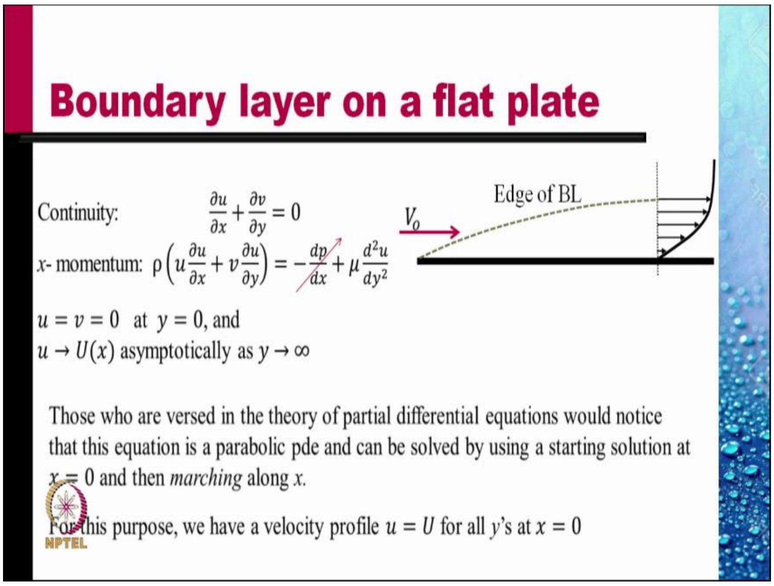
Fluid Mechanics & its Applications
Professor Vijay Gupta
Sharda University
Indian Institute of Technology, Delhi
Lecture 28A
Boundary Layer on a Flat Plate

(Refer Slide Time: 00:19)

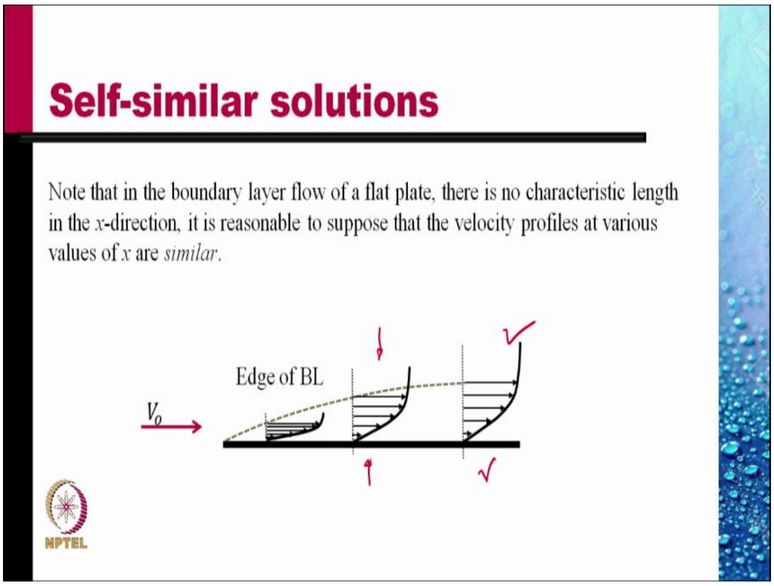


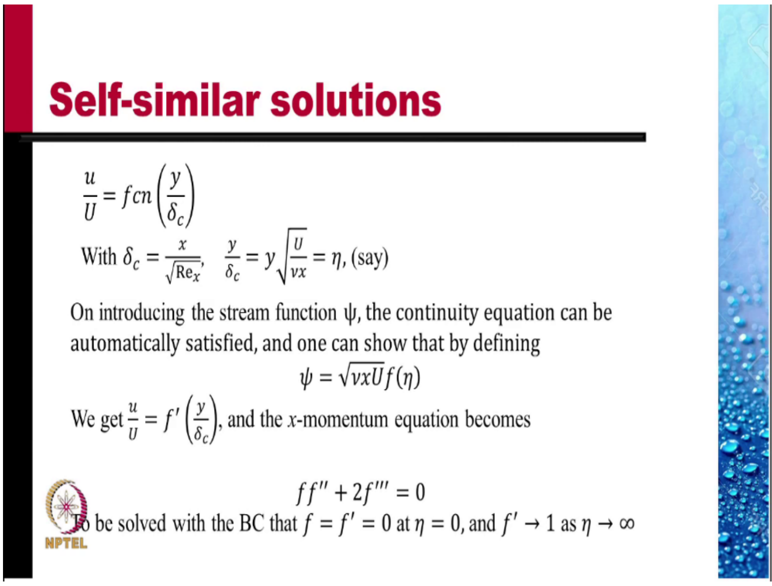
Now, let us consider the boundary layer on a flat plate. The first problem of boundary layer which was solved by a student of Prandtl by the name, Blasius, a fluid dynamicist of great repute by himself. For a flat plate, the continuity equation is $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$, and the x momentum equation within the boundary layer is $\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{dp}{dx}+\mu\frac{d^2u}{dy^2}$.

We have dropped $\mu\frac{d^2u}{dx^2}$, because of the properties of the boundary layer: that is, only in the y direction, the viscous stresses are significant. The x direction gradient $\mu\frac{d^2u}{dy^2}$ is insignificant. And this is to be solved with the boundary conditions that u and v, the velocities at the boundary y is equal to 0 should be 0, and that $u\to U(x)$ asymptotically as $y\to\infty$. $\frac{dp}{dx}$ of course is 0, flat plate, low pressure gradient, and this gives U as constant.

Those who are versed in the theory of partial differential equations would notice that this equation is a parabolic partial differential equations and can be solved by using a starting solution at $x=0$, and then marching along x. The solution at $x=0$ is that the horizontal component of velocity is equal to V_o, or U everywhere, so that we can march from there onwards. The marching means that after the solution x equals to 0, we go a little distance in the x direction and find out the profile there.

(Refer Slide Time: 03:12)





But an interesting thing to notice in this boundary layer on flat plate is that there is no characteristic length in the x direction. We start from $x = 0$, this plates extends to infinity. So, there is no characterizing length, and that is why the velocity profiles at different values of x should be similar in a certain sense. In the sense, that if we take velocity profile at one station, we can scale it properly to fit at any other station.

These are known as self-similar solutions. Profile at one x is similar to profile at another x. So, in this picture velocity profile at another x can be obtained by simply scaling the velocity profile at the first value of x. We have compressed this profile in the vertical direction to adjust for the value of delta there.

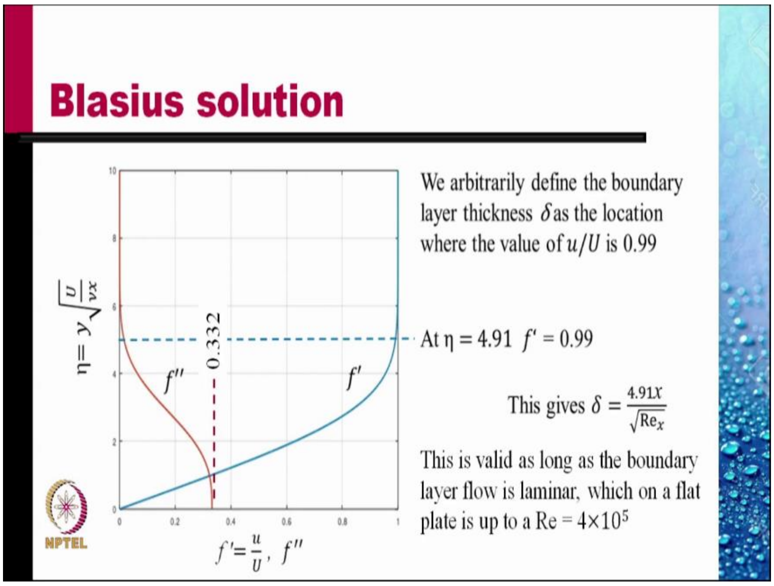
The only difference between the two profile is that the extent of the variations in the profile is now smaller. Delta is smaller at the first location than at the second location. We could scale it further, and obtain the velocity profile at the third location. I drew these second and third profiles simply by changing the scale in the vertical direction. In the horizontal direction the scale is the same, because the velocity at the edge of boundary layer is the same. These are, as I stated before, known as self-similar solution.

So, in these cases, we can treat it as if there is only one variable, space variable. And then u/U is a function only of y/δ_c. There is no x dependence except that the δ_c is a function of x. But if we use δ_c appropriate for a given location x, then the u/U becomes a function independent of x, with δ_c, like $\frac{x}{\sqrt{\mathrm{Re}_x}}$. $\frac{y}{\delta_c}$ becomes $= y\sqrt{\frac{U}{\nu x}}$, and we call it η.

So, η is now a composite variable that varies like $y/\sqrt{x}$. On introducing the stream function ψ, the continuity equation can be automatically satisfied. And one can show that by defining $\psi = \sqrt{\nu x U} f(\eta)$, we get $\frac{u}{U} = f'\left(\frac{y}{\delta_c}\right)$, where a prime stands for $df/d\eta$.

And the x momentum equation becomes simply an ordinary differential equation of third order $ff'' + 2f''' = 0$, to be solved with the boundary conditions that f is equal to 0. This is because the surface itself is a streamline. $f = 0$ at $\eta = 0$, and $f' = 0$ at $\eta = 0$. f' is like u. The horizontal component of velocity should be 0, the no slip condition. The no slip condition and the fact that the surface of the flat plate is impermeable, gives you $f = f' = 0$ at $\eta = 0$. And the far away boundary condition becomes $f' \to 1$, that is lowercase u/U is 1 as $\eta \to \infty$.

(Refer Slide Time: 09:16)



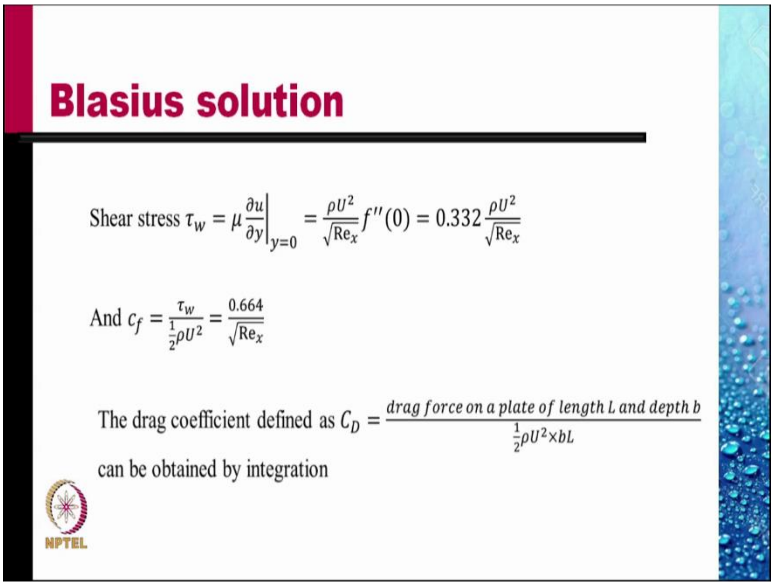
This equation is rather easy to solve. A small MATLAB program has been developed. We have been able to solve this. f' is shown to vary like this blue curve, 0 at the wall and tending to 1, asymptotically, as η becomes large. The red curve shows the plot of f''. f''relates to $\partial u/\partial y$. So f' relates to u, and f''relates to $\partial u/\partial y$, the gradient of velocity.

You notice that the velocity approaches one very quickly, but almost never reaches 1. We arbitrarily define the boundary layer thickness delta as the location with the value of u/U as 0.99, that is the streamwise velocity within the boundary layer is 99 percent of the inviscid velocity of the flat plate type. We define this as the boundary layer thickness.

And so, this is obtained at $\eta = 4.91$ where f' is 0.99. So, the boundary layer thickness is that a value of y for which η is 4.91. In classical literature, this value is normally taken as 5, instead of 4.91. This gives $\delta = \frac{4.91x}{\sqrt{\text{Re}_x}}$. Notice that the characteristic length across the boundary layer was taken $\frac{x}{\sqrt{\text{Re}_x}}$.

Another thing to note is the value of f'', which starts with a value of 0.332 at the wall. Remember, we said f'' is related to $\partial u/\partial y$, and this value is 0.332 at the wall. We will use it to estimate the shear stress at the wall. These calculations are valid as long as the boundary layer flow is laminar, which on a flat plate is up to a Reynolds number of about 4×10^5.

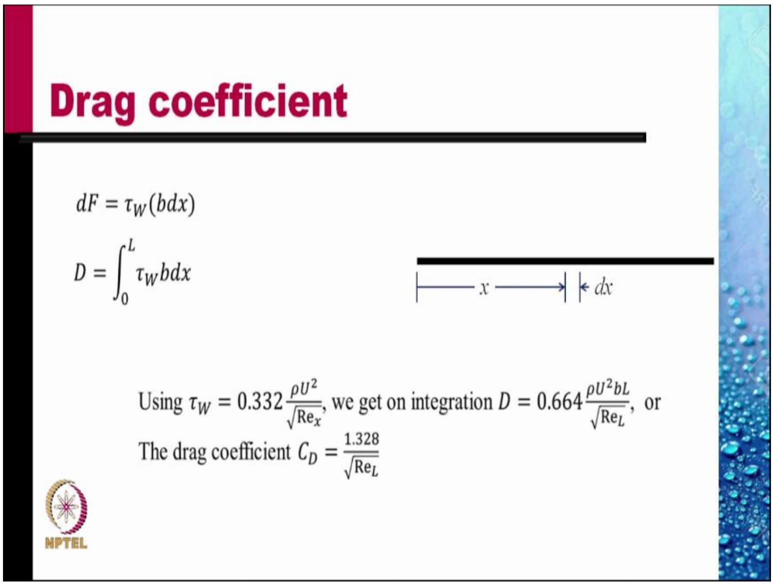
(Refer Slide Time: 12:47)



The shear stress τ_w at the wall is $\mu \frac{\partial u}{\partial y}\Big|_{y=0}$, that is at the wall, and it can be shown that, with our transformation, that becomes $\frac{\rho U^2}{\sqrt{\mathrm{Re}_x}} f''(0) = 0.332 \frac{\rho U^2}{\sqrt{\mathrm{Re}_x}}$ at wall. The value of f'' at wall, that is, at $\eta = 0$ was obtained at 0.332. And so, the shear stress of the wall is $0.332 \frac{\rho U^2}{\sqrt{\mathrm{Re}_x}}$. Re_x increases as x increases, and so the shear stress decreases as we go along the plate.

And the skin friction coefficient, which is defined as $\frac{\tau_w}{\frac{1}{2}\rho U^2}$, becomes $\frac{0.664}{\sqrt{\mathrm{Re}_x}}$. The skin friction also decreases as x increases. The drag coefficient on the flat plate defined as drag force on the plate of length L and depth b divided by $\frac{1}{2}\rho U^2 \times bL$, can be obtained by integration.

(Refer Slide Time: 14:38)

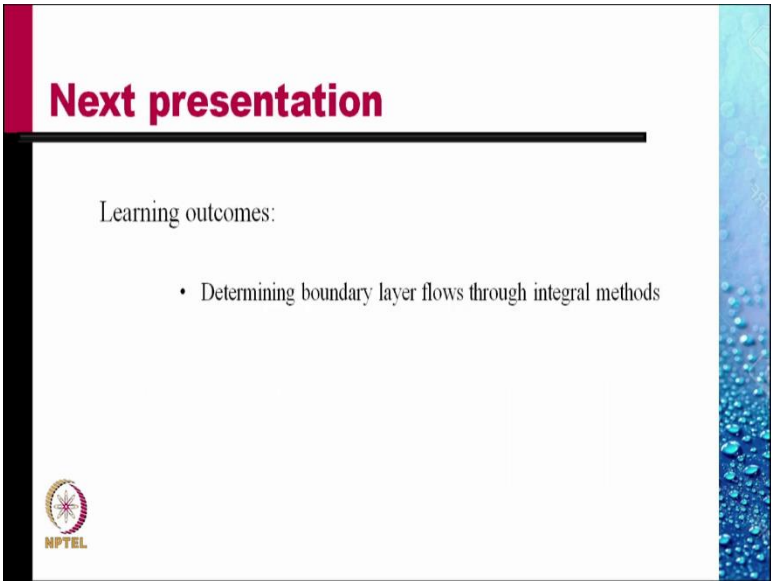


So, that dF the force, contribution to the drag, is the shear stress times the area of a small strip of width b, that is, of area bdx. And the total drag is obtained by integrating this from x equal to 0 to L, the length of the plate. And using the expression for the shear stress that we obtained earlier, we get the drag coefficient as $C_D = \frac{1.328}{\sqrt{\mathrm{Re}_L}}$. Notice that for large Reynolds numbers, this drag coefficient still small.

So, it does not lead to D'Alembert's paradox. We have neglected friction, we have neglected viscosity effects, and the results that we obtain would be approximate. The error that we should be contributing should be of the order of viscous effects. And since viscous effects are small, the error should be small. We predicted drag of 0 and we obtain a drag which is like $\frac{1}{\sqrt{\mathrm{Re}_L}}$ for large Reynolds numbers.

So, there is no paradox involved here, the drag is small. The paradox arises on bluff bodies, where though the prediction made by inviscid theory applicable for larger Reynolds numbers is still the drag should be 0. But the drag coefficient is of order 1. That is a paradox. How can the drag coefficient be that large? We will show later that it is because of the dynamics of this thin boundary layer due to Prandtl. And the dynamics leads to separation from the surface where the boundary layer does not remain thin, and the viscous effects penetrate in the main flow.

(Refer Slide Time: 17:23)



Thank you very much.