

Fluid Mechanics and Its Applications
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Lecture 23A
Irrotational Flows

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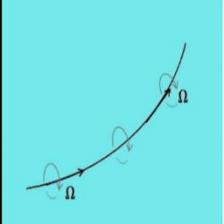
Irrotational flows

Irrotational flows have some special properties which render the equations governing them linear and, therefore, these flows have been studied in great details.

Before doing this, we introduce the concepts of vortex lines and circulation, which play important roles in irrotational flows.

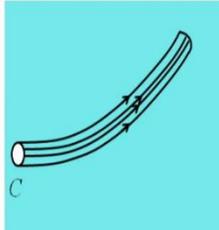


Vortex lines and vortex tubes



A vortex line is defined as a line which is tangent to the vorticity vector, $\Omega = \text{curl } V$ at every point

A vortex tube is a bundle of vortex lines passing through a closed circuit C within the flow field.



Irrotational flows have some special properties which render the equations governing them linear and therefore, these flows have been studied in great details. Before doing this, we introduce the concept of vortex lines and circulation. These two concepts play very important roles in irrotational flows.

A vortex line is defined just like a streamline. It is a line which is tangent to the vorticity vector. The vorticity vector is $\Omega = \text{curl } V$, and at every point the vorticity vector is tangent to a vortex

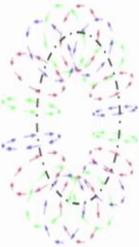
line. A vortex tube is a bundle of vortex lines passing through a closed circuit C . Imagine a circuit C as shown. So, from every point of this we pass a vortex line, and so these vortex lines would occupy a tubular region in space and this tubular region is termed as a vortex tube.

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Vortex rings

We shall later see this as an important property of a vortex tube: A vortex tube either closes on itself or extends to the boundaries of the flow.

The lateral surface of the vortex tube is a vortex surface with the vorticity vector tangent to the surface.

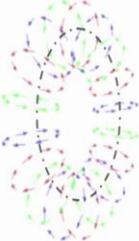


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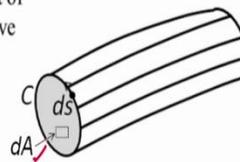
We shall later see that an important property of vortex tube is that a vortex tube either closes on itself, or extends to the boundaries of the flow. Here we have shown a smoke ring gun. It is made of a container in which smoke is filled, and there is a diaphragm at the back where we tap and a smoke ring comes out. The smoke rings are regions of vorticity, donut shaped vortex tubes. Shown in the right is the pattern of the flow in these. If we take the central axis of this donut, we see that there is a vorticity vector tangent to it at every point.

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Strength of a vortex tube

The *strength* of a vortex tube is defined by the product of the vorticity normal to the surface enclosed by the curve C .

$$\int_{\text{Area capping } C} \boldsymbol{\Omega} \cdot d\mathbf{A}$$



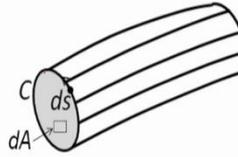
Thus, the *vortex tube strength* is equal to the normal component of vorticity $\boldsymbol{\Omega} (= \nabla \times \mathbf{V})$ integrated over the area of cross-section of the vortex tube.



Strength of a vortex tube

Gauss's theorem relates the integral of a vector quantity over a closed circuit to the surface integral of the curl of that quantity over a surface capping the circuit

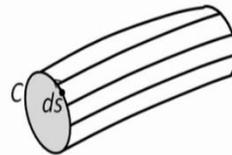
$$\int_{\text{Area capping } C} \boldsymbol{\Omega} \cdot d\mathbf{A} = \int_{\text{Area capping } C} (\nabla \times \mathbf{V}) \cdot d\mathbf{A} = \oint_C \mathbf{V} \cdot d\mathbf{s}$$



Thus, the *strength of a vortex tube* is equal to the integral of the tangential component of the velocity around a circuit enclosing the tube

Definition of circulation

$\oint_C \mathbf{V} \cdot d\mathbf{s}$, the line integral of the tangential component of velocity about a closed contour C is termed as the circulation Γ



$$\Gamma = \oint_C \mathbf{V} \cdot d\mathbf{s}$$

with integration proceeding in the counter-clockwise direction

The *circulation about a circuit* is equal to the *vortex strength of the vortex tube* formed by the vortex lines passing through each point of the circuit

The strength of the vortex tube is defined by the product of the vorticity normal to the surface enclosed by the curve C . So, in this curve C if we take a small area dA then the strength of the vortex tube is $\int_{\text{Area capping } C} \boldsymbol{\Omega} \cdot d\mathbf{A}$. Thus, the vortex tube strength is equal to the normal component of vorticity integrated over the area of cross section of the vortex tube. Some texts make a mistake and describes this vortex strength $\boldsymbol{\Omega} \cdot d\mathbf{A}$ as the flux of vorticity through the tube. It is wrong in the sense that this vorticity is not flowing through the tube. It does not flow through the tube, so there is no flux, it does not enter the tube. Those of you who are familiar with the calculus of vectors know that Gauss's theorem relates the integral of a vector quantity over a closed circuit to the surface integral of the curl of that quantity over a surface capping the circuit.

Thus, for the vortex tube shown the strength of the vortex, which is $\int_{Area\ capping\ C} (\nabla \times \mathbf{V}) \cdot d\mathbf{A}$, is equal to $\oint_C \mathbf{V} \cdot d\mathbf{s}$. That becomes the tangential component of velocity along the circuit. Thus, the strength of the vortex tube is equal to the integral of the tangential component of velocity around a circuit enclosing the tube.

This is interesting development, because there is a name for this integral over the closed circuit. And that name is circulation. $\oint_C \mathbf{V} \cdot d\mathbf{s}$ is termed as the circulation, Γ , with the integration proceeding in the counter clockwise direction. The circulation about the circuit equals to the vortex strength of the vortex tube formed by the vortex lines passing through each point of the circuit. So we start with the circuit C, then the circulation about the circuit C is equal to the vortex strength of the vortex tube formed by the vortex lines passing through each point of the circuit.

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Vorticity

There is no net component of vorticity through a closed surface.

This follows directly from the *divergence theorem*. Consider a volume \mathcal{V} bound by a closed surface S .

Then $\iint_{\text{bounding surface}} (\nabla \times \mathbf{V}) \cdot d\mathbf{A} = \iiint_{\mathcal{V}} \nabla \cdot (\nabla \times \mathbf{V}) d\mathcal{V}$

The right-hand side of this vanishes since $\nabla \cdot (\nabla \times \mathbf{V}) \equiv 0$, therefore, the right-hand side gives

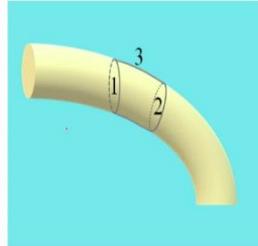
$\iint_{\text{bounding surface}} (\nabla \times \mathbf{V}) \cdot d\mathbf{A} = 0.$



Properties of vortex tubes

Circulation is the same in all circuits embracing the same vortex tube.

By the rule stated above the net component of vorticity through this closed surface is zero. Since the lateral surface is a vortex surface, there being no component of vorticity normal to it and, therefore, the net normal component of the vorticity across each cross-section is the same, inwards at one, and outwards at the other.



Strength of a vortex tube remains constant along its length

The above result implies that the strength of a vortex tube remains constant along its length. This also implies that a vortex tube cannot terminate within an inviscid fluid. It either closes on itself, like a vortex ring, or extends to the boundaries of the flow.

After persisting for a while, a smoke ring diffuses away. This is because of the viscous action over time.



Another important property of vorticity is there is no component of vorticity through a closed surface, that is, if Ω is the vorticity vector at any point, then $\iint_{\text{bounding surface}} \Omega \cdot d\mathbf{A} = 0$ if the flow is irrotational. And for this we require the divergence theorem from the calculus of vectors.

The divergence theorem states that if we consider a volume V bound by a closed surface S , then $\iint_{\text{bounding surface}} (\nabla \times \mathbf{V}) \cdot d\mathbf{A} = \iiint_V \nabla \cdot (\nabla \times \mathbf{V}) dV$. The right hand side of this expression vanishes, since the divergence of curl must be zero, curl is perpendicular to the divergence vector, and so divergence of curl must be zero identically. And therefore, we get the net component of vorticity through a closed surface as zero. We will use these facts shortly.

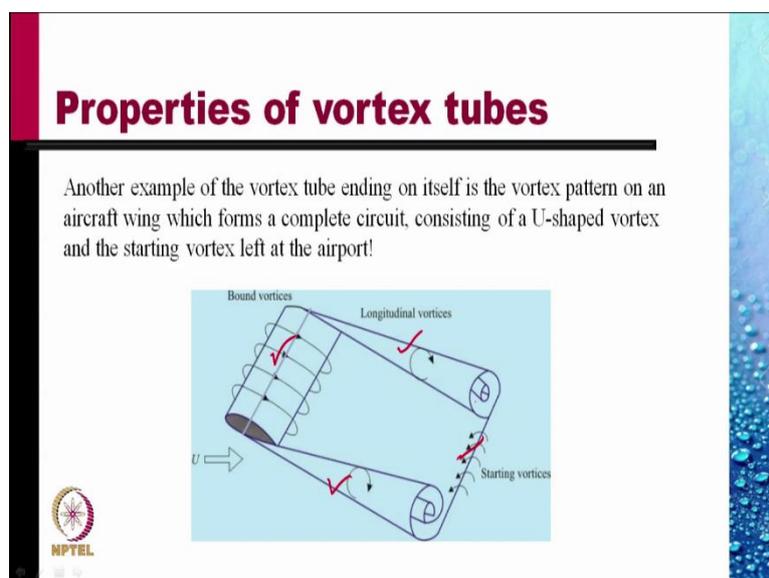
Now one of the important properties of vortex tubes is that the circulation is same in all circuits embracing the same vortex tube. Consider a volume which is part of a vortex tube shown. The enclosing surface of this volume is broken up into three parts. One and two, as the cross-section of the vortex tube, and third is a surface of the vortex tube itself. So, surfaces 1, 2 and 3 enclose a part of the vortex tube.

Now, the net vorticity at the surface, through the closed surface, must be zero. Since the surface 3 is tangential to the vortex tube, is a vortex surface, in the sense that the vortex vector is parallel to these, tangents at every point of the surface, so the vorticity through this surface is zero.

So, $\boldsymbol{\Omega} \cdot d\mathbf{A}$ over surface 1 must equal the $\boldsymbol{\Omega} \cdot d\mathbf{A}$ integrated over surface 2, that is the net component of vorticity across each cross section is the same in inward at one, and the outward at other, and since this normal component of vorticity is the strength of the vortex tube, and the strength of the vortex tube is a circulation. So the circulation is the same for all circuits embracing the same vortex tube.

In other words, the strength of the vortex tube remains constant along its length. And if this be so, this implies that a vortex tube cannot terminate within an inviscid fluid. Thus, if I have a vortex tube, it cannot end within an inviscid fluid, because what happens to the vorticity after that? So it must either close on itself, like a vortex ring, or extend to the boundaries of the flow. After persisting for a while a smoke ring diffuses away. This is because of the viscous action over time.

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Properties of vortex tubes

Another example of the vortex tube ending on itself is the vortex pattern on an aircraft wing which forms a complete circuit, consisting of a U-shaped vortex and the starting vortex left at the airport!



The trailing vortices



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The trailing vortices



Another example of a vortex tube ending on itself is the vortex pattern on an aircraft wing which forms a complete circuit consisting of a U-shaped vortex and the starting vortex left at the airport. This is a typical pattern of vortices on an aircraft wing. We assume a bound vortex on the wing, in fact it is this vorticity that is supposed to produce the lift. The Prandtl's theory of aircraft wings is based on this assumption that there is a bound vortex with the wing.

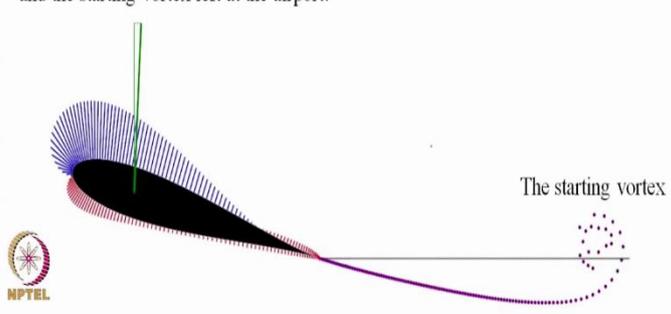
There are two trailing vortices and a starting vortex. I will show a little bit later the development of the starting vortex. But then this bound vortex, and two longitudinal vortices which are called the trailing vortices, and this starting vortex form a closed loop of vorticity of vortex tubes.

This picture shows the vorticity trailing from the edges of the aircraft wing. Another view of these trailing vortices. We are looking behind the aircraft and the smoke is swirling around showing two distinct vortices at the two wing tips.

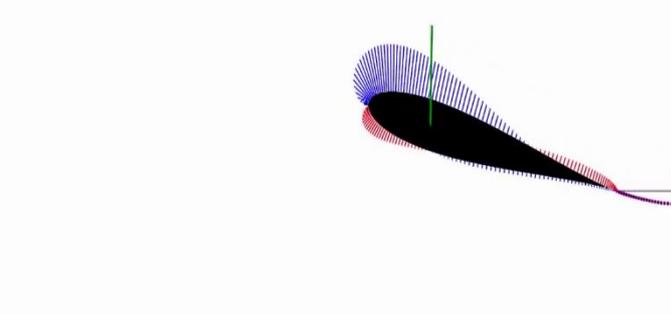
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Properties of vortex tubes

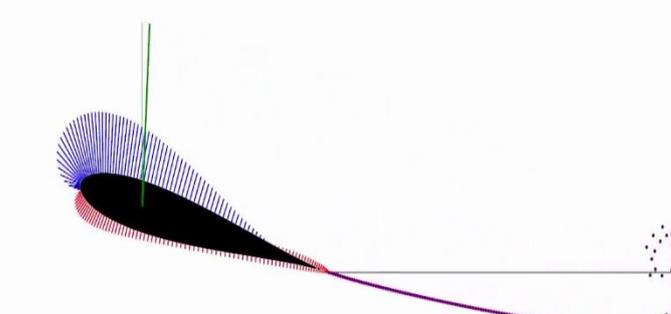
Another example of the vortex tube ending on itself is the vortex pattern on an aircraft wing which forms a complete circuit, consisting of a U-shaped vortex and the starting vortex left at the airport!



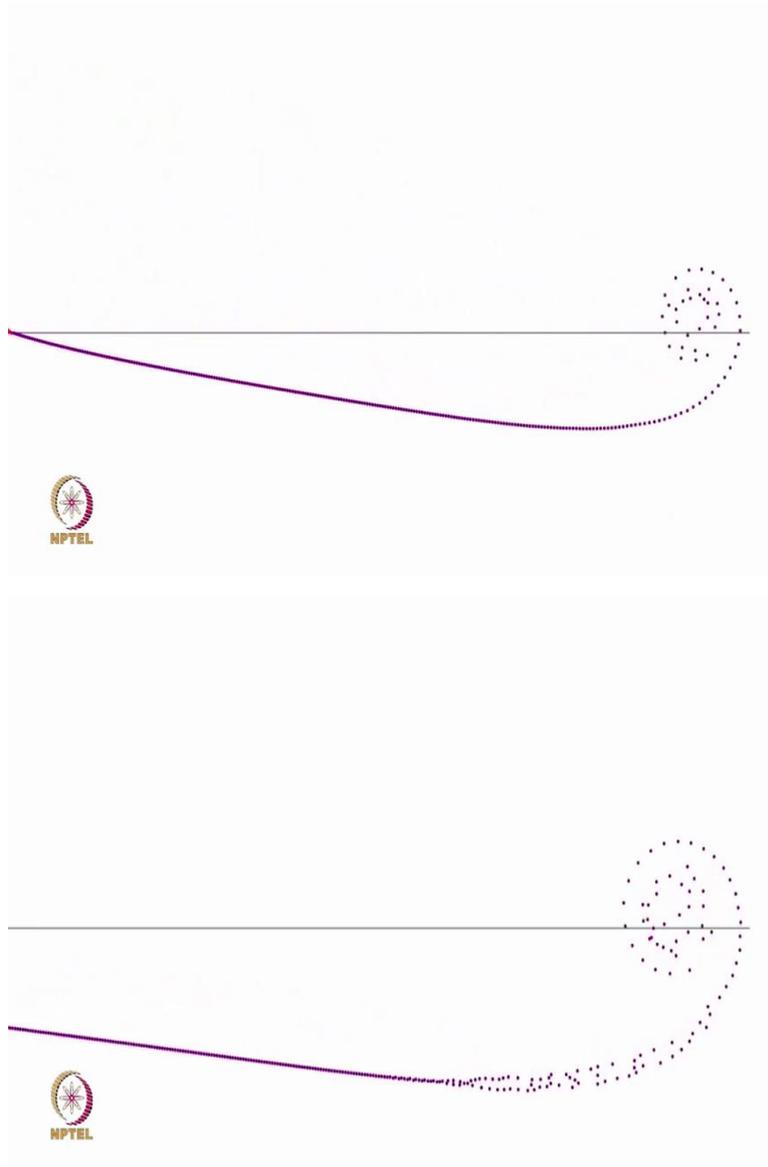
The diagram shows a black airfoil with a green vertical line through its leading edge. A purple vortex tube originates from the leading edge, wraps around the airfoil to form a U-shape, and then extends as a dotted line to a circular starting vortex on the right. The text 'The starting vortex' is placed near this circular vortex. The NPTEL logo is in the bottom left corner.



This diagram is identical to the one above, showing the vortex tube on the airfoil and the starting vortex. The NPTEL logo is in the bottom left corner.



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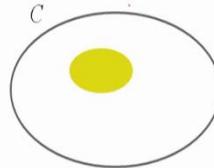
And there is a starting vortex, as the aircraft wing moves a vortex is formed which is bound with the wing, and there is a vortex which is termed as a starting vortex and is left behind at the airfield.

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Properties of vortex tubes

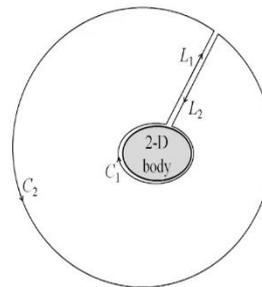
The relationship between circulation and the area integral of the vorticity is only valid if the region enclosed by the contour C is simply connected. This means that the region must be such that it is possible to shrink the contour C to a point without leaving the region. So, a region with a hole, like a bathtub drain is not simply connected and the contour is not reducible.

Similarly, a flow field about 2-dimensional body is not simply connected.



Properties of vortex tubes

The circulation about any circuit enclosing a 2-D body within an irrotational flow is the same.



Further properties of vortex tubes: the relationship between circulation and the area integral of the vorticity is only valid if the region enclosed by contour C is simply connected. This means that the region must be such that it is possible to shrink the contour C to a point without leaving the region.

So a region with a hole like a bathtub drain is not simply connected, since any contour that we take around it cannot shrink to a point, the bathtub drain would come in between. Similarly, a flow about a two dimensional body is not simply connected. This circuit C shown here cannot shrink to a point, the presence of this yellow body would forbid that.

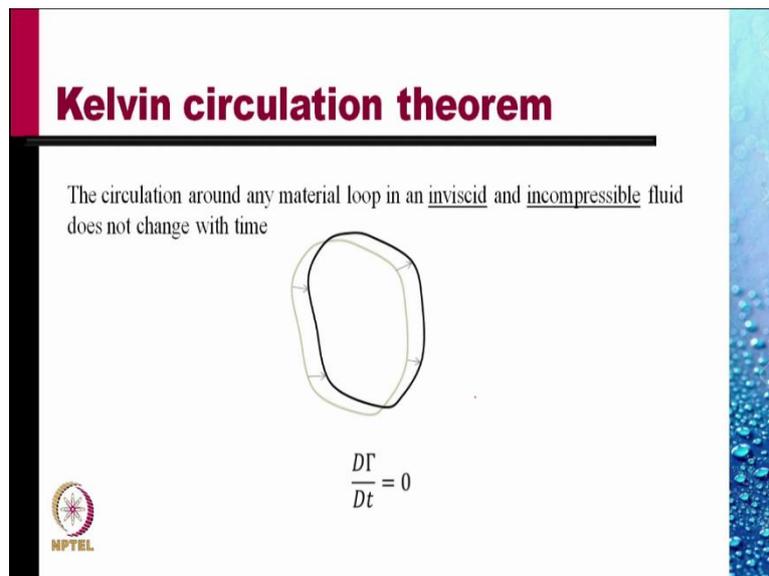
Of course, this is possible only in two dimensions. In three dimensions, if the yellow body was a three-dimensional body, we could have a capping surface on C , which avoids the solid body, and this contour could shrink along the capping surface.

Another property: this circulation about any circuit enclosing a 2 D body within a irrotational flow is the same. This is easy to show.

Consider the two bodies in two circuits C1 and C2. Neither C1 nor C2 can be shrunk to a point, but we can artificially create a composite circuit. Slit the space between the two circuits by two straight lines L1 and L2, and connect them to C1 and C2 in the manner shown. Now the circuit which consists of L1, C2, L2 and C1 is a circuit which is reducible, and which lies totally within an inviscid flow.

We can shrink this to a point, and that point would lie within the inviscid flow. It avoids the 2 D body completely, and so the circulation about this should be zero. Since, this lies wholly within the inviscid flow, there is no vorticity which is enclosed. Now, the circulation produced by the components L1 and L2 cancel out, because they are in the opposite directions. So this shows that the circulation about C1 and about C2 must be the same. Circulation about a circuit that lies within inviscid flow is the same for all circuits going around a 2D body.

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Kelvin circulation theorem

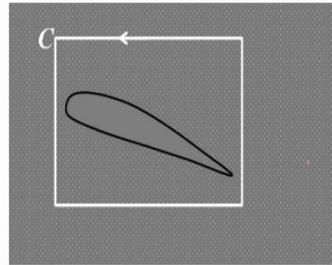
The circulation around any material loop in an inviscid and incompressible fluid does not change with time

$$\frac{D\Gamma}{Dt} = 0$$

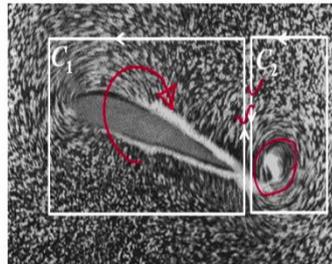
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The slide features a diagram of a material loop, which is a closed curve with a dashed line inside it, representing the path of fluid particles. The slide has a red header bar, a black vertical bar on the left, and a blue vertical bar on the right with a water droplet pattern.

Vortex system of a wing



Vortex system of a wing



There is a famous theorem called Kelvin circulation theorem, we will not prove it here because it takes a little higher level of vector calculus, but the theorem is easy to understand. The circulation around any material loop in an inviscid and incompressible fluid does not change with time.

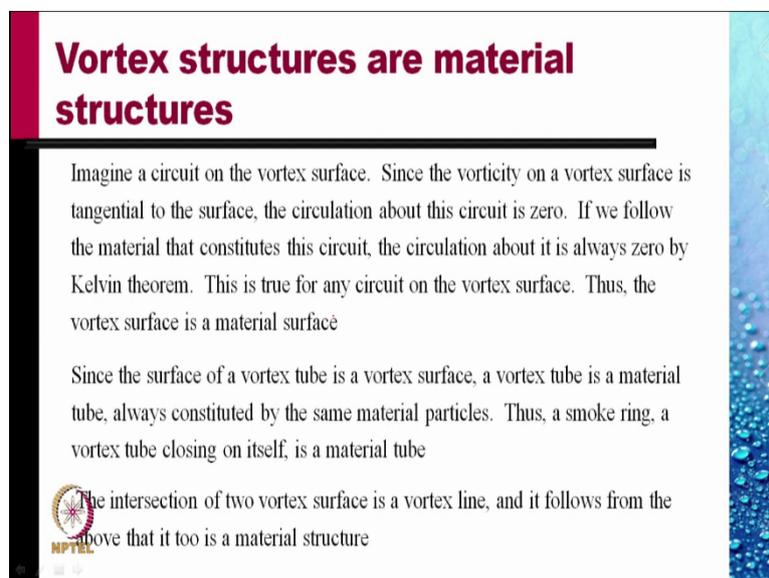
Consider this circuit as a material loop. This is the position at a given time t . At a time t plus dt the same particles occupy this new location. These are the same material particles, so we say the circuit is a material circuit. Then, whatever be the circulation of this loop of this circuit at any given time remains constant with time: $\frac{D\Gamma}{Dt} = 0$.

Note that if this being a material circuit is important, and this gives an interesting conclusion. Consider an aerofoil sitting in an inviscid fluid. There is no velocity anywhere, I draw a circuit C . Clearly since the velocity is zero the circulation around this circuit is zero. Now, let us

assume that the fluid starts moving to the right. As the fluid starts moving to the right, this circuit would stretch, and would occupy a position like this. I replaced the aerofoil by a picture, a photograph that was originally produced by Prandtl himself in his laboratory, and in this photograph you see a vortex out here.

So there is vorticity now that is induced there. The science of aerodynamics as developed by Prandtl talks about the origin of this vortex which is termed as the starting vortex, which we talked about earlier. Now, the circulation about this white circuit that we show here, would still be zero, because it is the material circuit. So how about this vorticity? Divide the circuit into two circuits C_1 and C_2 . The original circuit C was nothing but C_1 plus C_2 . The contribution to circulation about these two arms cancel one another out. They are close together. Around circuit C_2 there is a positive circulation, because the flow around the starting vortex is counterclockwise, so there is a positive C_2 . Now about the circuit C_1 then there should be an equal and opposite circulation because the net circulation must still be zero. This circulation is due to what is termed as the bound vortex with the wing. So the wing has a circulation going clockwise. This is the bound vortex which is one arm of the closed vortex circuit behind an aircraft wing.

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Vortex structures are material structures

Imagine a circuit on the vortex surface. Since the vorticity on a vortex surface is tangential to the surface, the circulation about this circuit is zero. If we follow the material that constitutes this circuit, the circulation about it is always zero by Kelvin theorem. This is true for any circuit on the vortex surface. Thus, the vortex surface is a material surface

Since the surface of a vortex tube is a vortex surface, a vortex tube is a material tube, always constituted by the same material particles. Thus, a smoke ring, a vortex tube closing on itself, is a material tube

The intersection of two vortex surface is a vortex line, and it follows from the above that it too is a material structure

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Properties of vortex tubes

For the special case of a vortex tube with small area, the vorticity Ω can be considered as approximately uniform across the tube cross-section A , and the strength of a vortex tube can be written as $\Gamma = \Omega A$.

Therefore, the vorticity along a vortex tube varies inversely as the cross-sectional area of the tube.



Properties of vortex tubes

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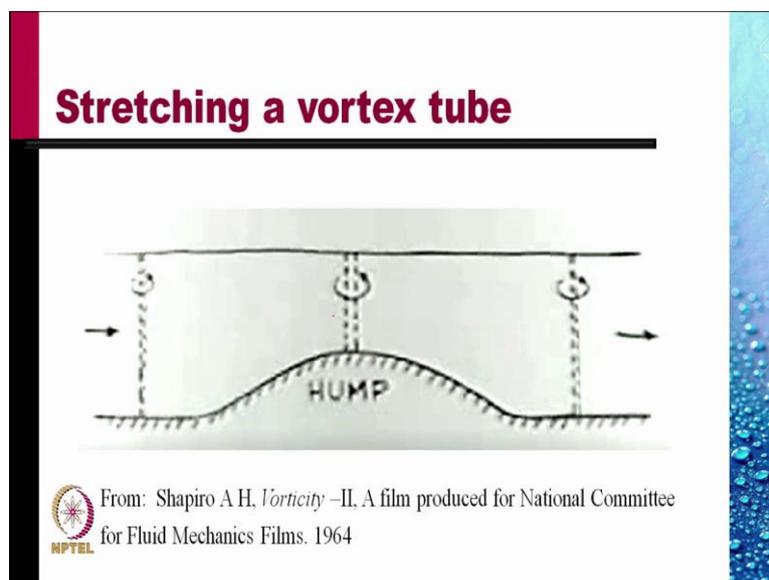
Imagine a circuit on the vortex surface. Since the vorticity on the vortex surface is tangential to the surface, the circulation about this circuit is zero. If we follow the material that constitutes the circuit, the circulation about it is always zero by Kelvin theorem. This is true for any circuit on the vortex surface.

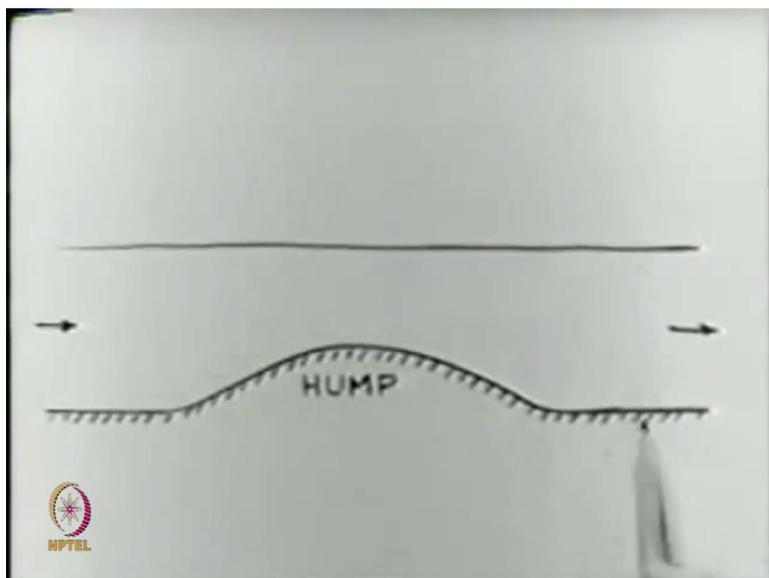
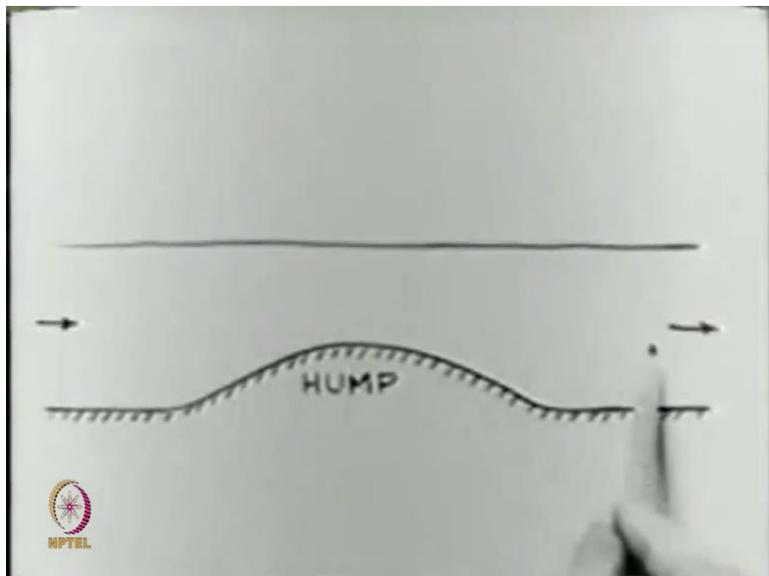
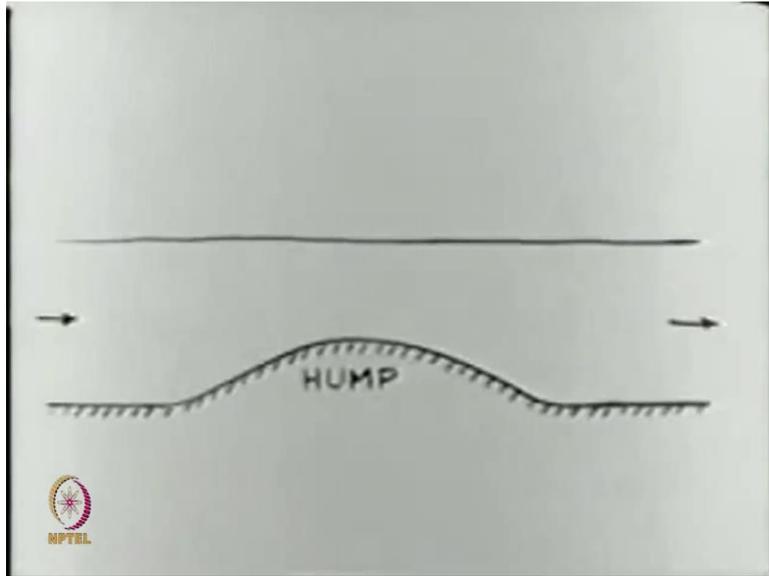
Thus, the vortex surface is a material surface. So, whatever is a vortex surface once, will always be a vortex surface in any rotational flow. Since the surface of a vortex tube is a vortex surface, a vortex tube is a material tube, always constituted by the same material particles. Thus, a smoke ring, a vortex tube, closes on itself, is a material tube. That is why we see as a tube of smoke, the same smoke, always. The intersection of two vortex surfaces is a vortex line and it follows from the above that this too should be a material structure. Thus, a vortex line is a material line.

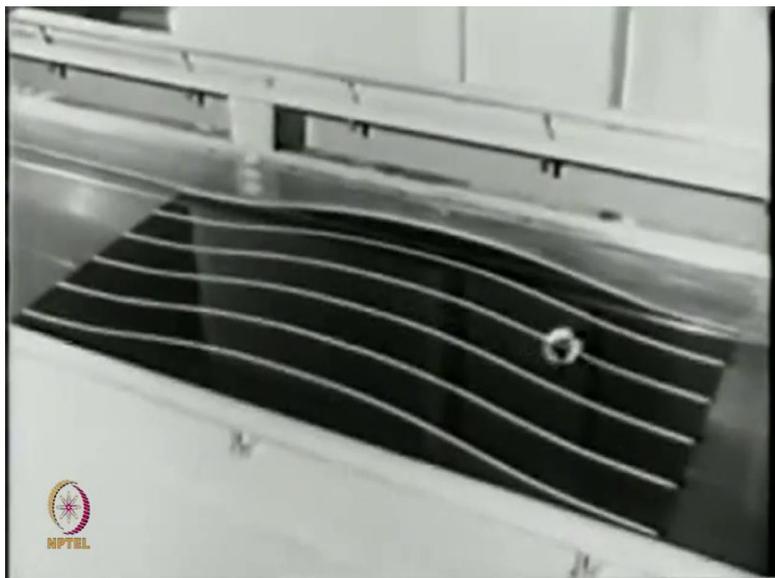
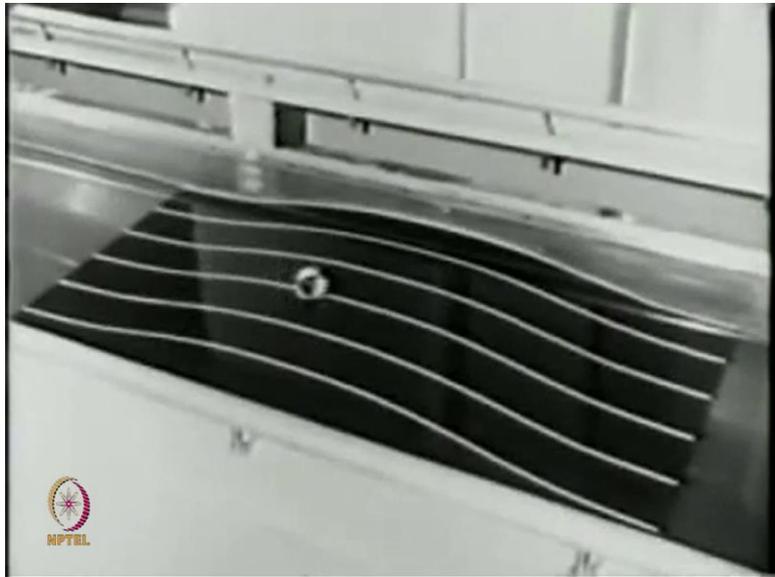
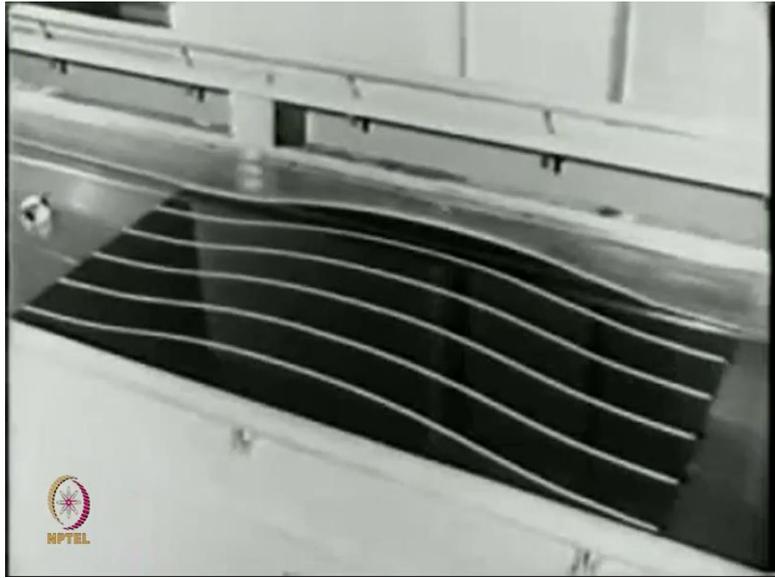
For the special case of a vortex tube with small area, the vorticity Ω can be considered as approximately uniform across the tube cross section A , and the strength of the vortex tube can be written as $\Gamma = \Omega A$. Therefore, if we stretch the vortex tube, and since it is a material tube it consists of the same material, if we stretch it its cross-sectional area will decrease, and so this vorticity along a vortex tube varies inversely as the cross-sectional area of the tube.

This clip shows a ballerina. The ballerina is spinning around, and as she stretches herself by raising her hands, the speed of rotation goes up. That is exactly how the vorticity along a stream tube increases.

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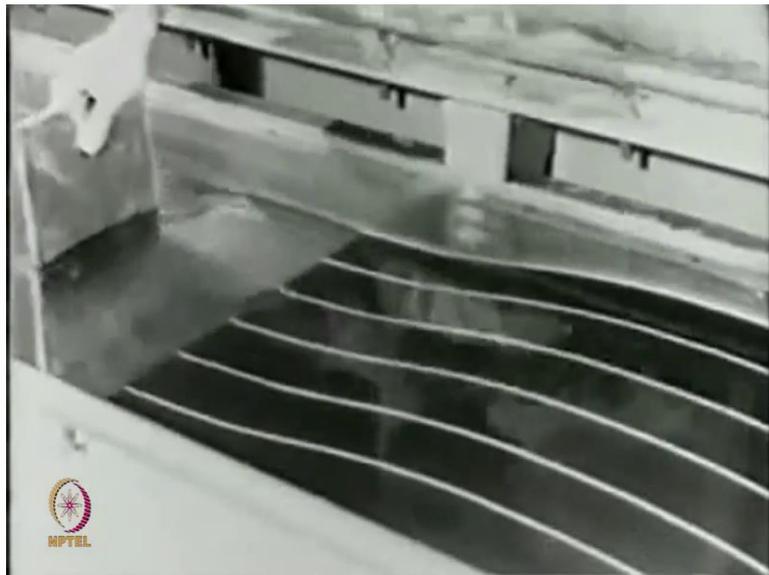


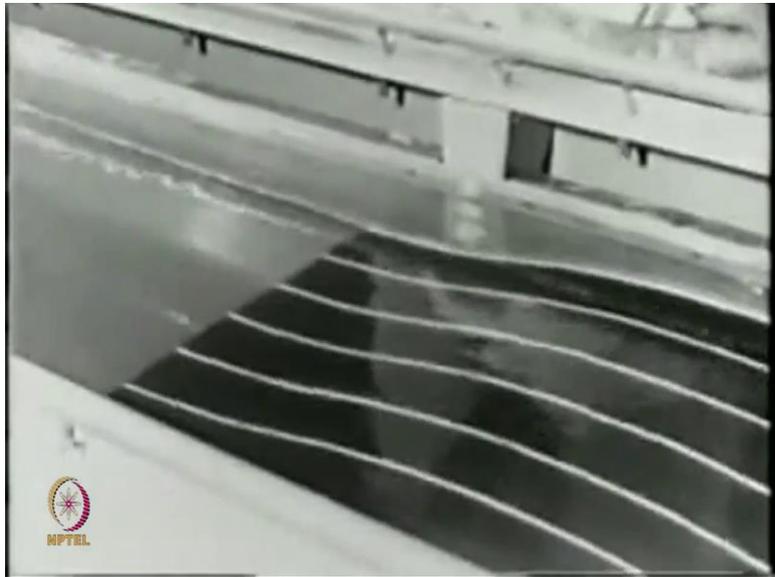


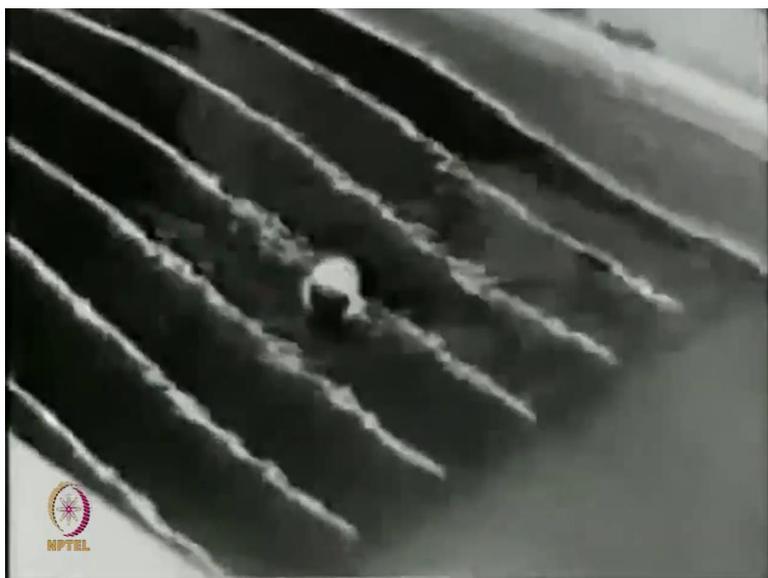
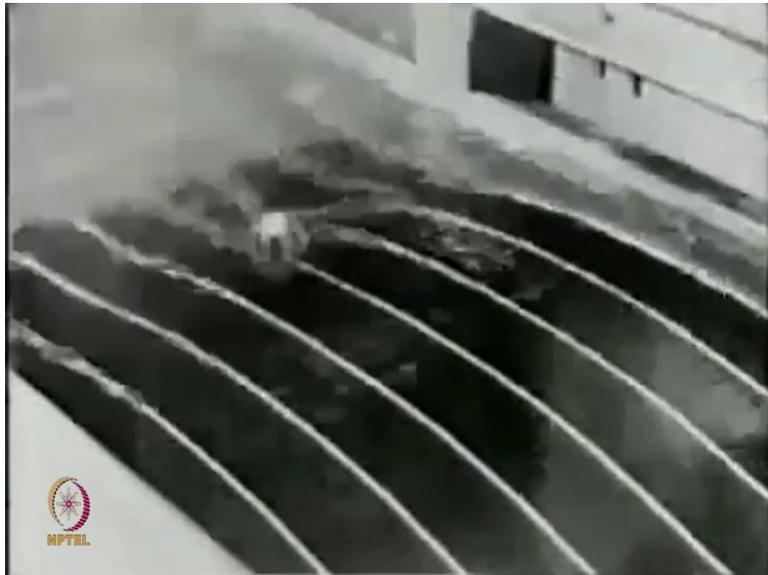
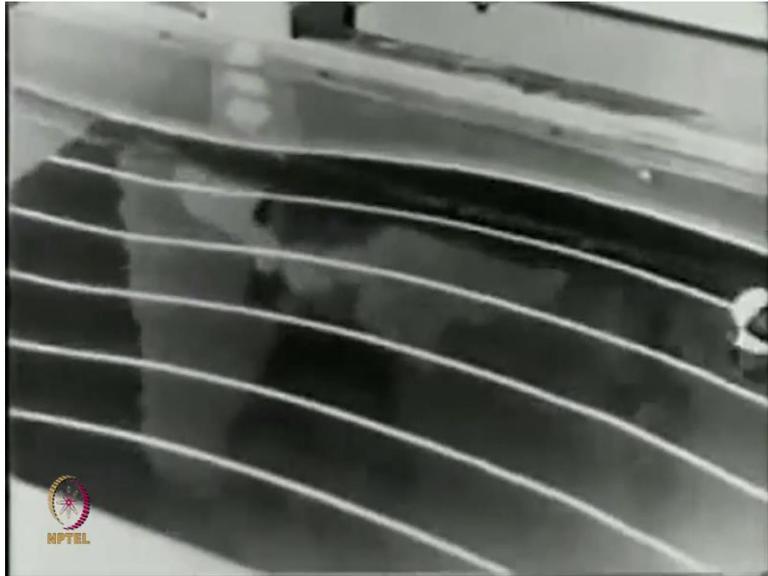
This has been illustrated very nicely by professor Shapiro of MIT in a movie, Vorticity – II, produced by him for the National Committee of Fluid Mechanics Film in 1964.

To show the effect of stretching vortex lines, we have an apparatus in which water flows over a hump. The water depth first decreases and then increases. The speed of the water is inversely proportional to the depth, and so the flow accelerates as it goes up the hump and then decelerates as it goes down the hump.

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Next presentation

Learning outcomes:

- Potential flows about 2-D bodies



Now, I am making a vortex with a sharp edged plate. This vortex has its axis vertical. As it goes up the hump the length of the vertical vortex line decreases, and so the vorticity decreases as you can see by the spin. When the vortex line goes down the hump, the length of the line increases and the rate of spin also increases.

Thank you very much.