Fluid Mechanics and Its Applications
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Lecture 24
Irrotational Flows

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Irrotational flows

A flow is said to be irrotational when the vorticity Ω is zero everywhere.

The circulation around any arbitrary loop in an irrotational flow is zero (provided that the loop is reducible).

If an inviscid fluid is initially irrotational then it remains irrotational at all subsequent times.

Irrotational flows

Euler equation: $\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -g\nabla z - \nabla\left(\frac{p}{\rho}\right) = -\nabla\left(gz + \frac{p}{\rho}\right)$

Using the vector identity $\mathbf{A} \times (\nabla \times \mathbf{B}) = \mathbf{A} \cdot \nabla\mathbf{B} - (\mathbf{A} \cdot \nabla)\mathbf{B}$,

we get $\mathbf{V} \times \Omega = \mathbf{V} \cdot \nabla\mathbf{V} - \frac{1}{2}\nabla V^2$ for $\mathbf{A} = \mathbf{B} = \mathbf{V}$,

or $\quad \mathbf{V} \cdot \nabla\mathbf{V} = \mathbf{V} \times \Omega + \frac{1}{2}\nabla V^2$

So, $\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \times \Omega = -\nabla\left(gz + \frac{p}{\rho} + \frac{V^2}{2}\right)$

Bernoulli equation holds EVERYWHERE for steady irrotational flows, and NOT just on a streamline.

Irrotational flows

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \times \mathbf{\Omega} = -\boldsymbol{\nabla}\left(gz + \frac{p}{\rho} + \frac{V^2}{2}\right)$$

Taking the curl of this equation, we get
$$\frac{\partial \mathbf{\Omega}}{\partial t} + (\mathbf{V} \cdot \boldsymbol{\nabla})\mathbf{\Omega} = (\mathbf{\Omega} \cdot \boldsymbol{\nabla})\mathbf{V}$$

or, $\frac{D\mathbf{\Omega}}{Dt} = (\mathbf{\Omega} \cdot \boldsymbol{\nabla})\mathbf{V}$

This is known as *vorticity evolution equation*.

Thus, if $\mathbf{\Omega} = 0$ initially in an inviscid flow, then $D\mathbf{\Omega}/Dt = 0$, and, consequently, $\Omega = 0$ at all subsequent times.

Welcome back.

In the last lecture we studied the irrotational flows. A flow is said to be irrotational when the vorticity Ω that is the curl of velocity is zero everywhere. The circulation around any arbitrary loop in an irrotational flow is zero, provided that the loop is reducible. If an inviscid fluid is initially irrotational then it remains irrotational at all subsequent times. We had seen that the governing equation for irrotational flow is the Euler equation given as $\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}.\boldsymbol{\nabla})\boldsymbol{V} = -g\boldsymbol{\nabla}z - \boldsymbol{\nabla}\left(\frac{p}{\rho}\right)$.

The last two term combine to give $-\boldsymbol{\nabla}\left(gz + \frac{p}{\rho}\right)$. Using the vector identity $\boldsymbol{A} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = \boldsymbol{A} \cdot \boldsymbol{\nabla}\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{\nabla})\boldsymbol{B}$, we get $\boldsymbol{V} \times \boldsymbol{\Omega} = \boldsymbol{V}.\boldsymbol{\nabla}\boldsymbol{V} - \frac{1}{2}\boldsymbol{\nabla}V^2$ for $\boldsymbol{A} = \boldsymbol{B} = \boldsymbol{V}$. Then we get from this $\boldsymbol{V}.\boldsymbol{\nabla}\boldsymbol{V} = \boldsymbol{V} \times \boldsymbol{\Omega} + \frac{1}{2}\boldsymbol{\nabla}V^2$.

And using this in the governing equation, we get $\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V} \times \boldsymbol{\Omega} = -\boldsymbol{\nabla}\left(gz + \frac{p}{\rho} + \frac{V^2}{2}\right)$. Thus, if the flow is irrotational, that is, if Ω is equal to 0, and the flow is steady, the left hand side is 0, and we get $-\boldsymbol{\nabla}\left(gz + \frac{p}{\rho} + \frac{V^2}{2}\right) = 0$.

Thus, the Bernoulli equation holds everywhere in a steady irrotational flow, and not just on a streamline. We can take points one and two for Bernoulli equation anywhere in the flow field, and not just on a streamline as was required for general inviscid flow which was not necessarily irrotational.

We again start with the last form of the Bernoulli equation and taking the curl of this equation. We get $\frac{\partial \boldsymbol{\Omega}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{\Omega} = (\boldsymbol{\Omega} \cdot \boldsymbol{\nabla})\boldsymbol{V}$. The first two terms, that are the terms on the left, are combined to give you the substantive derivative of Ω with time, $\frac{D\boldsymbol{\Omega}}{Dt} = (\boldsymbol{\Omega} \cdot \boldsymbol{\nabla})\boldsymbol{V}$. This is known as the vorticity evolution equation.

If Ω is 0 initially in an inviscid flow, then $\frac{D\boldsymbol{\Omega}}{Dt} = 0$. This means that the substantive derivative of Ω is 0, and since Ω was 0 to begin, with the value of Ω is 0 at all subsequent times, that is, any irrotational flow remains irrotational flow at all subsequent times.

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The vorticity Ω which is the curl of V in an irrotational flow is zero, and the circulation Γ about any reducible circuit lying entirely in the irrotational field is also zero.

Let us consider a circuit as shown, the $\oint_{ABCDA} V \cdot ds$ is the integral of $V \cdot ds$ over the arm ABC plus the integral over the arm CDA.
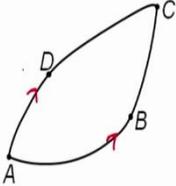
We can reverse the direction from CDA to ADC to get the circulation about the whole circuit as integral of $V \cdot ds$ over ABC minus the integral over the path ADC. Therefore, the integral about this path ABC is the same as the integral about ADC. Since, we have taken paths arbitrarily, this means that the integral of $V \cdot ds$ between any two points A and C is independent of the path that we take.

Thus, they depend only on the locations of the end points A and C, so that we can write $\int_A^C V \cdot ds = \phi(C) - \phi(A)$. It depends only on the locations of the endpoints not on the path. This is possible only if $V \cdot ds$ is dϕ that is $\nabla\phi \cdot ds$, that is, we can write the velocity as the gradient of function ϕ so that the velocity in any direction is $\frac{\partial \phi}{\partial s} = \nabla\phi \cdot ds$. The function ϕ is called the velocity potential, and the velocity components in the various coordinate systems can be found out.

The velocity components in rectangular coordinates are given by $u = \partial\phi/\partial x$ and v = $\partial\phi/\partial y$. Also given in this table are the functions are the velocity coordinates u and v in terms of the gradient of the stream function psi that we did earlier, u is equal to $u=\partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. Similarly, we can write the velocity components in the cylindrical polar coordinates.

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Governing equations

Continuity equation: $\nabla \cdot V = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

Two components of the Euler equation for incompressible and steady flows are

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \rho f_x - \frac{\partial p}{\partial x}$$

and $\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \rho f_y - \frac{\partial p}{\partial y}$

Subject to $V_n = V \cdot n = 0$ at walls

Velocity potential

If the inviscid incompressible flow is irrotational, as well, the continuity equation results in

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

In vector notation, this is

$\nabla^2 \phi = 0$ Laplace equation to be solved with $\frac{\partial \phi}{\partial n} = 0$ at the walls

No longer coupled to Euler equation.

We have done this for two dimensional flows. These can be done for three dimensional flows as well. The continuity equation is $\boldsymbol{\nabla} \cdot \boldsymbol{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. The two components of Euler equation for the incompressible steady flows are given below, and these are to be solved subject to boundary condition that the normal component of velocity is zero at the wall. As was discussed earlier, we cannot apply the no slip condition at the wall.

If the inviscid incompressible flow is irrotational, the continuity equation above results in $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$. This is obtained by substituting $\partial \phi / \partial x$ for u, and $\boldsymbol{\partial \phi / \partial y}$ for the velocity component v.

In vector notation this becomes $\nabla^2 \phi$, the Laplacian of ϕ is equal to 0. This is the well-known Laplace equation to be solved with $\frac{\partial \phi}{\partial n} = 0$ at the walls. Notice that this equation, though difficult to solve, is no longer coupled to the Euler equation. The flows in which we can define the velocity potentials, and which are therefore called potential flows, can be obtained simply as the solution of the continuity equation, which reduces to the Laplace equation, subject to the boundary condition that the normal gradient at the boundary is zero. Physically this means that there is no flow into the wall or out of the wall. The wall is impervious.

Velocity potential

Lines of constant velocity potential ϕ are called potential lines of the flow. In two dimensions

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$

$$= udx + vdy$$

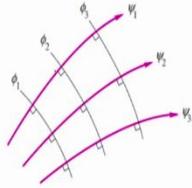Thus, the slope of a line of constant potential is

$$\frac{dy}{dx} = -\frac{u}{v}$$

MPTEL

Velocity potential

Recall that streamlines are lines everywhere tangent to the velocity, such that the slope of a streamline is given by

$$\frac{dy}{dx} = \frac{v}{u}$$

So the potential lines (the lines of constant ϕ) are perpendicular to the streamlines (the lines of constant ψ), except at a stagnation point

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}$$

Cauchy-Riemann conditions for analytical function of complex variables

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The lines of constant velocity potentials are called potential lines of the flow. In two-dimensional, $d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$ and recognizing that $\frac{\partial \phi}{\partial x}$ is u and $\frac{\partial \phi}{\partial y}$ is v, we get $d\phi = udx + vdy$.

Thus, a slope of a line of constant potential is $\frac{dy}{dx} = -\frac{u}{v}$, the negative of the x component of velocity divided by the y component of velocity. Recall that the streamlines which are the lines everywhere tangent to the velocity, the slope of those stream lines is given by $\frac{dy}{dx} = \frac{v}{u}$. Since the slope of potential lines has been found to be $-u/v$, this means that the potential lines, which are the lines of constant potential ϕ, are perpendicular to streamlines which are the lines of constant ψ, except at this stagnation point. Thus, the streamlines are

perpendicular to the potential lines at every point of the flow field. This is called flow net in some texts.

And from this we get $\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$ and $\frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}$. If we define a complex variable phi plus i psi, real part phi and then the imaginary part psi the stream function, then these two conditions are nothing but the Cauchy-Riemann conditions applicable for analytical functions of a complex variable.

Therefore, psi and phi, when they have combined to form a complex variable $\phi + i\psi$, this complex variable is an analytic function of z, which is $x + iy$. This property of the potential and the stream functions, which is given the name complex potential, is exploited in finding out the solutions of the Laplace equation. We will not do this here in this elementary course.

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Velocity potential and Stream function

	Potential Function	Stream Function
Definition	$V = \nabla\phi$	$V = \nabla \times \psi$
Requirement of continuity	$\nabla^2\phi = 0$	Built-in
Requirement of irrotationality	Built-in	$\nabla^2\psi = 0$
Velocity components in rectangular coordinates	$u = \partial\phi/\partial x$ $v = \partial\phi/\partial y$	$u = \partial\psi/\partial y$ $v = -\partial\psi/\partial x$
Velocity components in cylindrical coordinates	$V_r = \partial\phi/\partial r$ $V_\theta = \partial\phi/r\partial\theta$	$V_r = \partial\psi/r\partial\theta$ $V_\theta = -\partial\psi/\partial r$

To summarize, the velocity potential and stream function once again, the definition of potential function is that V is equal to $\nabla\phi$, for the stream function, it is V is equal to $\nabla \times \psi$, then continuity requires the potential function to be $\nabla^2\phi = 0$. This requirement is built in into the stream function. The requirement of irrotationality is built in in the potential function, but for the stream function it becomes $\nabla^2\psi = 0$. So, both ϕ and ψ satisfy Laplace equation. Given below are the velocity components in the rectangular coordinates and in the cylindrical-polar coordinates same as before.

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Potential flows

Let us point out some interesting consequences of the fact that the velocity field is determined fully by the Laplace equation.

First, the equation and the boundary condition are valid for *both steady and unsteady flows* even though the time variation does not enter the equations explicitly.

Potential flows

If the boundary conditions are changing with time, the solution also changes with time. The rate of change of the boundary condition does not affect the solution at a given instant.

This means that the solution is *quasi-steady* with every point in the flow field responding immediately to any changes in the boundary conditions. This is a consequence of the flow being incompressible which requires that the disturbances propagation speed is infinite

Potential flows

The second interesting consequence is that the velocity field can be obtained independently of the pressure field, but not *vice versa*.

Thus, the pressure gradient adjusts itself depending upon the inertia of the fluid (measured by the density ρ) to give the desired velocity gradients
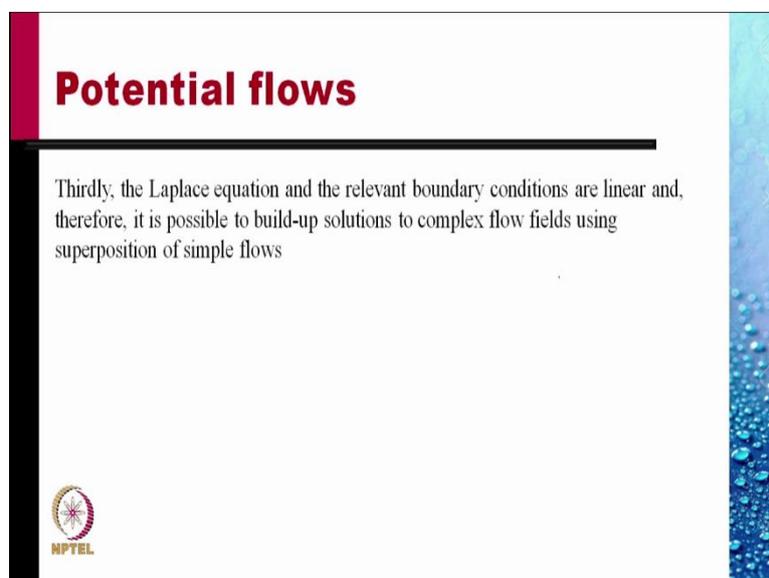
Let us point out some interesting consequence of the fact that the velocity field of a potential flow is determined fully by the Laplace equation. First, the equation and boundary conditions are valid for both steady and unsteady flows, even though the time variation does not enter the equations explicitly, interesting.

If the boundary conditions are changing with time, the solution also changes with time. The rate of change of boundary condition does not affect the solution at a given time. This means that the solution is quasi-steady with every point in the flow field responding immediately to any change in the boundary conditions. This is the consequence of flow being incompressible, which requires that the disturbance propagate at infinite speed.

The second interesting consequence is that the velocity field can be obtained independently of the pressure field, but not vice versa. We cannot solve the Euler equation to obtain the pressure field without knowing the velocity field, but we can solve the Laplace equation to obtain the velocity field without dealing with the pressure field. The Laplace equation for potential flow is purely a kinematic equation, while the Euler equation is a dynamic equation. Thus, the pressure gradient adjusts itself depending upon the inertia of the fluid measured by the density rho to give the desired velocity gradients.

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Potential flows

Thirdly, the Laplace equation and the relevant boundary conditions are linear and, therefore, it is possible to build-up solutions to complex flow fields using superposition of simple flows

The third important property of the potential flow is that the Laplace equation and the relevant boundary condition are linear and, therefore, it is possible to build up solutions to complex flow fields using superposition of simple flows. So, the equation as well as boundary conditions are linear. We can take a number of elementary flows and simply add the potentials of those flows and the sum would satisfy the Laplace equation.