

**Fluid Mechanics and its Application**  
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**Indian Institute of Technology, Delhi**  
**Lecture: 24A**  
**Some Simple 2-D Potential Flows**

Let us next study some simple two-dimensional potential flows that could be used to construct more complex flows.

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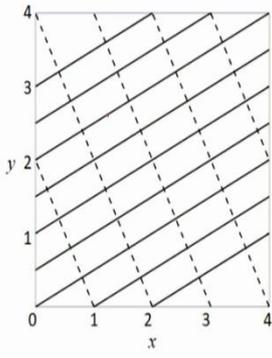
## Uniform flow

$U = V_o \cos \theta$ , and  $V = V_o \sin \theta$

$\phi = Ux + Vy = V_o x \cos \theta + V_o y \sin \theta$

and  $\psi = -Vx + Uy = -V_o x \sin \theta + V_o y \cos \theta$

Not very interesting in itself, but will be used for building up flows past bodies



The first flow we consider is a uniform flow where the velocity at every point is the same  $V_o$ , and the velocity vector is inclined at an angle  $\theta$  to the x-axis. So, that the velocity in the x-direction  $V_x$  is  $V_o \cos \theta$ , and we give it a symbol capital U, and the velocity  $V_y$  in the y direction is  $V_o \sin \theta$  which is given a symbol V.

Then  $\phi$ , the velocity potential is simply  $Ux + Vy$ , that is  $V_o x \cos \theta + V_o y \sin \theta$ . And the corresponding stream function is  $-V_o x \sin \theta + V_o y \cos \theta$ .

This is the flow net. The solid lines represent the streamlines the lines of constant  $\psi$ , and the broken lines are the potential lines, the lines are constant  $\phi$ , constant velocity potentials. This is not very interesting in itself, but will be used for building up flows past bodies in 2-dimensions.

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## Source or sink flows

Laplace equation  $\nabla^2\phi = 0$  in 2-D cylindrical polar coordinates is

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0$$

For axially symmetric flows, it reduces to

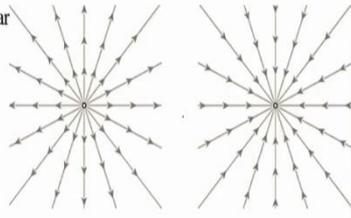
$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) = 0$$

with  $\phi = C_1 \ln r + C_2$  as its solution.

We arbitrarily set  $C_2 = 0$  to obtain

$$\phi = C \ln r$$

as the fundamental solution of the Laplace equations for two-dimensional axi-symmetric flows



The next flow we take up is the source or the sink flow. Laplace equation  $\nabla^2\phi = 0$  in two-dimensional cylindrical polar coordinates is  $\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0$ . For axially symmetrical flows, the derivative with respect to  $\theta$  are 0 and so, this equation reduces to  $\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) = 0$ , which is readily integrated to give a solution  $\phi = C_1 \ln r + C_2$ , where  $C_1$  and  $C_2$  are constants.

Just like in stream functions, in velocity potential we can add an arbitrary constant to any of the velocity potentials and stream functions, because it is only the difference of the stream functions or the gradient of velocity potential that matter. So, we can set  $C_2$  as 0, and then, the velocity potential is constant times  $\ln r$ . This is the fundamental solution of Laplace equation for two-dimensional axi-symmetric flows.

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## Source or sink flows

$\phi = C \ln r$  gives

$$V_r = \partial\phi/\partial r = \frac{C}{r},$$

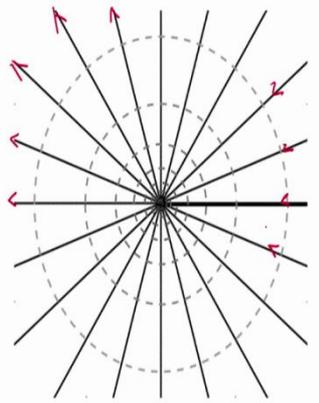
and

$$V_\theta = \partial\phi/r\partial\theta = 0$$

Velocity is radial: outwards for  $C$  positive, and inwards for  $C$  negative

$\int_0^{2\pi} V_r r d\theta = 2\pi C = q$ , the source strength, and so

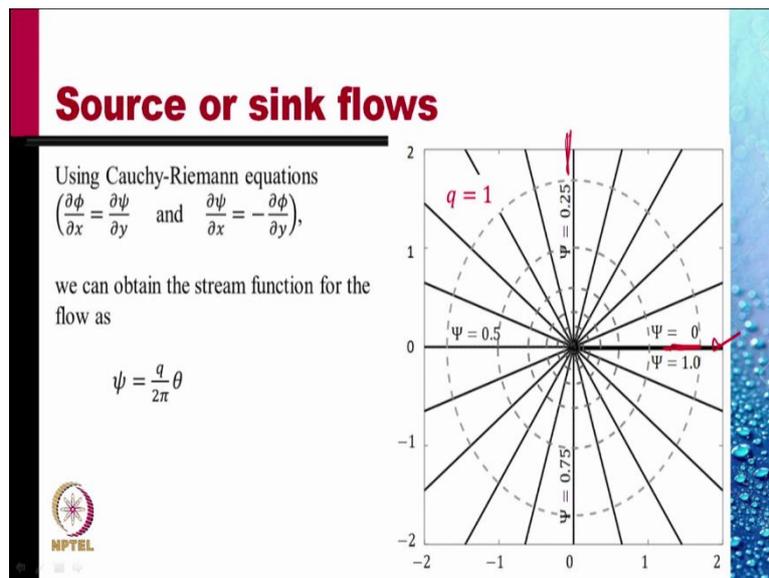
$\phi = \frac{q}{2\pi} \ln r$  is the velocity potential for a source of strength  $q$



Now, this potential function gives you the  $r$  velocity component as  $V_r = \partial\phi/\partial r = \frac{C}{r}$ , varying inversely as radius, and  $V_\theta = \partial\phi/r\partial\theta = 0$ . So, velocity is radial, outwards for  $C$  positive, and inwards for  $C$  negative. So, the velocity vectors are like these. For source flow when  $C$  is positive and inwards if  $C$  is negative.

If we find out the total flow, total volume flow, crossing a circle of radius  $r$ , we can obtain it by integrating the  $r$  component of velocity multiplied by  $r d\theta$  over  $\theta$  from 0 to  $2\pi$ , and we get this value as  $2\pi C$ . We can treat this total volume flowing out as the source strength  $q$ , such that  $C$  becomes  $\frac{q}{2\pi}$ , and the velocity potential  $\phi$  can be given as  $\frac{q}{2\pi} \ln r$  as the velocity potential for a source of strength  $q$  per unit depth for two-dimensional flows.

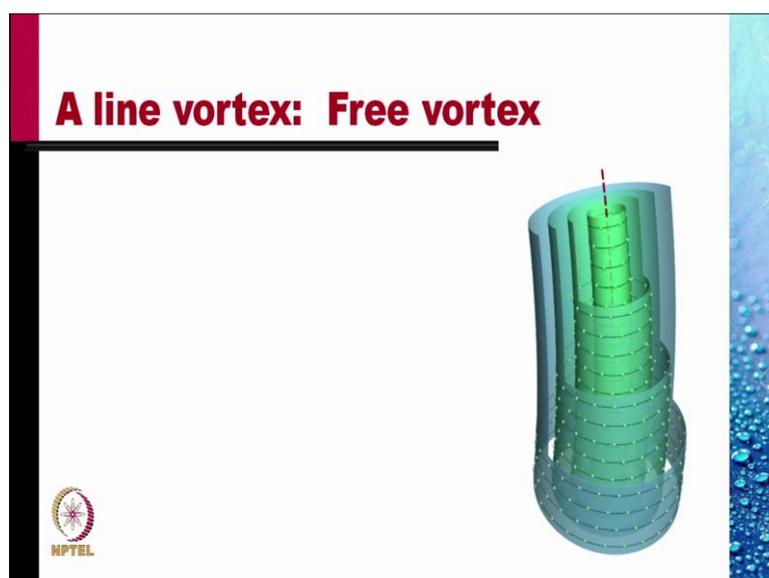
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using Cauchy-Riemann conditions, that is  $\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$  and  $\frac{\partial\psi}{\partial x} = -\frac{\partial\phi}{\partial y}$ , we can obtain the stream function for the flow as  $\psi = \frac{q}{2\pi}\theta$ . These are the streamlines that are plotted for  $q$  is equal to 1. That is for unit source strength.

This line here is for  $\psi$  is equal to 0. We go around we reach here, this is for  $\phi$  is equal to 0.25, and we come back, we go the whole circle. We have this line which represents  $\psi$  is equal to 0 as well as  $\psi$  is equal to 1.0. These are the streamlines for the source flow, velocity outwards for a source, and the velocity inward for a sink, when the value of  $q$  would be minus 1.

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## A line vortex: Free vortex

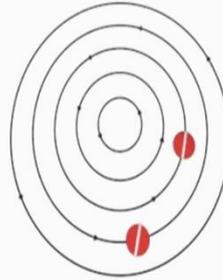
The radial component of velocity  $V_r$  is zero, and the swirl component  $V_\theta$  of velocity is a function of  $r$  alone.

The Laplace equation for the velocity potential ( $\nabla^2 \phi = 0$ ) in 2-D cylindrical polar coordinates reduces to

$$\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

The fundamental solution of this equation is  $\phi = C\theta$  giving the velocity component  $V_\theta$  as

$$V_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = \frac{C}{r}$$



Let us next do what is called a line vortex, or a free vortex. We have seen earlier that free vortex is the name given to a swirling flow in which there is no rotation of the fluid elements. Two particles are shown with same orientation throughout the motion. The particles are not rotating about their own axis, but the different velocities in the two circles, both the angular velocity as well as the linear velocity. The radial component of velocity  $V_r = 0$  in this case, and the swirl component  $V_\theta$  is a function of  $r$  alone.

So, swirl component is not a function of  $\theta$ . The Laplace equation for the velocity potential in 2-d cylindrical polar coordinates reduces  $\frac{\partial^2 \phi}{\partial \theta^2} = 0$ . And the fundamental solution is  $\phi = C\theta$ , up to an arbitrary constant, and which gives the velocity component  $V_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = \frac{C}{r}$ . So, this swirl component of velocity varies inversely as  $r$ . Fastest is at the centre and slowest outside.

(Refer Slide Time: 09:51)

## A line vortex

We evaluate the circulation about a circular circuit of radius  $r$  to get

$$\Gamma = \oint \mathbf{V} \cdot d\mathbf{s} = \int_0^{2\pi} V_\theta \cdot r d\theta = \int_0^{2\pi} \frac{C}{r} \cdot r d\theta = 2\pi C,$$

Circulation  $\Gamma$  of a free vortex is independent of the circuit you use, and therefore, can be used to designate the strength of the free vortex.

We can, thus, write

$$\begin{aligned} \phi &= \frac{\Gamma\theta}{2\pi}, \\ V_r &= 0, \\ V_\theta &= \frac{\Gamma}{2\pi} \cdot \frac{1}{r} \end{aligned}$$

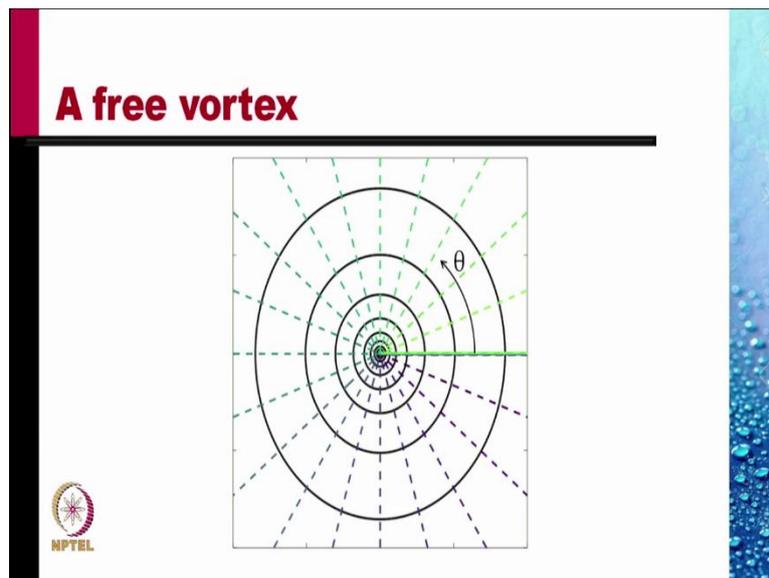
By using Cauchy-Riemann equations, we can obtain the stream function for the flow as

$$\psi = -\frac{\Gamma}{2\pi} \ln r$$


We evaluate the circulation about a circular circuit of radius  $r$  in this flow, and we get the circulation is equal to twice pi  $C$ , independent of the radius of the circuit. We have chosen circular circuit of radius  $r$  and find out that  $r$  cancels out, and the circulation  $\Gamma$  is obtained as independent of  $2\pi C$ .

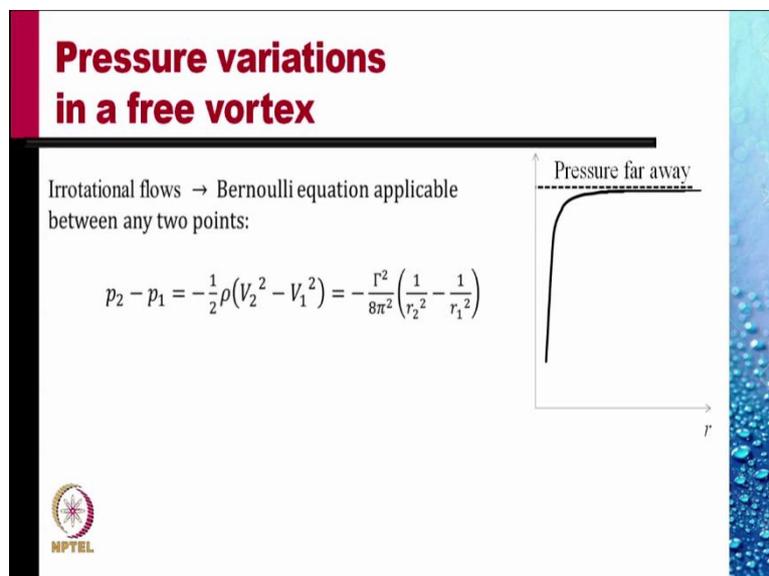
Therefore,  $\Gamma$  can be used as a strength of the free vortex. So, that in the expression of the velocity potential that we have  $C\theta$ , we replace  $C$  by  $\frac{\Gamma}{2\pi}$ , and if we do so, the velocity potential becomes  $\frac{\Gamma\theta}{2\pi}$ , the radial component of velocity is 0, the swirl component velocity is  $\frac{\Gamma}{2\pi} \cdot \frac{1}{r}$ ,  $C$  by  $r$ . By using Cauchy Riemann equations, we can obtain the stream function, and stream function  $\psi = -\frac{\Gamma}{2\pi} \ln r$ .

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So, these are the streamlines and potential lines for a free vortex. The solid lines are the streamlines. The velocity tangent to the streamlines, counter clockwise for positive value of  $\Gamma$ , and clockwise for the negative values of  $\Gamma$ . The radial lines are the equipotential lines or the potential lines.

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Now, we have seen earlier that in irrotational flows, the Bernoulli equation is applicable between any 2 points in the flow field. And therefore, for a free vortex if we take a point 1 at radius  $r_1$  and a point 2 at radius  $r_2$ , then the pressure difference  $p_2$  minus  $p_1$  is obtained as

$$p_2 - p_1 = -\frac{1}{2}\rho(V_2^2 - V_1^2) = -\frac{\Gamma^2}{8\pi^2}\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right).$$

If  $r_2$  is less than  $r_1$ , then this quantity is negative. That means, pressure  $p_2$  at the inner point is much less than pressure at the outer point. This is the shape of the pressure distribution. Pressure far away, at  $r$  tending to infinity, is shown by the broken line, and then the pressure decreases as  $r$  decreases.

(Refer Slide Time: 13:32)

### Free vortex in nature



The Maelstrom of Saltstraumen is Earth's strongest maelstrom. It is located close to the Arctic Circle in Norway. It is estimated that 400 million cubic metres (110 billion US gallons) of water passes the narrow strait during this event. The water is creamy in colour and most turbulent during high tide. It reaches speeds of 40 km/h (25 mph). Its impressive strength is caused by the world's strongest tide occurring in the same location during the new and full moon



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The vortices are very common in nature. They are found in oceans as whirlpools, in the atmosphere the tornadoes. They are huge vertices in the ocean, one of the largest one, is located close to the Arctic Circle in Norway. It is estimated that 400 million cubic meters of water passes in a narrow state during this event. This occurs regularly because of the lower pressure towards the centre of this vortex, the water level decreases. So, it produces a kind of a hole.

This is an underwater view of the same vortex. What you see here is an air cone that seems to be pointing downwards towards the ocean floor.

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## Free vortex in nature

A tornado is a violently rotating column of air, in contact with the ground underneath a cumuliform cloud, and often visible as a funnel cloud.

The intense low pressure caused by the high wind speeds and rapid rotation water vapour in the air to condense into cloud droplets due to adiabatic cooling. This results in the formation of a visible funnel cloud or condensation funnel.




In the atmosphere, these whirlpools appear as tornadoes. A tornado is a voluntary rotating column of air in contact with the ground underneath a cumuliform cloud, and is often visible as a funnel cloud. The pressures are very low in this funnel, and so, it sucks the debris, cars and human beings up. The intense low pressure caused by the high speed wind and the rapid rotation, the water vapour in the air condenses into cloud droplets due to adiabatic cooling. This results in the formation of a visible funnel cloud or a condensation funnel.

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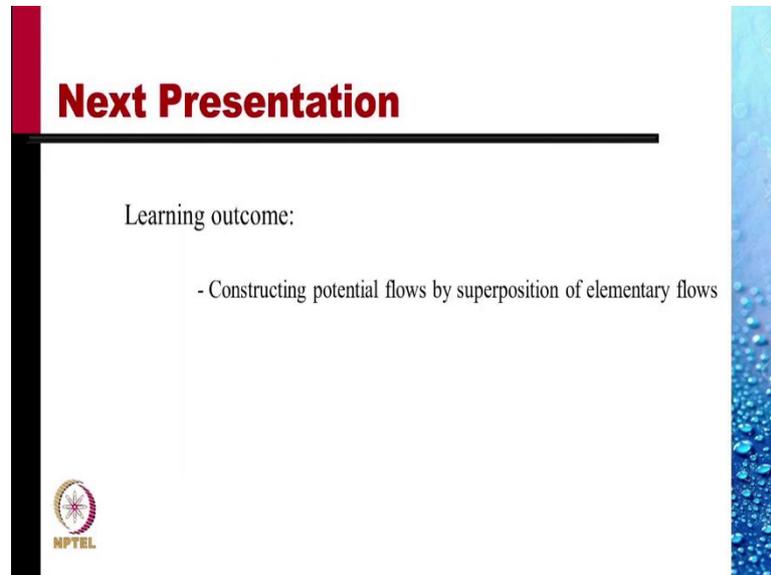
## Summary of basic potential flow solutions

Flow	Potential function, $\phi$	Stream function, $\psi$
Uniform flow	$V_0 x \cos \theta + V_0 y \sin \theta$	$-V_0 x \sin \theta + V_0 y \cos \theta$
Source	$\frac{q}{2\pi} \ln r$	$\frac{q}{2\pi} \theta$
Free vortex	$\frac{\Gamma}{2\pi} \theta$	$-\frac{\Gamma}{2\pi} \ln r$



So, let us summarize the three basic potential flow solutions that we studied so far. A uniform flow, a source flow, and a free vortex. We will build up the solutions of more complex flows from these basic flows in the next lectures.

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## Next Presentation

Learning outcome:

- Constructing potential flows by superposition of elementary flows



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Thank you.