Fluid Mechanics & its Applications
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Lecture - 28

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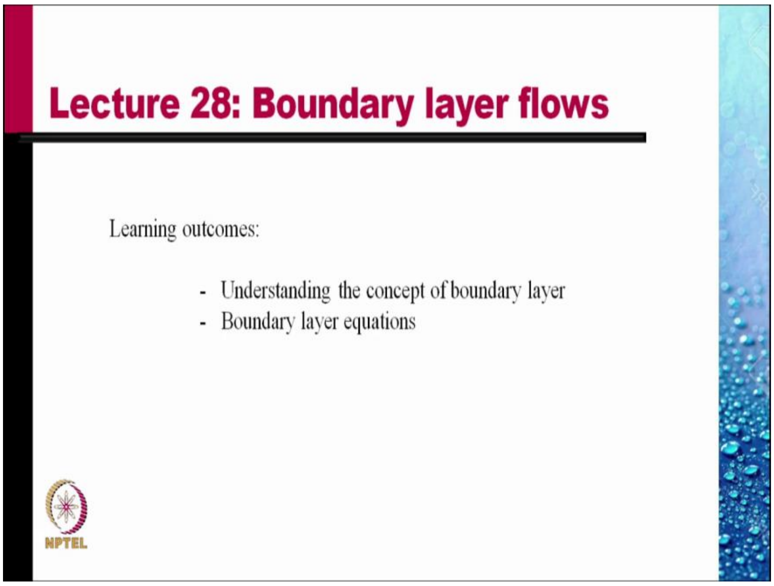

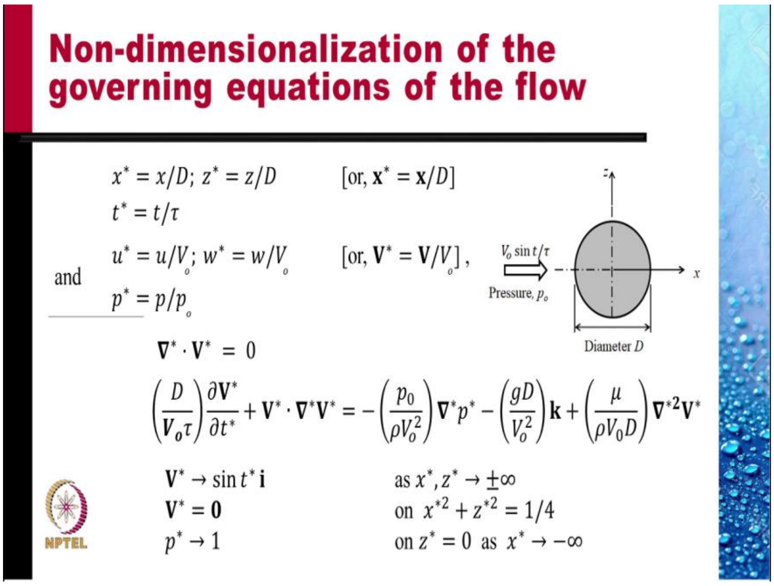

Welcome back.

In this lecture, we will cover the introduction to boundary layer flows. When we non-dimensionalize the governing equations of a flow, and for the variable x and z, the spatial coordinates, we non-dimensionalize with the characteristic length of the body. In the case of flow pass a cylinder, we use the diameter of the cylinder as the characteristic length and define $x^* = x/D$, and $z^* = z/D$. We use τ as a characteristic time and define $t^* = t/\tau$.

The velocity component u and w are non-dimensionalized with respect to the amplitude of the free stream velocity V_0, the pressure is non-dimensionalized with p_0, the free stream pressure, and when we do this the governing equations take these forms. We have simplified the momentum equation by making the coefficient of the convective acceleration as one. Then these are to be solved subject about the condition, $\boldsymbol{V}^* \to \sin t^* \boldsymbol{i}$ as $x^*, z^* \to \pm\infty$. $\boldsymbol{V}^* = \mathbf{0}$ on the surface of the cylinder, and $p^* \to 1$, as $x^* \to -\infty$ on $z^* = 0$.

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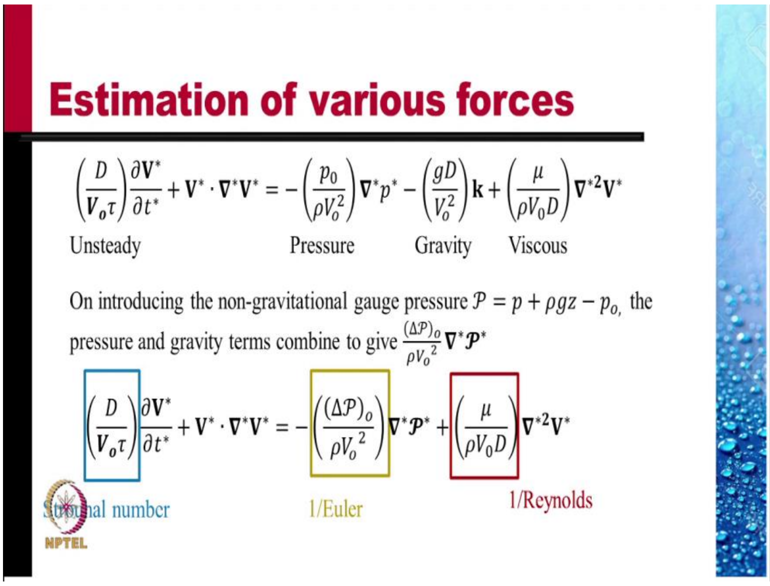

The first term in this is the unsteady acceleration term, the second term is the convective acceleration term, and on the right-hand side, the first term is the pressure term, the second term is the gravity force term, and the third term is the viscous terms, as we have discussed before. On introducing the non-gravitational gauge pressure $\mathcal{P} = p + \rho g z - p_o$, the pressure and gravity terms combine to give $\frac{(\Delta\mathcal{P})_o}{\rho {V_o}^2}\boldsymbol{\nabla}^*\boldsymbol{\mathcal{P}}^*$. And then the momentum equation acquires this form. The coefficient of the first term is this Strouhal number, the coefficient of the first term on the right-hand side is 1/Eu, and the coefficient of the last term is recognized as 1/Re.

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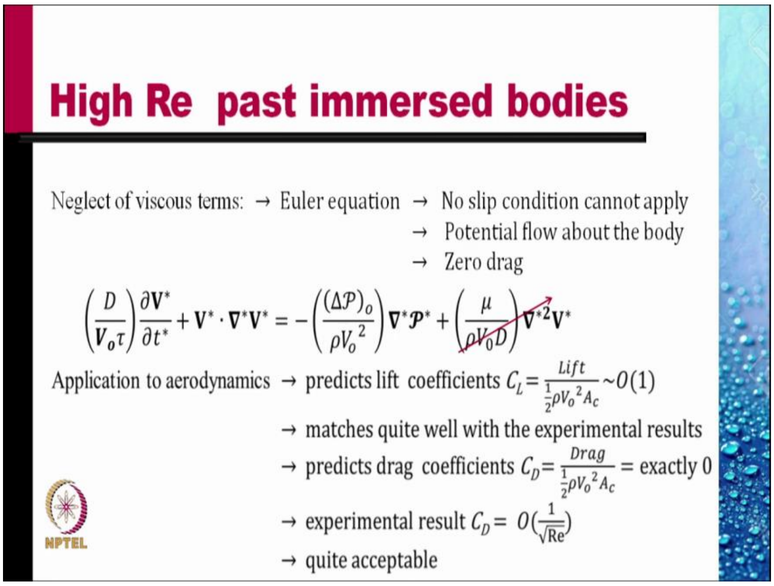

If Reynolds number is large, that is, if we are considering a high Reynolds number flow past an immersed body, then we can neglect the viscous terms, and that leads us to the Euler equation. The Euler equation can be solved relatively easily. We cannot apply the no slip condition at the boundary. This gives potential flow about the body, and as seen before, this leads to zero drag about a two-dimensional body. This Euler equation has been applied in aerodynamics, and when we apply it to aerodynamics, we are able to predict lift coefficients defined as C_L is called $= \frac{Lift}{\frac{1}{2}\rho {V_o}^2 A_c}$ where A_c is area that characterizes the airfoil, and we find that this is of order one.

This matches quite well with the experimental results. This predicts the drag coefficient C_D to be exactly 0. Since the experimentally obtained drag coefficient is of the order of $\sqrt{1/}$Re, this is quite acceptable result. And therefore, Euler equations is applied routinely to obtain the first estimates in aerodynamics.

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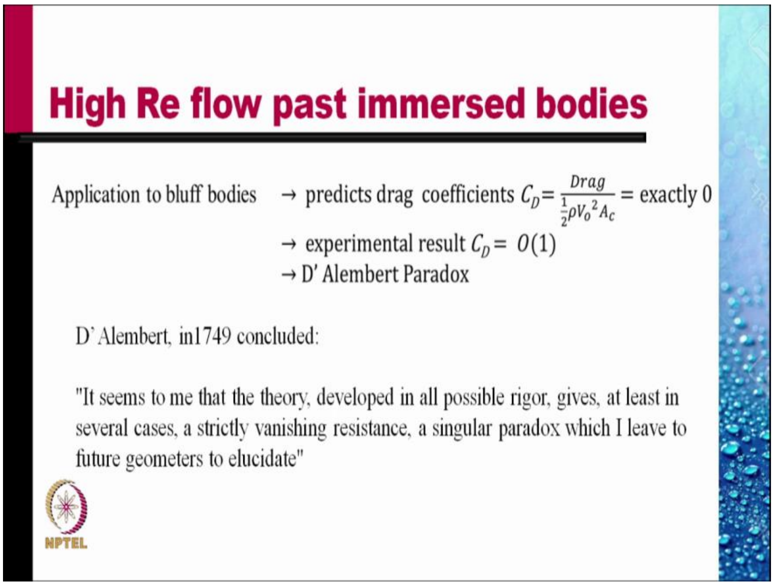

However, when you apply it to bluff bodies, it again predicts drag coefficient to be exactly zero, but experimentally we obtain the drag coefficient of order 1. This leads to a paradox that is known as the D'Alembert's Paradox. D'Alembert, in 1749 concluded: "it seems to me that the theory developed in all possible rigor gives at least in several cases, a strictly vanishing resistance (or drag), a singular paradox which I leave to future geometers to elucidate".

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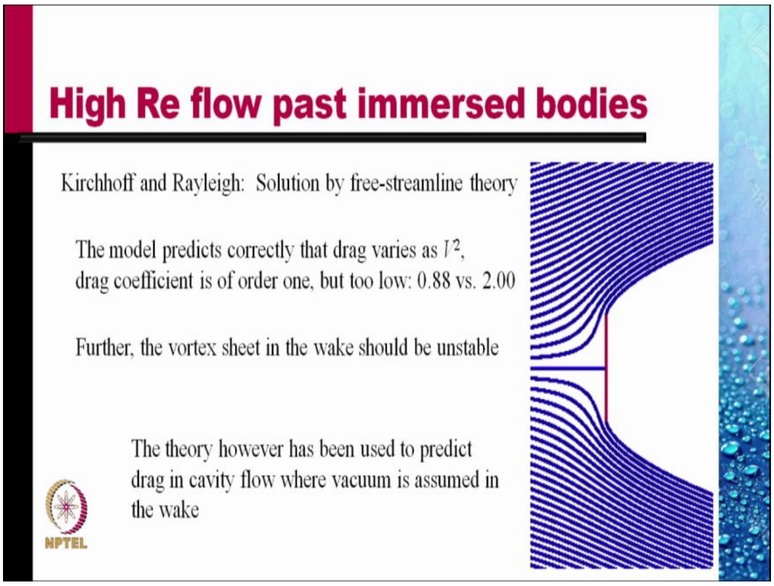

Kirchhoff and Rayleigh applied the free streamline theory to solve flow past bluff bodies. The model predicts correctly that the drag varies like the square of the velocity, and obtains the drag coefficient of order 1, but the value is too low. The value they obtained for a flat plate held normal to the flow is 0.88 versus 2.0 as obtained by experiments. Further, the vortex sheet in the wake should be unstable. This theory however, has been used to predict drag in cavity flows, where vacuum is assumed in the wake.

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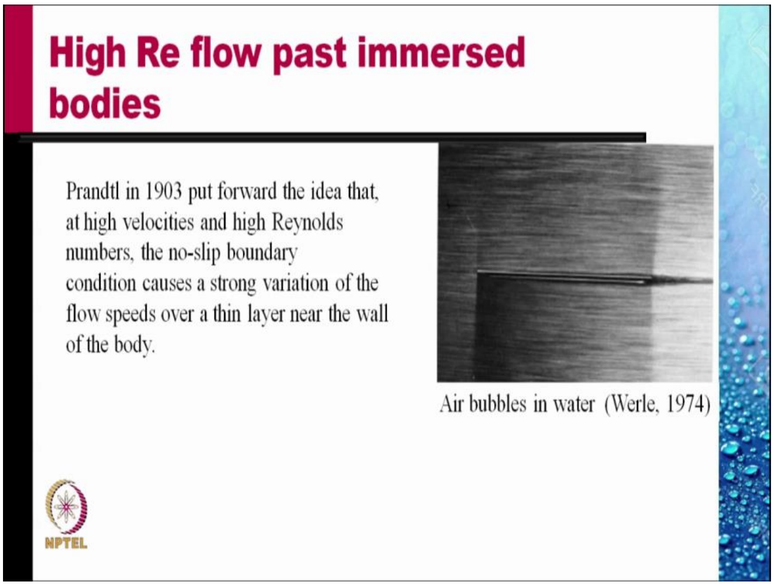

Prandtl in 1903 put forward the idea that at high velocities and high Reynolds numbers, the no slip boundary conditions causes a strong variation of the flow speeds over a thin layer near the wall of the body. In this picture, we have water flowing past a flat plate held parallel to the flow. Hydrogen bubbles are produced upstream by passing current through our electric wire. The tiny hydrogen bubbles make the flow lines visible, and we see very close to the plate in a thin region relative absence of hydrogen bubbles.

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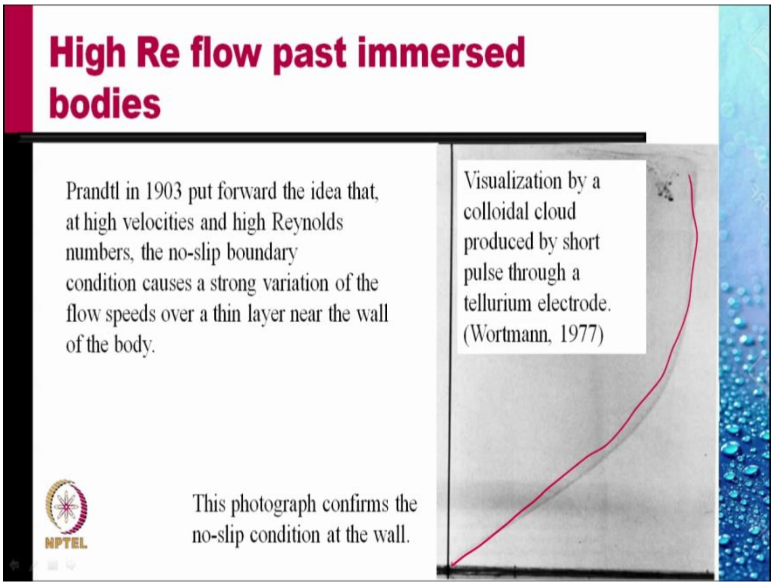

Then in 1977, Wortmann used a tellurium electrode in water. When the current passes through the tellurium electrode, a colloidal cloud is produced, and it is swept down with the velocity of flow very close to the plate. You obtain the picture of the flow after a small time. So that this curve that you see represents the velocity profile of the flow past a flat plate. This photograph confirms that the no slip condition at the wall is met even in very high Reynolds number flows.

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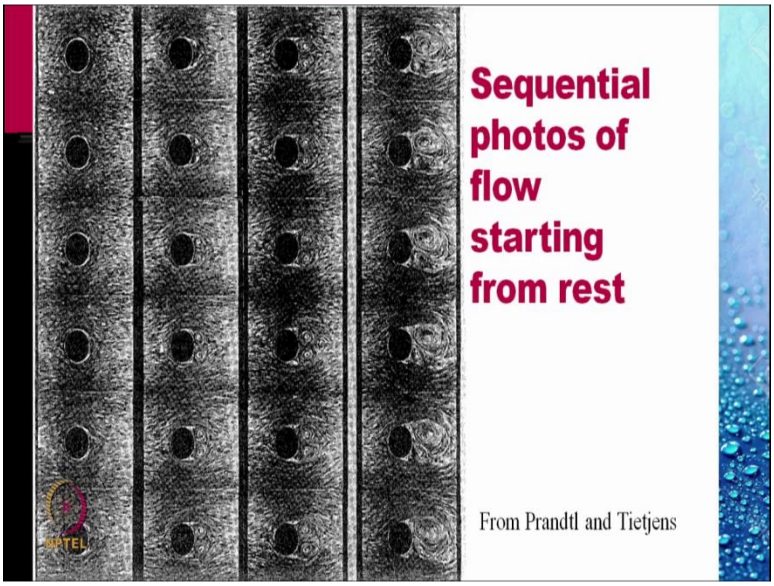

Prandtl has also taken a sequence of photographs for flow past a cylinder in water. The flow starts from rest starting from the top left going down, we see just as the flow starts. The flow picture is very close to the picture of an ideal flow, that is, a picture of the potential flow. It is only as the time develops, the boundary layer grows around the cylinder, and starts separating at about this picture, that we get the flow that we see as steady flow past a circular cylinder. This established that there truly is a boundary layer near the front of the cylinder, and that it separates from the cylinder in the latter half of the cylinder.

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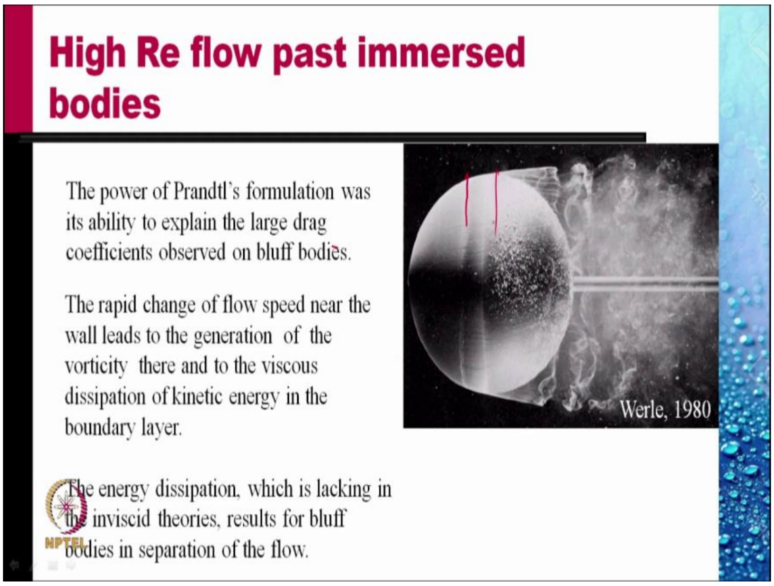

The power of Prandtl's formulation was in its ability to explain the large drag coefficients observed on bluff bodies. The rapid change of flow speed near the wall lead to generation of the vorticity there, and to the viscous dissipation of kinetic energy in the boundary layer. The energy dissipation, which is lacking in the inviscid theories, results for bluff bodies in separation of flow. In this picture of flow past a sphere, the fluid is water and we use an electrode around here, wrapped around the sphere.

As the electric current passes through this electrode, a cloud of hydrogen bubbles is created, and we take a picture of the cloud as it forms. It clearly shows the separation of the flow at around this location, as discussed earlier. The flow here is laminar. This separation results in complete disturbance of flow in the wake of the sphere, and leads to large drag, as will be explained in this lecture.

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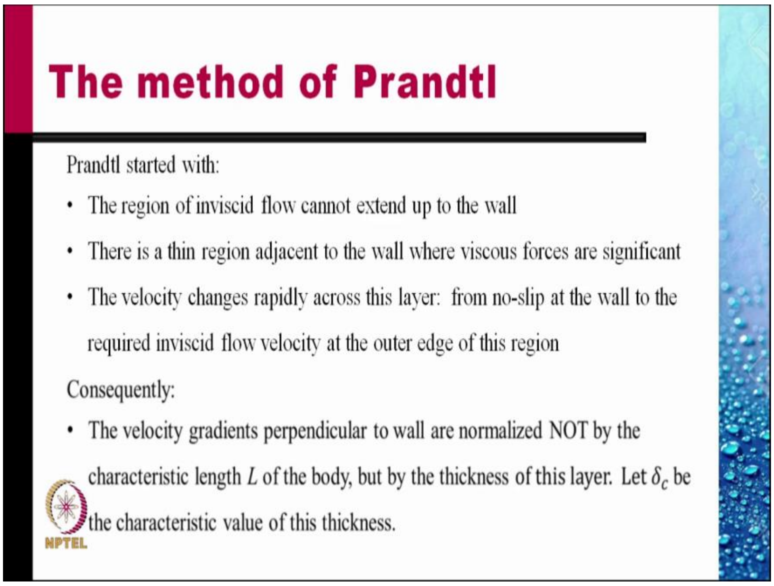

Prandtl started his development by stating that the region of inviscid flow cannot extend right up to the wall. He formulated a thin region adjacent to the wall, where viscous forces are not negligible and are significant. The velocity changes rapidly across this thin layer from the no slip at the wall, to the required inviscid flow velocity at the outer edge of this region. Consequently, the velocity gradient perpendicular to the wall should be normalized, not by the characteristic length of the body, but by the thickness of this layer. Let δ_c be the characteristic value of this thickness.

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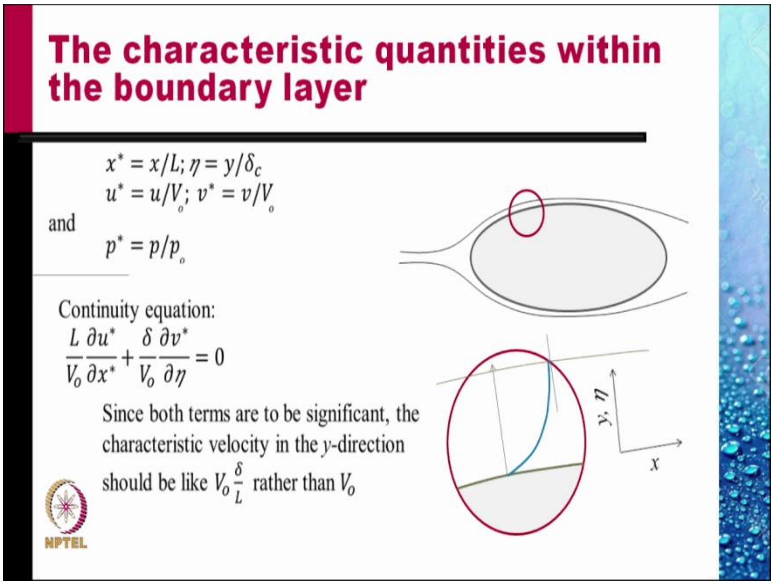

A thin boundary layer is developed around the body. The thickness shown here is quite exaggerated, but to see clearly what is happening, we take a small element of the surface and enlarge it. Typically, the velocity changes from 0 at the wall to a value close to what is predicted by the inviscid flow in the outer region, outside the boundary layer. Let the coordinate along with the flow direction be x and perpendicular to this point at the surface is y, then while non-dimensionalizing we use L, the dimension of the body as the characterizing length for the x coordinate. But for the y coordinate, we use δ_c, the length characterizing the thickness of the boundary layer, so that we define a non-dimensional coordinate $\eta = y/\delta_c$.

The u component of the velocity is characterized by V_0, as is the v component. And if we apply this non-dimensionalization to the continuity equation, we get $\frac{L}{V_o}\frac{\partial u^*}{\partial x^*} + \frac{\delta}{V_o}\frac{\partial v^*}{\partial \eta} = 0$. Since both terms are to be significant in this continuity equation, we cannot have $\frac{\partial u^*}{\partial x^*}$ tending to 0, as well as $\frac{\partial v^*}{\partial \eta}$ tending to 0. The characteristic velocity in the y direction should be like $V_o\frac{\delta}{L}$, rather than V_0. So, that the two terms become of the same order.

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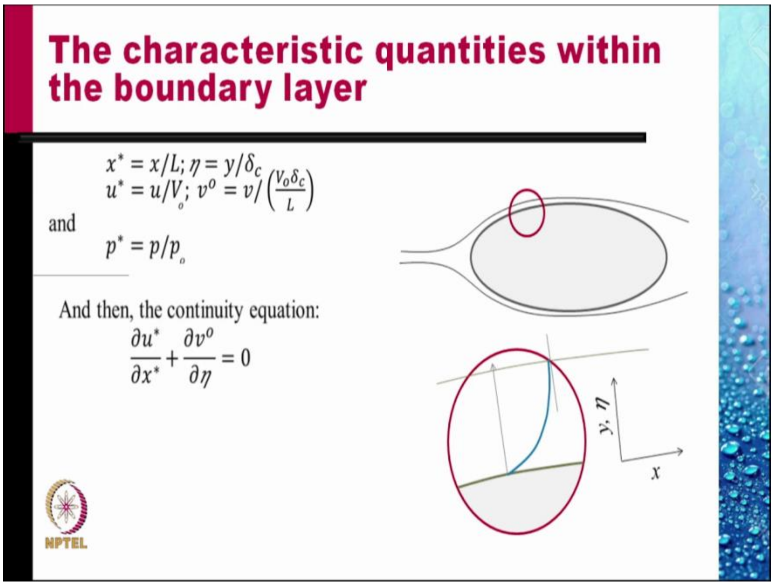

If we do this, and use v^o as the non-dimensionalized velocity in the y direction, $v/\left(\frac{V_o\delta_c}{L}\right)$. We use a different symbol for the non-dimensionalize vertical velocity v, because we do not want it to be confused with the characteristic velocity V_0. So that, now, the continuity equation gives $\frac{\partial u^*}{\partial x^*} + \frac{\partial v^o}{\partial \eta} = 0$. Thus, with this non-dimensionalization of velocity and the vertical space coordinate, we get the proper form of the continuity equation, where both terms are significant.

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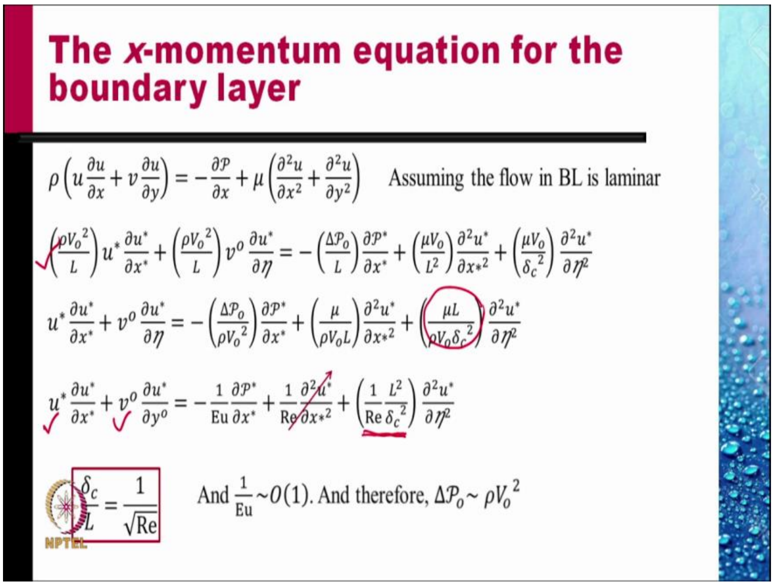

Let us now work with the x momentum equation for the boundary layer, assuming that the flow in boundary layer is laminar, so that we can use the appropriate form of the Navier Stokes equation for steady flows. And if we non-dimensionalize it using δ_c for the distances normal to the surface, and $V_O\frac{\delta}{L}$, as the velocity in the normal direction, this is the equation that we get. We simplify by dividing across by $\frac{\rho V_O{}^2}{L}$, and we get this equation. Here we have $\frac{\Delta \mathcal{P}_O}{L}$ as the characteristic pressure difference in the x direction.

We notice that $\frac{\Delta \mathcal{P}_O}{\rho V_O{}^2}$ is like 1/Eu. $\frac{\mu}{\rho V_O L}$ is like 1/Re, and the coefficient of the last term is simplified to $\left(\frac{1}{\mathrm{Re}}\frac{L^2}{\delta_c{}^2}\right)$. We notice here that 1/Re is very low compared to 1, which are the coefficients of the convective acceleration terms, and can be neglected from this equation. Of course, this term needs to be retained, because if this term also is neglected, then we will get back, the Euler equation, and that will not suffice for the boundary layer flow.

We can retain this equation because δ_c is as yet unknown. We choose the value of δ_c such that the coefficient of this last term is of the same order as the convective acceleration terms, which is order one. So, letting the order of this last term to be 1, we get δ_c/L to be like $\frac{1}{\sqrt{\mathrm{Re}}}$, that is, the boundary layer is characterized by a thickness which is $\frac{1}{\sqrt{\mathrm{Re}}}$ times the length scale of the body for large Reynolds number. This could be a small fraction of the body length and that is why we say that the boundary layer is thin.

This is an important result. We have been able to obtain an estimate of the boundary layer thickness with very little mathematics, and without solving any equation. We also conclude, that since 1/Eu should be order 1, therefore, the pressure differences in the x direction should be of order $\rho {V_o}^2$, the same as the pressure differences we obtain in the Euler equations for the inviscid flows.

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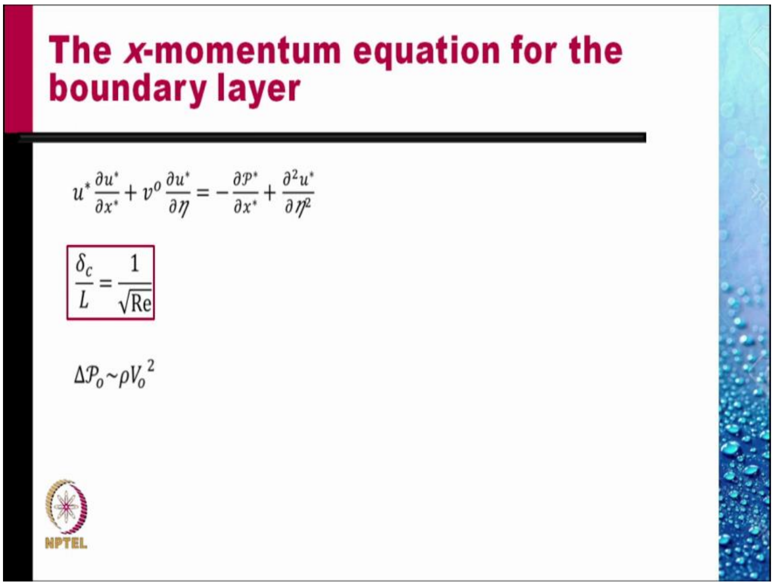

This, to repeat the result, we get $\frac{\delta_c}{L} \sim \frac{1}{\sqrt{\mathrm{Re}}}$, and the pressure differences are of order $\rho {V_o}^2$.

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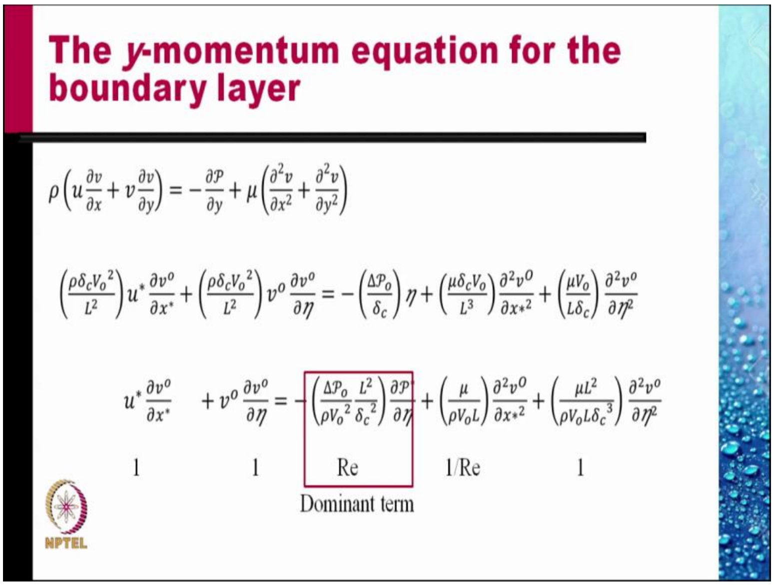

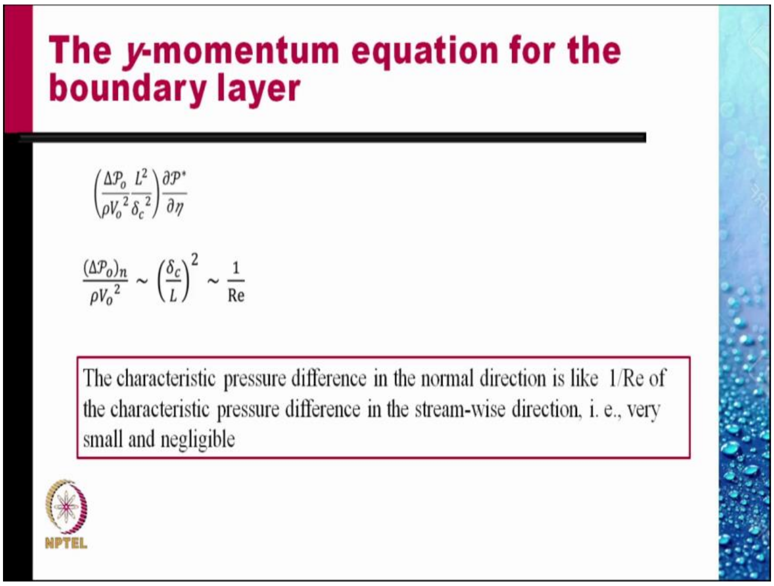

Now, let us work with the y-momentum equation for the boundary layer, and if we non-dimensionalize using the same definition of the non-dimensional variable as before, we get this equation, and when we simplify this equation, we get this equation where the first term and the second term, the convective acceleration terms, have been rendered of order one. Notice that the first term on the right hand side is of order Reynolds number, because $\frac{\Delta\mathcal{P}_o}{\rho V_o{}^2}$, and if we do this, then $\frac{L^2}{\delta_c{}^2}$ is like Reynolds number.

The second term on the right hand side is about 1/Re, and a third term the last, is of order 1. Thus, for large Reynolds number, it is the pressure term that is dominant. And from this we can conclude that $\frac{(\Delta\mathcal{P})_n}{\rho V_o^2}$ should be of order $(\delta_c/L)^2$, that is, like 1/Re. That is, if we use $\Delta\mathcal{P}_o$ as the $\Delta\mathcal{P}_o$ we obtained earlier in the x direction, then this term becomes too big, and the $\frac{\partial\mathcal{P}^*}{\partial\eta}$ is 0.

So, to the order of $\rho V_o{}^2$, pressure changes in the normal direction are negligible. But, the characteristic pressure difference in the normal direction, $\Delta\mathcal{P}_{o,\mathrm{normal}}$, is like 1/Re of the pressure differences in the stream-wise direction, and is $\rho V_o{}^2$. Thus, the characteristic pressure difference in the normal direction is like 1/Re of the characteristic pressure difference in the stream-wise direction, that is, very small and negligible. Prandtl neglects this pressure difference totally and so, the y-momentum equation disappears.

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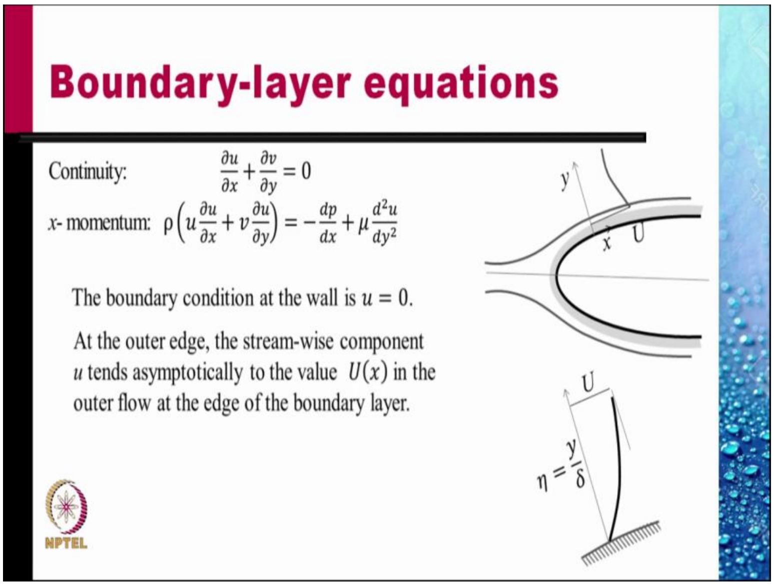

And $\frac{\partial \mathcal{P}}{\partial x}$ in the x momentum equation is replaced by minus $\frac{dp}{dx}$, assuming that within the boundary layer the pressure varies only in the x direction. There is no variation of pressure in the y direction. It is an assumption on which the theory of aerodynamics is built up. The boundary condition for these equations is that at the wall $u = 0$.

At the outer edge, this stream-wise component a u tends asymptotically to the value $U(x)$, that is, the velocity in the outer floor at the edge of the boundary layer. This velocity, $U(x)$, is obtained from the Euler equation. So, this is the solution of the inviscid potential flow in the outer layer the flow region, outside the boundary layer.

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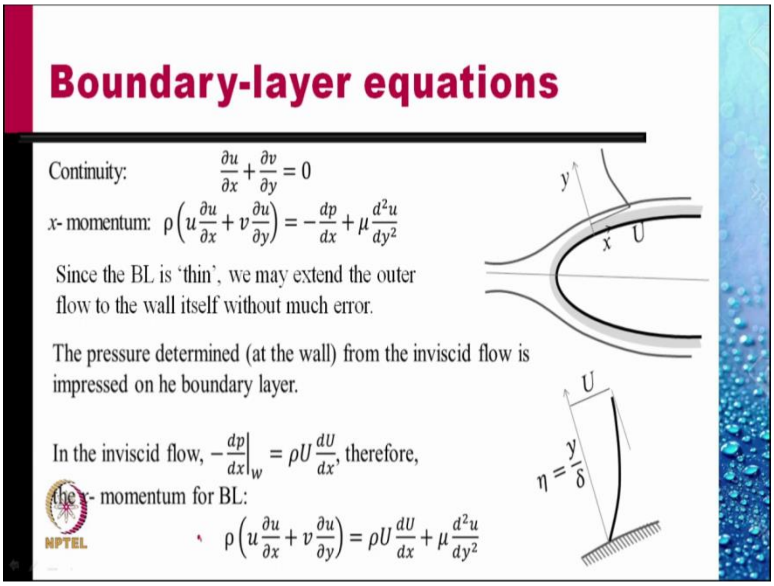

Since the boundary layer is thin, we may extend the outer flow to the wall itself without much error. The pressure determined at the wall from the inviscid flow is impressed on the boundary layer. In the inviscid flow $-\frac{dp}{dx}\Big|_w$ at the wall is like $\rho U \frac{dU}{dx}$, and therefore, the x momentum equation for the boundary layer becomes this.

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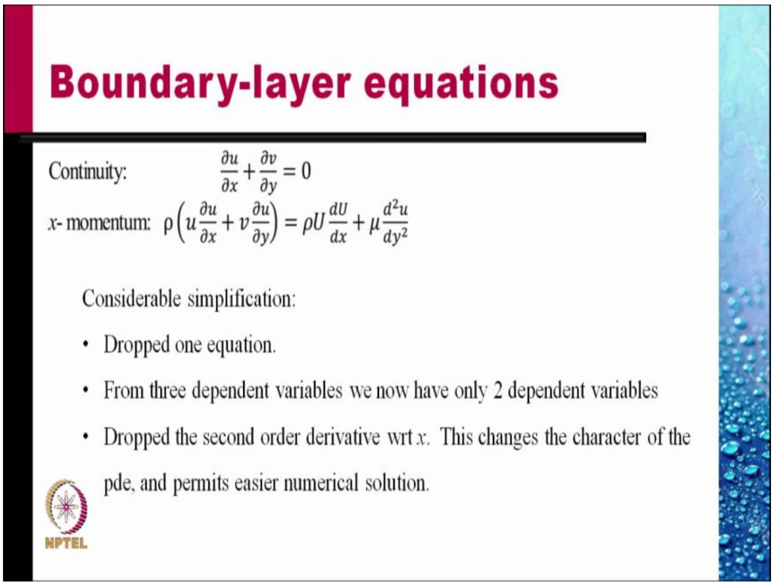

So, the equations for the boundary layers are the continuity equation in the x momentum, and this represents considerable simplification. We have dropped one equation, so instead of three equations, we have now two equations within the boundary layer. Now, there are only two unknowns, u and v. The pressure has been solved from the inviscid flow equation, and is put there. So, instead of three dependent variables, we have only two dependent variables. Also, and this is a great importance, is that we have dropped the second-order derivative with respect to x in the x momentum equation.

This changes the character of the partial differential equation. Those of you who are familiar with the theory of partial differential equation, will realize that this change in the character of this equation is a very significant change. The equation has become parabolic instead of an elliptic equation. Solving of parabolic equation is relatively easier than the solution of elliptical equation, and this is what was achieved by many students of Prandtl, who did heavy calculations of calculating the boundary layers. A parabolic equation is like a heat equation which can be solved by marching in time. In this case, in marching along x, starting with the flow at x equal to 0, or at initial point.

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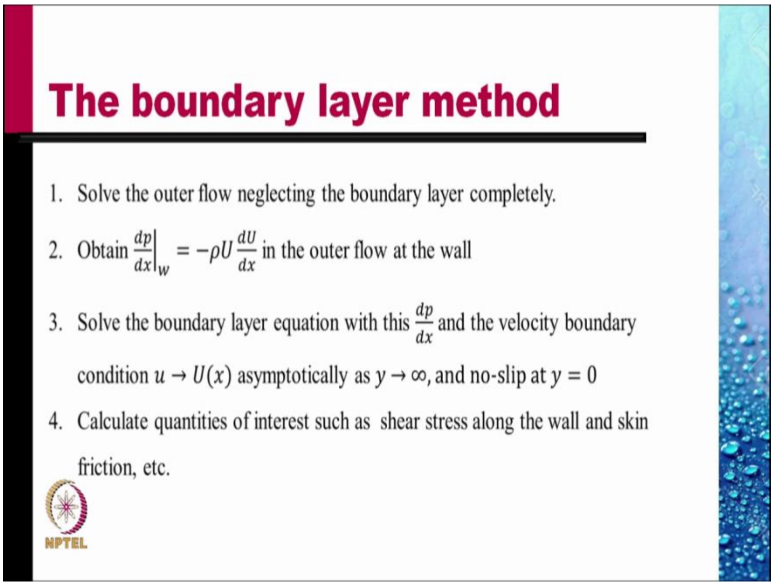

So, the process is: we solve the outer flow neglecting the boundary layer completely. Though the outer floor is only up to the edge of the boundary layer, but since the boundary layer is thin, while doing this calculation, we calculate the outer flow right up to the boundary of the body.

Once we have solved for this outer flow, we obtain $\frac{dp}{dx}$ at the wall as $-\rho U \frac{dU}{dx}$ in the outer flow at the wall. U represents the flow-wise velocity at the surface of the body after neglecting the boundary layer.

Now, this $\frac{dp}{dx}$ is now impressed on the boundary layer itself. We imagine that across the boundary layer, the pressure does not change in the y direction, in the normal direction, and therefore, whatever be the dp by dx obtained as the edge of boundary layer, is the $\frac{dp}{dx}$ across the boundary layer.

Now, we solve the boundary layer with the condition that $u \to U(x)$, the inviscid velocity, the potential velocity, asymptotically as y within the boundary layer tends to infinity. And the other boundary condition is that, there is no slip at the wall, $y = 0$. Once we can solve for this, we can calculate for the quantities of interest such as shear stress along the wall and skin friction, etc.

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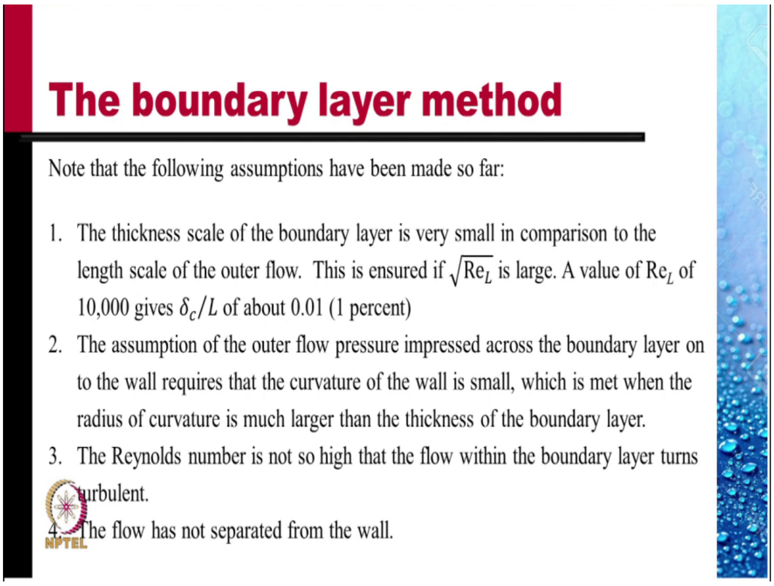

Note that the following assumptions have been made so far. The thickness scale of the boundary layer is very small in comparison to the length scale of the outer flow. This is ensured if Reynolds number based on the scale of the body is large. A value of the Reynolds number of 10,000 gives δ_c/L of about 0.01. That is only 1 percent. In most flows of interest, the Reynolds number is higher than 10,000. So, δ_c/L is smaller than 1 percent.

The second assumption that we made is that the outer flow pressure is impressed across the boundary layer up to the wall, and this requires that the curvature of the wall to be small. If the curvature of the wall is small, then we can neglect that curvature, and this condition is met when the radius of curvature is much larger than the thickness of the boundary layer.

So, except for a very small radius, we can neglect the curvature of the surface and treat boundary layer as on a flat surface. We also assumed that the flow is laminar, so that we can use the form of the Navier stokes equation which is valid for laminar flows, that is, with Stokes hypothesis for the stresses in the body.

If the Reynolds number is too high, the boundary layer turns turbulent, and this formulation would not be valid. If we also assumed that the flow has not separated from the wall. It is an important condition because once the flow separates from the wall, it is no longer thin. The boundary layer is no longer thin, and the total boundary layer assumptions break down.