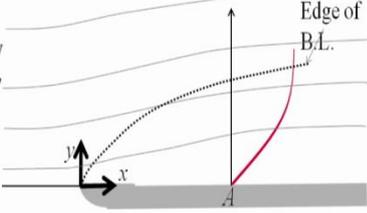
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Momentum thickness

Momentum flux at A: $\int_0^{\infty} \rho u^2 dy$

Momentum flux if the actual mass flow had the inviscid velocity U: $\int_0^{\infty} \rho u U dy$

Resultant deficit of momentum flux:

$$\int_0^{\infty} \rho u (U - u) dy$$


This can be accounted for if we displace the boundary upwards (in inviscid flow) by δ_2 given by

$$\delta_2 = \frac{1}{\rho U^2} \left[\int_0^{\infty} \rho u (U - u) dy \right] = \int_0^{\infty} \frac{u}{U} \left(1 - \frac{u}{U} \right) dy$$

NPTTEL

This is another concept that is used routinely in boundary layer theory and that is the concept of momentum thickness. Now the momentum flux at A is $\int_0^{\infty} \rho u^2 dy$. Here, the mass flux in a strip of thickness dy , $\rho u dy$, and the stream-wise velocity is u . So, the x momentum is $\rho u^2 dy$ in a strip of thickness dy at y . And the total momentum is flux is obtained by integrating this across the boundary layer.

Now, the momentum flux, if the actual mass flow had the inviscid velocity capital U , would have been $\int_0^{\infty} \rho u U dy$, that is, the mass flow rate $\rho u dy$ multiplied by the inviscid velocity U . This results in a deficit of moment of flux of $\int_0^{\infty} \rho u (U - u) dy$.

This can be accounted for if we displace the boundary, that is the plate, upwards. Then the inviscid flow by an amount δ_2 given by $\delta_2 = \frac{1}{\rho U^2} \left[\int_0^{\infty} \rho u (U - u) dy \right]$, which can be written in this manner. Recall that u/U within the boundary layer on a flat plate, which is self similar, is a function of η , which is y/δ , which is a function of η . If we express this as integral in terms of η , then this will be independent of x , and that is what we do next.

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Momentum thickness

$$\delta_2 = \frac{1}{\rho U^2} \left[\int_0^\infty \rho u (U - u) dy \right] = \int_0^\infty \frac{u}{U} \left(1 - \frac{u}{U} \right) dy$$

With $\frac{y}{\delta_c} = y \sqrt{\frac{U}{\nu x}} = \eta$

For self-similar flows, we can write $\frac{u}{U} = f'(\eta)$

$$\delta_2 = \sqrt{\frac{\nu x}{U}} \int_0^\infty f'(1 - f') d\eta$$

0.664 for flat plate

For flat plate, $\frac{\delta_2}{x} = \frac{0.664}{\sqrt{Re_x}}$



So, with $\frac{y}{\delta_c}$, the characteristic thickness in the y direction, as $\sqrt{\frac{\nu x}{U}}$, we get $y \sqrt{\frac{U}{\nu x}} = \eta$ as introduced by Blasius. And then $\frac{u}{U} = f'(\eta)$, and we get $\delta_2 = \sqrt{\frac{\nu x}{U}} \int_0^\infty f'(1 - f') d\eta$. This clearly is independent of x. For a flat plate, the value of this is 0.664. So that we can write $\frac{\delta_2}{x} = \frac{0.664}{\sqrt{Re_x}}$. So δ_2 also varies like \sqrt{x} .

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Boundary layer on a flat plate

Boundary layer thickness, δ	$\frac{\delta}{x} = \frac{4.91}{\sqrt{Re_x}}$
Displacement thickness, δ_1	$\frac{\delta_1}{x} = \frac{1.72}{\sqrt{Re_x}}$
Momentum thickness, δ_2	$\frac{\delta_2}{x} = \frac{0.66}{\sqrt{Re_x}}$
Skin friction coefficient, c_f	$c_f = \frac{\tau_w}{\frac{1}{2} \rho U^2} = \frac{0.664}{\sqrt{Re_x}}$

Some authors use δ^* and θ to denote the displacement thickness and momentum thickness, respectively, instead of δ_1 and δ_2 used here



In this table, we summarize some results for a flat plate boundary layer. The boundary layer thickness δ defined as the location where 99 percent of the inviscid flow velocity is reached is given by 4.91 divided by under root Reynolds number. The displacement thickness delta 1

is 1.73 divided by under root Reynolds number based on x, and notice this is about one-third of the boundary layer thickness.

The momentum thickness is another third lower, $\frac{0.66}{\sqrt{Re_x}}$, and the skin friction coefficient, which is defined as $\frac{\tau_w}{\frac{1}{2}\rho U^2}$, is obtained as $= \frac{0.664}{\sqrt{Re_x}}$. This skin friction is obtained here as $\mu \frac{du}{dy}$, which we obtained in the last class as $\frac{0.664}{\sqrt{Re_x}}$ times $\frac{1}{2}\rho U^2$.

Note that some authors use δ^* and θ to denote the displacement thickness and momentum thickness, respectively, instead of δ_1 and δ_2 used here. So, do not get confused if you follow some other authors.

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Shape factor

The ratio δ_1/δ_2 of a boundary layer is often used as a measure of the shape of velocity profile.

The value of the shape factor on the flat plate aligned to the flow, that is, for the Blasius profile has a value of $H = 1.721/0.664 = 2.59$

Since δ_2 is related to the shear stress at the wall, a high value of this ratio indicates a decreased shear stress.

It will be seen that at the point of separation the shear stress tends to vanish. Thus, an increasing value of $H = \delta_2/\delta_1$ indicated that the separation is imminent

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The slide includes a diagram of a flat plate with several boundary layer profiles. The profiles are shown as dashed lines. The shear stress at the wall is indicated by red arrows pointing downwards from the wall. The diagram shows the shear stress increasing from left to right, and then decreasing to zero at the point of separation, where the flow reverses direction. A red circle highlights the point of separation.

The ratio δ_1/δ_2 of a boundary layer is often used as a measure of the shape of the velocity profile. The value of the shape factor on a flat plate aligned to the flow, like we have been discussing so far, that is, for a Blasius profile, has a value of H, the shape factor is 1.721 divided by 0.664, that is 2.59.

Since δ_2 is related to the shear stress on the wall, a high value of H, that is, of the ratio δ_1/δ_2 , indicates a decreased value of shear stress. It will be seen later that at the point of separation of boundary layer from the wall, the shear stress tends to vanish. Here on a wall we have shown different boundary layers profiles. At this location du/dy at the wall is 0. This is the point of division between the attached flow here and the separated flow here, which is indicated by the reversal of flow near the plate and this is where the shear stresses 0 or the

value of H, the shape factor, tends to become very large. And so, the increase in value of H indicates that we are approaching separation.

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Drag

Steady flow

by mass balance: $\dot{m} = \rho U \delta' - \int_0^{\delta'} \rho u dy$

Momentum efflux at the top:
 $\dot{m}U = \rho U^2 \delta' - \int_0^{\delta'} \rho u U dy$

By momentum equation:

$$\int_0^{\delta'} \rho u^2 dy + \left[\rho U^2 \delta' - \int_0^{\delta'} \rho u U dy \right] - \rho U^2 \delta' = -D' \text{ Shear force, } D'$$

$$\int_0^{\delta'} \rho u(u - U) dy = -D';$$

or $D' = \rho U^2 \int_0^{\delta'} \frac{u}{U} \left(1 - \frac{u}{U} \right) dy = \rho U^2 \delta \int_0^1 \frac{u}{U} \left(1 - \frac{u}{U} \right) d\left(\frac{y}{\delta}\right)$

Let us relate drag to these various thicknesses that we obtained. Let us consider a steady flow on a flat plate, the boundary layer grows as shown. The streamlines are as shown. Consider a control volume starting from $x = 0$, the leading edge of the flat plate, to the location A. And spanning a distance a little more, then the boundary layer thickness at A.

Outside the boundary layer, the dotted line, the velocity horizontally is equal to U. Inside the dotted line, the velocity varies with y. We assume constant pressure all around. There would be a flow that would be leaving the upper surface of this control volume. Let the rate at which the mass leaves this upper surface of the control volume be \dot{m} , so the momentum it carries is $\dot{m}U$, the inviscid velocity the x momentum. And what is \dot{m} ? \dot{m} is obtained as the mass entering this left control surface and leaving at the right control surface. The mass entering the left control surface is $\rho U \delta'$. But u here is capital U, the inviscid velocity, and the mass leaving this is obtained by integration of the velocity profile there.

So, clearly by mass balance, \dot{m} is $\rho U \delta'$ minus $-\int_0^{\delta'} \rho u U dy$ across the boundary layer at location A. Then let us consider the momentum balance. The momentum flux at the top, and it is an influx, is that $\dot{m}U$, and with the expression \dot{m} obtained, this is given by $\rho U^2 \delta' - \int_0^{\delta'} \rho u U dy$.

There is a momentum flux at the top surface. And the momentum influx at the left surface is $\rho U^2 \delta'$, because everywhere the velocity is the inviscid velocity U . At the right surface the momentum flux is obtained by integration of $\rho u^2 dy$ and so, by momentum equation, we get the net flux of momentum must be the net horizontal force applied on the control volume towards the right.

If the drag force on this surface is D' per unit depth, then this force is $-D'$, because the drag is applied to the left, because of force is to the right as positive. so, we use $-D'$. Simplifying, we get $D' = \rho U^2 \delta \int_0^1 \frac{u}{U} \left(1 - \frac{u}{U}\right) d\left(\frac{y}{\delta}\right)$. You notice, this is like momentum thickness.

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Relation to momentum thickness

While deriving the relation for momentum thickness, we obtained

$$\checkmark \delta_2 = \sqrt{\frac{\nu x}{U}} \int_0^1 \frac{u}{U} \left(1 - \frac{u}{U}\right) d\left(\frac{y}{\delta}\right)$$

so that we can write

$$D' = \rho U^2 \delta \int_0^1 \frac{u}{U} \left(1 - \frac{u}{U}\right) d\left(\frac{y}{\delta}\right) = \rho U^2 L \delta_2,$$

and drag coefficient $C_D = \frac{D'}{\frac{1}{2} \rho U^2 L} = \underline{2\delta_2(L)}$

 NPTEL

Momentum thickness δ_2 was defined as this. So that we can write D' , the drag per unit depth of the flow, as $\rho U^2 L \delta_2$, where L is the length up to the location where your control volume ends. And the drag coefficient C_D is, this drag divided by $\frac{1}{2} \rho U^2$ times the area of the plate for unit depth. The area is just the length of the plate L .

And so, the drag coefficient is twice the displacement thickness at the location $x = L$. A very important result. That is the importance of momentum thickness, the deficit created in the momentum flow because of the velocity profile in the boundary layer.

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Shear stress on the wall

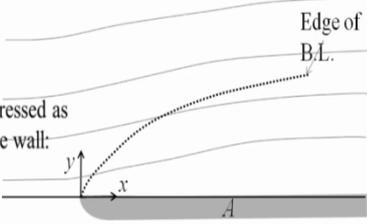
$$D' = \rho U^2 \delta \int_0^1 \frac{u}{U} \left(1 - \frac{u}{U}\right) d\left(\frac{y}{\delta}\right)$$

D' , the drag per unit depth, can be expressed as the integral of the shear stress τ_w at the wall:

$$D' = \int_0^x \tau_w dx, \quad \text{or } \tau_w = \frac{dD'}{dx}$$

Thus,

$$\tau_w = \rho U^2 \frac{d}{dx} \left[\delta \int_0^1 \frac{u}{U} \left(1 - \frac{u}{U}\right) d\left(\frac{y}{\delta}\right) \right] \text{ and } c_f = \frac{\tau_w}{\frac{1}{2} \rho U^2} = 2 \frac{d\delta_2}{dx}$$



Now, let us express this in terms of shear stress on the wall. This is D' that we obtained. D' is the drag per unit depth. And can be expressed as the integral of the shear stress τ_w . $D' = \int_0^x \tau_w dx$, or by differentiating, we get τ_w , the shear stress at the wall, as $\frac{dD'}{dx}$.

And since D' is now in terms of momentum thickness, the shear stress is obtained as the derivative of the momentum thickness. Last we obtain C_f as twice the derivative of momentum thickness with respect to x . This result which will be used shortly to obtain interesting results by much easier method than was used by Blasius.