Fluid Mechanics & its Application
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Lecture 22

Welcome back.

In this lecture we will cover flows at low Reynolds numbers.

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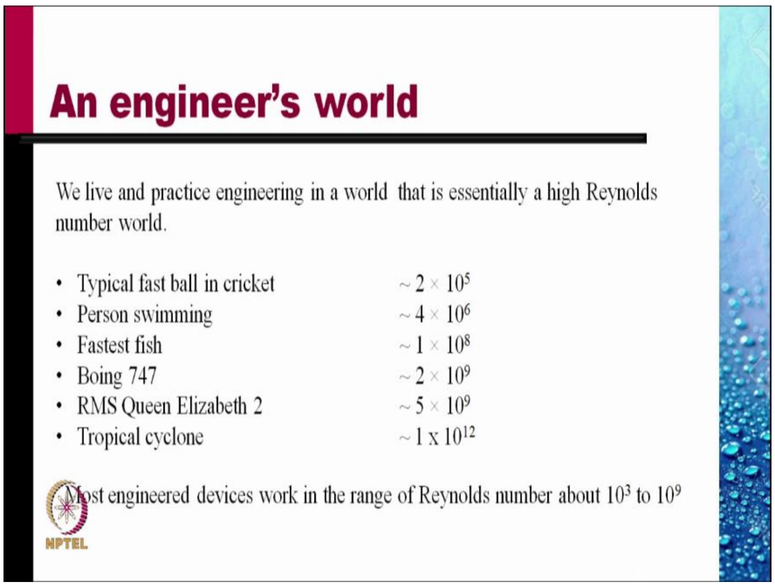

We live and practice engineering in a world that is essentially a high Reynolds number flow. Typical fast ball in cricket is at a Reynolds number about 2×10^5. A person swimming in water swims at about 4×10^6. The fastest fish is at about 10^8. A Boeing 747 airplane flies at 2×10^9. The RMS Queen Elizabeth II sailed at 5×10^5 Reynolds number. A tropical cyclone would have Reynolds number up to 10^{12}. Most engineer devices work in the range of Reynolds number about 10^3 to 10^9.

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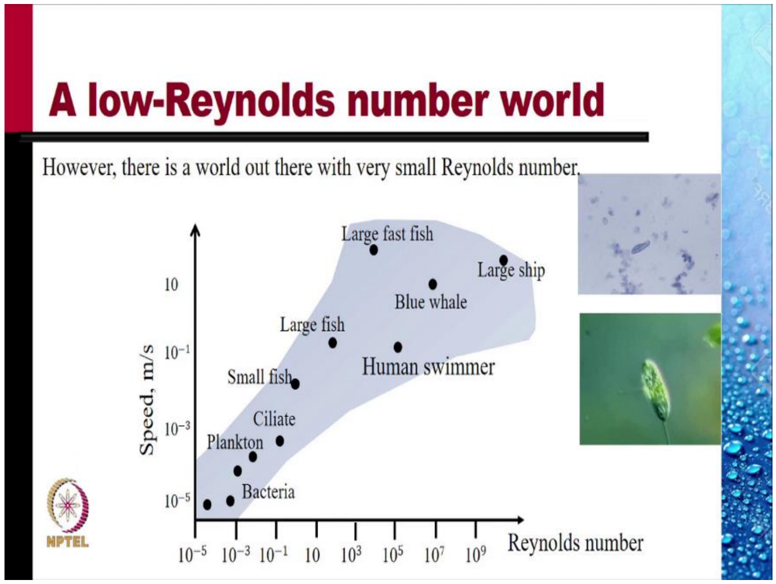

However, there is a world out there with very small Reynolds numbers. This is the region where the Reynolds number is less than 1 in this and included bacteria, plankton, ciliates, and other such small particles whose Reynolds number are less than 1.

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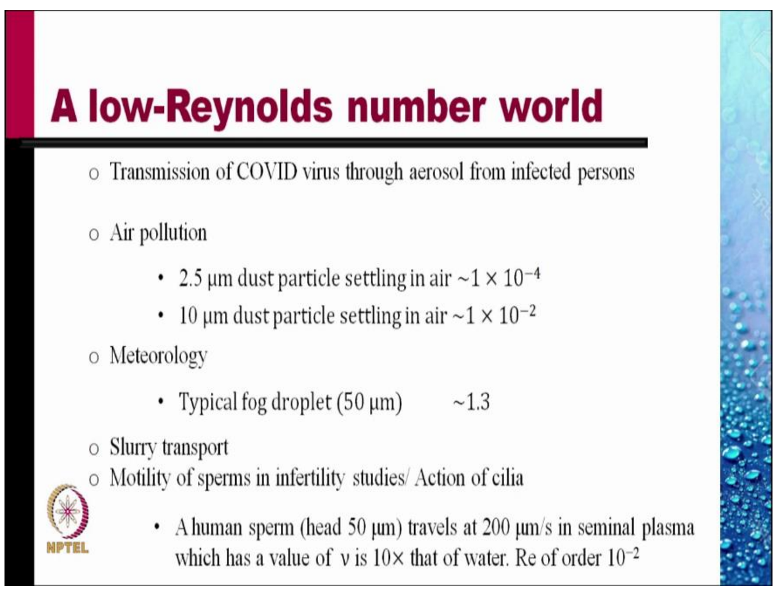

Transmission of COVID virus through aerosols from infected persons is a low Reynolds number phenomenon. Air pollution, the 2.5 µm dust particles settle in air with Reynolds number about 10^{-4}, while a 10 µm dust particle settles at a Reynolds number of 10^{-2}. In meteorology, the typical fog droplets of size 50 µm settle at a Reynolds number of 1.3.

Slurry transport in chemical engineering is one phenomenon where low Reynolds number prevail. Motility of sperms in infertility studies, and action of cilia are also very low Reynolds number flows. A human sperm with the head of 50 µm, travels at 200 µm /s in seminal plasma, which has a value of kinematic viscosity 10 times that of water. The Reynolds number is order 10^{-2}.

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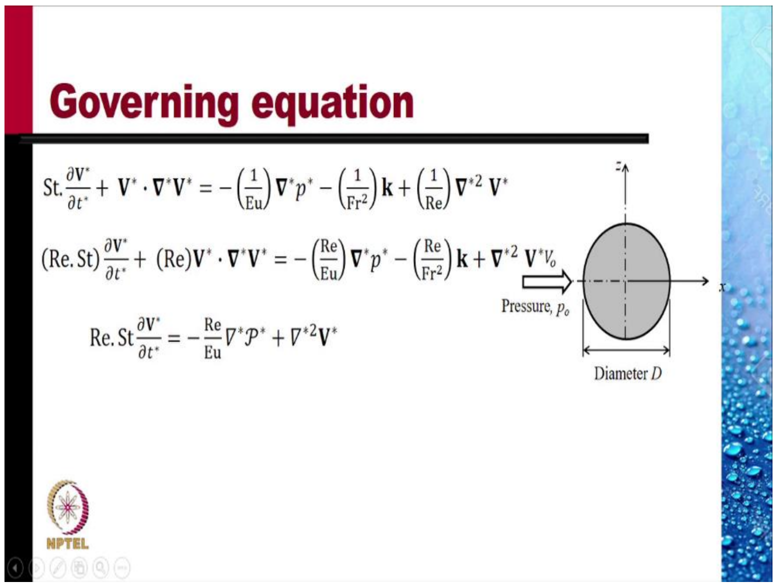

Let us study the low Reynolds number flows in details. We had non-dimensionalized the governing equations the Navier-Stokes equation, and the non-dimensionalized governing equations in normalized variable looks like this.

In this the coefficient of the convective acceleration term has been made unity, and then the coefficient of the unsteady acceleration term is Strouhal number, of the pressure gradient is 1/Eu, of the gravity term is $1/\mathrm{Fr}^2$, and of the viscous for 1/Re. If Reynolds number is low, the viscous forces would have a coefficient which is much larger than 1. It is not convenient to us. So, we multiply this equation across by the Reynolds number to make the coefficient of the viscous term as 1. This is the resulting equation.

Now in this equation, the coefficient or the convective acceleration term is Reynolds number. And if the Reynolds number is very low compared to 1, this term can be neglected, at least in comparison with the viscous term. And if we neglect it, the resulting equation is this. Here we have also done additional manipulation, where we combined the pressure term in the gravity terms by defining a non-gravitational pressure $\mathcal{P}$, and then the coefficient of the gradient of the $\mathcal{P}$ is Reynolds number by Euler number.

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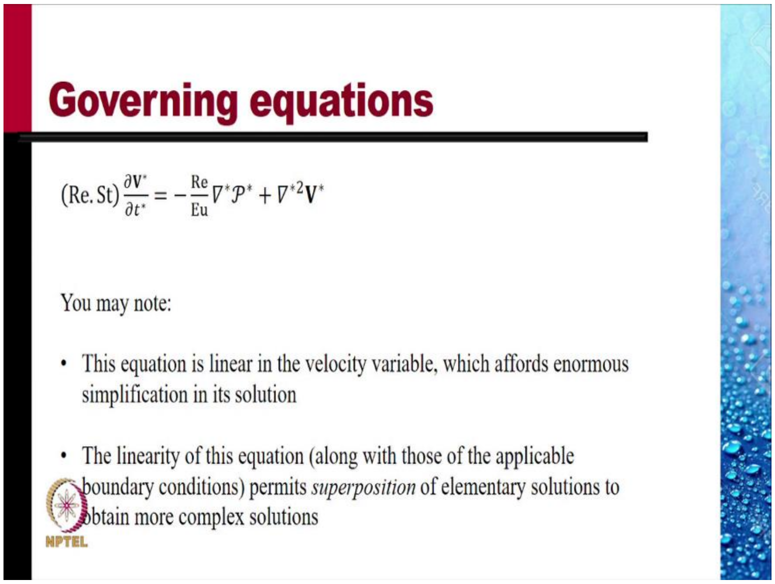

You may note that this equation is linear in the velocity variable, which affords enormous simplification in its solution. The linearity of this equation along with those of the applicable boundary conditions, permits superposition of elementary solutions to obtain more complex solutions.

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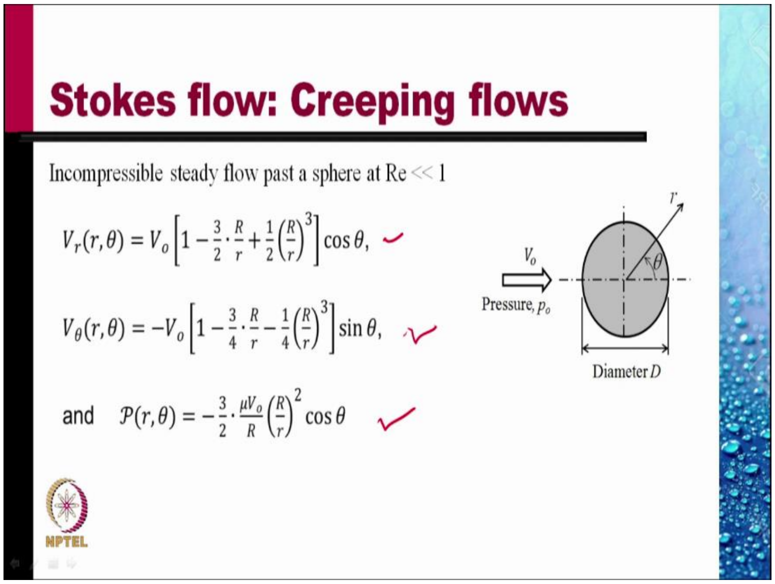

Stokes solved this equation in spherical polar coordinate for flows about a spherical body. Then, he got the velocity components V_r, the radial velocity component given by this equation, where capital R is the radius of the sphere. The angle θ measured from the horizontal axis, and r is the position vector. The θ component the velocity is this, and the pressure on the sphere is given by this equation. We give these equations without proof, because the mathematics involved is a little more complex for students at the first course level

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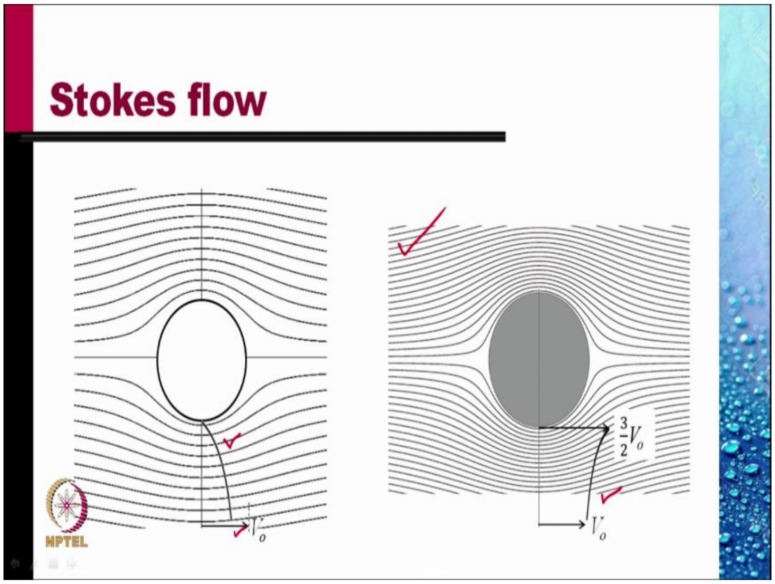

If you plot the streamlines, the streamlines look like this. This shows the velocity profile. At the shoulder of the sphere, this velocity V_o is the freestream velocity, the velocity with which the flow approaches the sphere. Notice that almost a diameter away from the sphere the velocity is still quite small compared to V_o.

We give here the flow about the same sphere in a high Reynolds number flow. In fact, in a infinite Reynolds number flow with no viscosity. Look at the velocity profile in this. It is quite different. Here the velocity at the shoulder of the sphere is large $\frac{3}{2}V_o$ compared to 0 velocity in the Stokes flow, and the velocity at the distance of about a diameter has already approached the velocity V_o. Remember, this flow solution is not a physical solution. This is for an idealized fluid with 0 viscosity, no such fluid exists. This is only to illustrate the mathematical nature of the two equations.

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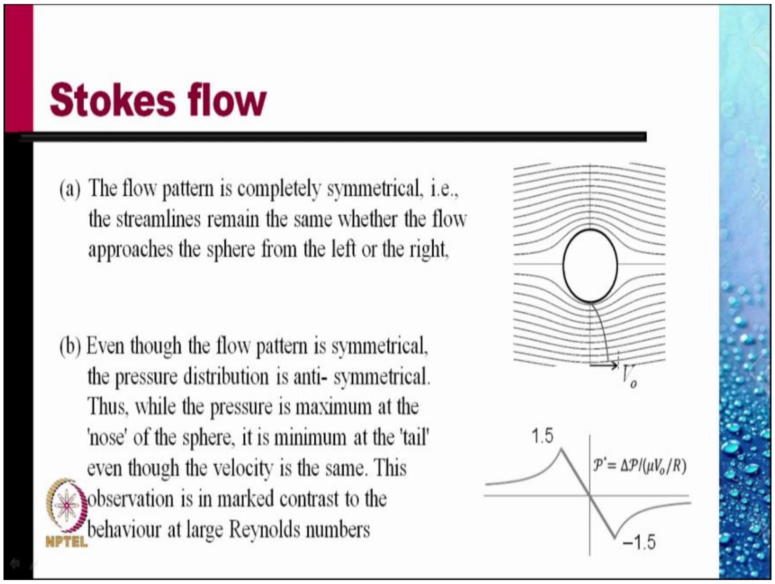

Let us, study some distinguishing properties of the Stokes flow. The flow pattern is completely symmetrical, that is, the streamlines remain the same whether the flow approaches the sphere from left or from right. Even though the flow pattern is symmetrical, the pressure distribution is anti-symmetrical. Thus, while the pressure is maximum at the nose, it is minimum at the tail, even though the velocity is the same at two points. This observation is in marked contrast to the behaviour at large Reynolds numbers, where pressure at the nose and the tail are exactly the same.

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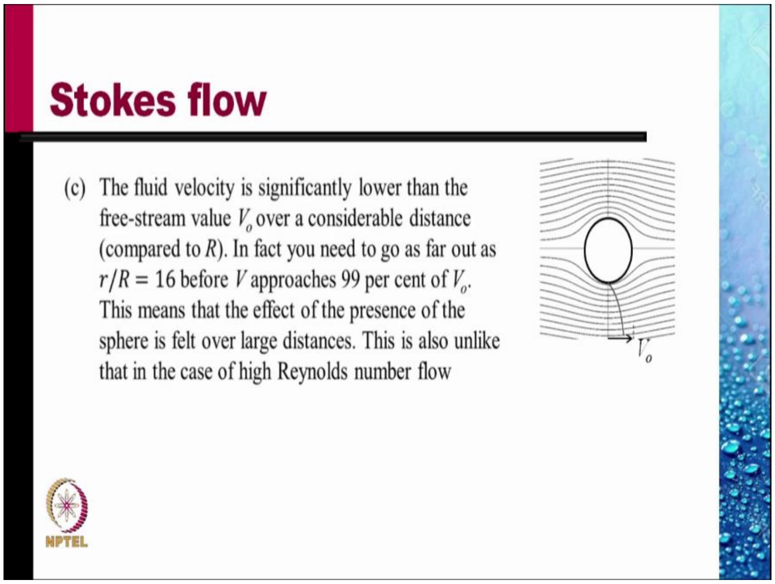

The fluid velocity is significantly lower than the freestream value V_o over a considerable distance compared to the radius, or the diameter, of the sphere. In fact, you need to go as far out as $\frac{r}{R} = 16$ before V approaches 99 percent of V_o. This means, that the effect of the presence of the sphere is felt over large distances. This is also unlike that in the case of high Reynolds numbers, where the flow velocity approaches the freestream velocity within less than two diameters from the centre of the sphere.

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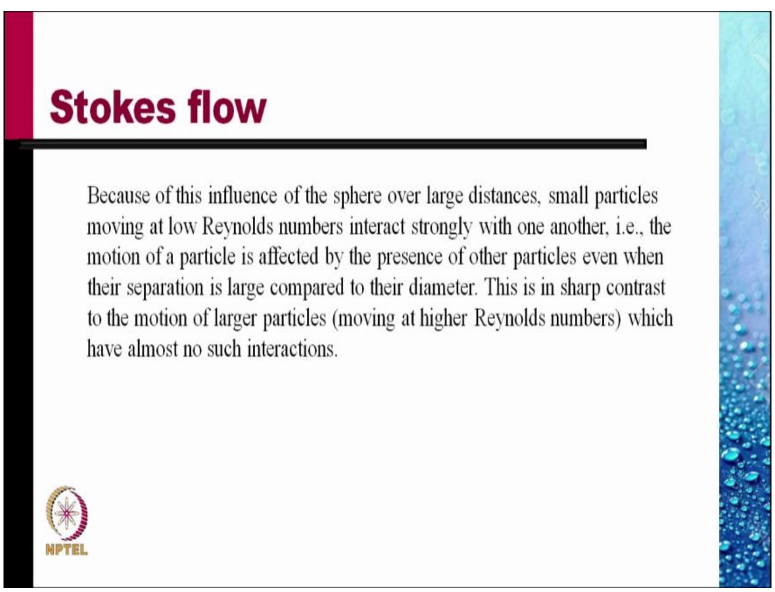

Because of the influence of the sphere over large distances in lower Reynolds number flows, small particles moving at low Reynolds numbers interact strongly with one another, that is, the motion of a particle is affected by the presence of other particles even when their separation is large compared to their diameter. This also is in sharp contrast with the motion of larger particles moving at a higher Reynolds numbers which have almost no such interactions.

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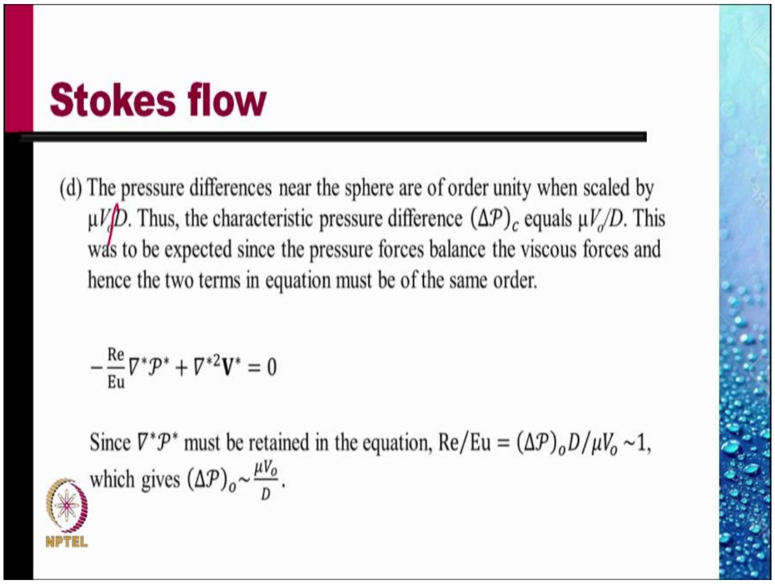

Another property, the pressure differences near this sphere are of order unity when scaled by $\mu V_o/D$. Thus, the characteristic pressure difference $(\Delta\mathcal{P})_c = \mu V_o/D$. This was to be expected, since the pressure force balances the viscous force, and hence the two terms in the equation must be of the same order. This also is in contrast with the high Reynolds number flows, where the characteristic pressure difference is $\frac{1}{2}\rho {V_o}^2$, not dependent upon the viscosity at all.

In this steady flow, the governing equations acquires the form $-\frac{\mathrm{Re}}{\mathrm{Eu}}\nabla^*\mathcal{P}^* + \nabla^{*2}\boldsymbol{V}^* = 0$. Since $\nabla^*\mathcal{P}^*$ must be retained in the equation along with $\nabla^{*2}\boldsymbol{V}^*$, Reynolds number by Euler number should be equal to 1, or of order 1, and that is why $(\Delta\mathcal{P})_o \sim \frac{\mu V_o}{D}$.

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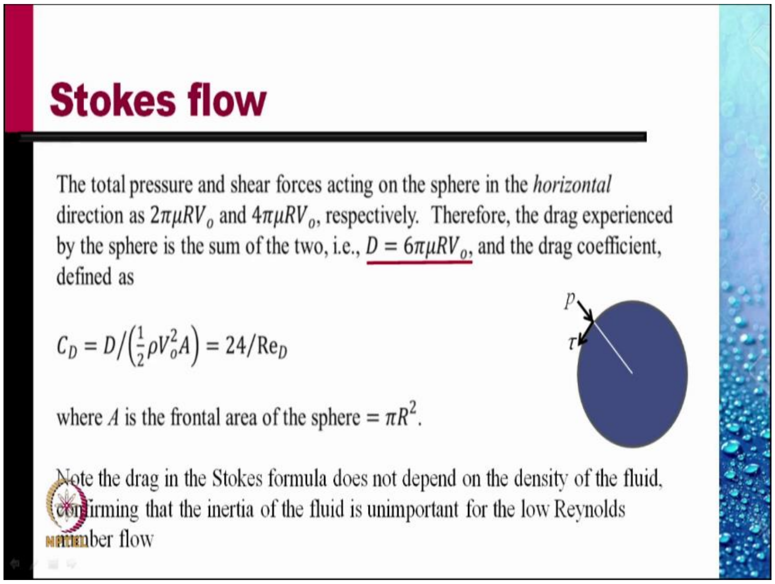

The total pressure and shear forces acting on the sphere in the horizontal directions are obtained as $2\pi\mu RV_o$ and $4\pi\mu RV_o$, respectively. These are obtained by integrating the horizontal component of the pressure forces and the shear forces over the entire surface of the sphere.

Therefore, the total drag experienced by the sphere is the sum of the two, that is, the drag is equal to $6\pi\mu RV_o$, and the drag coefficient defined as $C_D = D/\left(\frac{1}{2}\rho V_o^2 A\right) = 24/Re_D$, and A is the frontal area of the sphere, πR^2. Note that this definition of drag coefficient is not appropriate for low Reynolds number flows. But we do this in this manner because it is conventional in the high Reynolds number flow, and just to compare the drag coefficients, we have defined the drag coefficient in the low Reynolds number flow to in the same manner. When Reynolds number is very small, this drag coefficient become quite large.

This drag, $6\pi\mu RV_o$, is what you learned in a high school as Stokes law. It was not mentioned there that this is only at very low Reynolds number. It was given as the drag on a sphere. But now we know that this refers to drag on very small particles moving at very small velocities, like dust particles in air. Note that the drag in the Stokes formula does not depend upon the density of the fluid, confirming that the inertia of the fluid is unimportant for low Reynolds number flow. An interesting observation.

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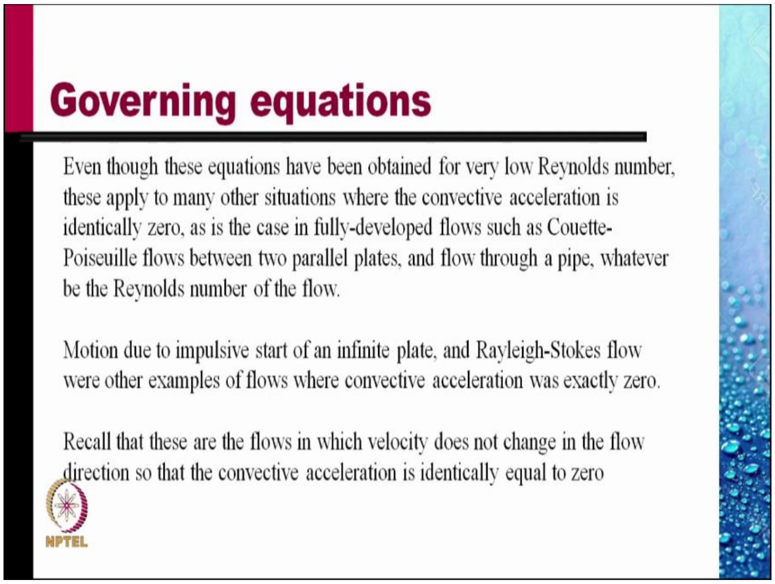

Even though the governing equation for the low Reynolds number flows has been obtained by assuming the Reynolds number is low, these are applied to many other situations, where the convective acceleration identically zero, as is always the case in fully-developed flow, such as the Couette and the Poiseuille flows between two parallel plates, and in flow through a pipe whatever be the Reynolds number of the flow.

There, the convective acceleration is identically equal to zero and so, it is only the pressure forces that balance the viscous forces, and so, the equation is the same as the equation for the low Reynolds number flow, even though the Reynolds number could be very high in those flows. Motion due to impulsive start of an infinite plate where again the convective accelerations are zero and the Rayleigh Stokes flows are other examples of flows where convective acceleration was exactly zero. Recall that these are the flows in which velocity does not change in the flow direction, so that the convective acceleration is identically equal to zero.

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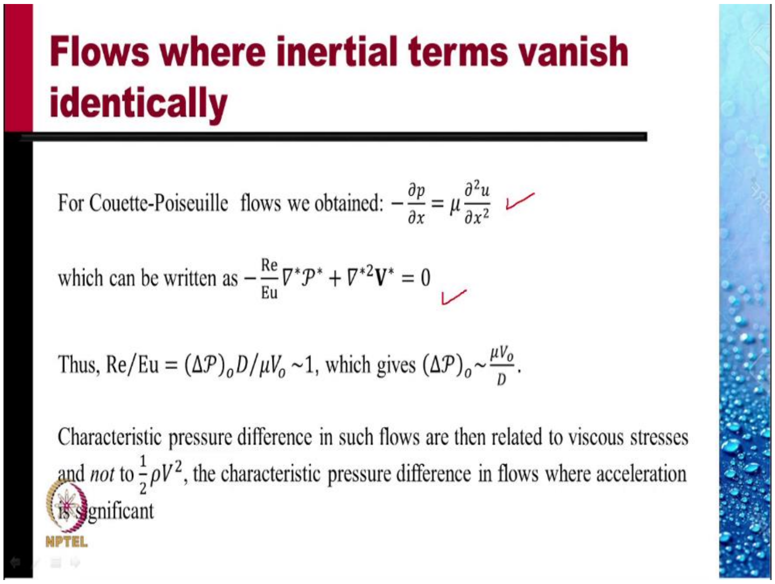

For Couette-Poiseuille flow we obtained this as the equation, and this could be written as this on non-dimensionalization, exactly the same equation that we are dealing with in low Reynolds number flows. Thus, Re/Eu is of order 1 and $(\Delta \mathcal{P})_o \sim \frac{\mu V_o}{D}$. The characteristic pressure differences in such flows are then related to the viscous stresses and not to $\frac{1}{2}\rho V_o^2$, the characteristic pressure difference in flows where acceleration is significant.

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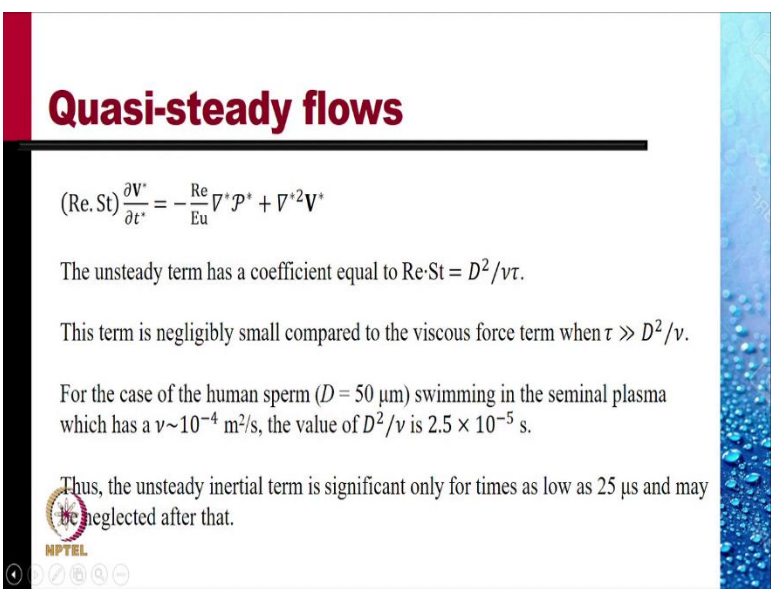

Now, let us talk about quasi-steady flows. The governing equation in an unsteady flow at low Reynolds number was obtained like this. The unsteady term here has a coefficient equal to Re·St, that is $D^2/\nu\tau$ where ν is the kinematic viscosity of the fluid which is μ/ρ, dynamic viscosity divided by the density. This term is negligibly small compared to the viscous term when τ is much greater than D^2/ν.

For the case of a human sperm with D is equal to 50 μm swimming in the seminal plasma which has a ν of 10^{-4} m^2/s. The value of D^2/ν is 2.5×10^{-5} s. Thus, the unsteady inertial term is significant only for times as low as 25 μs and maybe neglected after that.

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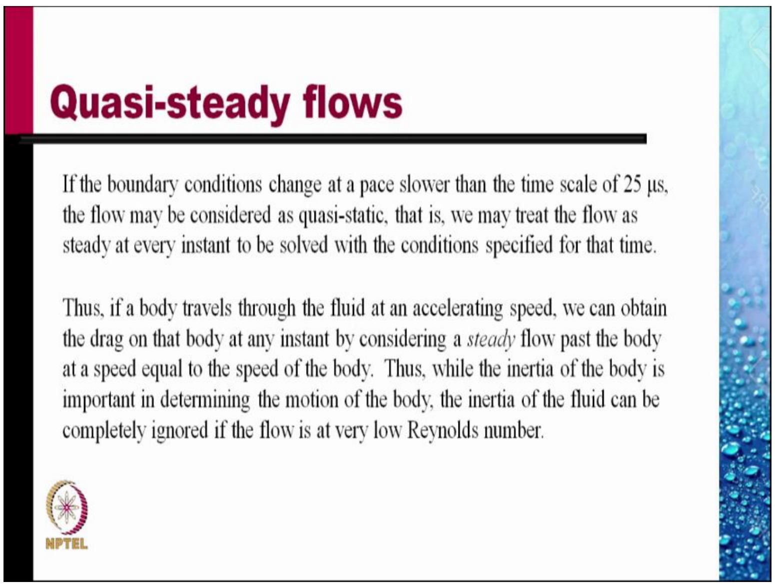

Also, if the boundary conditions change at a pace slower than the timescale of 25 μs, the flow may be considered as quasi-steady. That is, we may treat the flow as steady at every instant to be solved with the conditions specified for that time. It is exactly the same picture as we did earlier when we have a flow above a plate which was oscillating. Then we saw that when the motion of the plate was slow, the time period was large, the velocity profile in the flow was like that for steady flow with the boundary condition varying with time.

Thus, if a body travels through the fluid at an accelerating speed, we could obtain the drag on that body at any instant by considering a steady flow past the body at a speed equal to the speed of the body. Thus, while the inertia of the body is important in determining the motion of the body, the inertia of the fluid can be completely ignored, if the flow is at very low Reynolds number.

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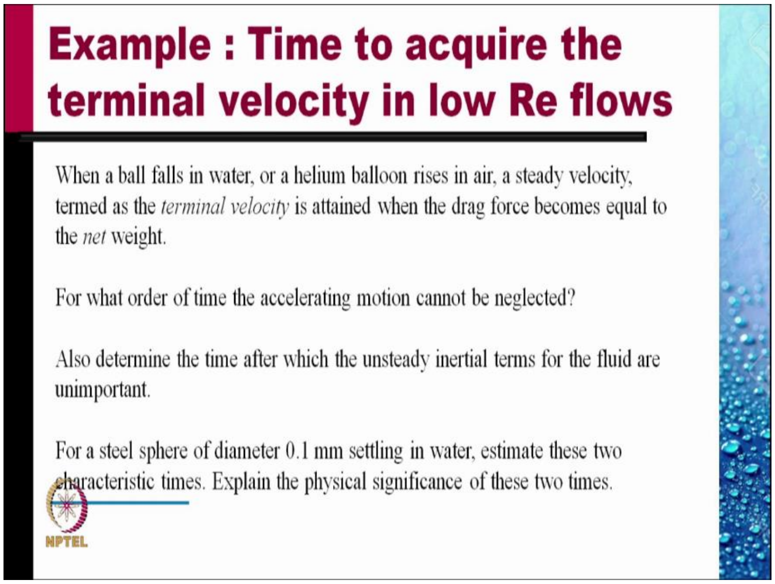

Let us, do an example. Let us, calculate the time to acquire the terminal velocity in low Reynolds number flows. When a ball falls in water or a helium balloon rises in air, a steady velocity term as the terminal velocity is attained when the drag force becomes equal to the net weight, that is, the weight minus the buoyancy. For what order of time the accelerating motion can be neglected? Also determine the time after which the unsteady inertial term for the fluids are unimportant.

For a steel sphere of diameter 0.1 mm settling in water, estimate these two characteristic times and explain the physical significance of these two times.

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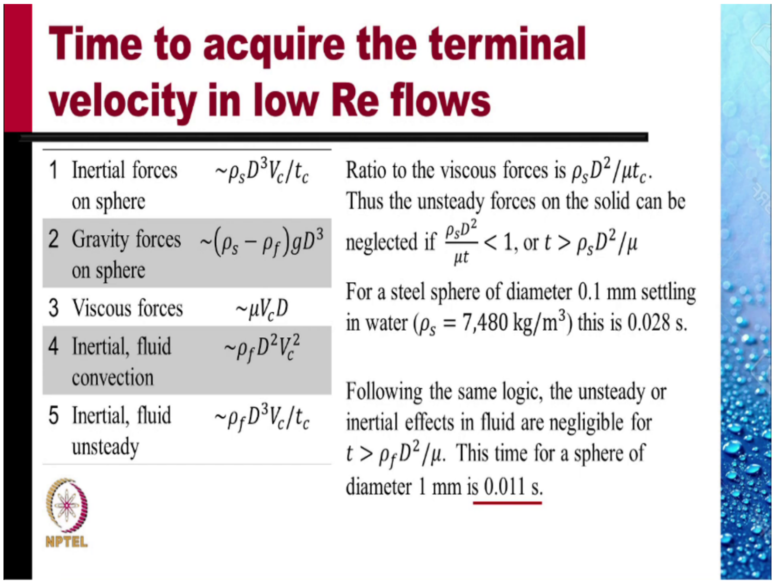

Let us see what are the forces, and let us estimate the forces. The inertial forces on the sphere would be like mass, which is like ρ for the solid, the ρ for steel, times D^3, which is like volume multiplied by V_c/t_c, which would be the characteristic acceleration.

The gravity forces on the sphere which is the wet weight is the density difference between the solid density and the liquid density to account for the buoyancy force, times g, times the volume, which is like D^3. The viscous forces a like $\mu V_c D$. Inertial forces of fluid convection, we have done that a number of times earlier is minus rho of fluid. It is like $\rho_f D^2 V_c^2$.

And the unsteady inertial fluid forces would be like $\rho_f D^3$, which is like mass of the fluid, times acceleration, V_c/t_c. The ratio of the viscous forces to inertial forces of sphere are like $\rho_s D^2/\mu t_c$. Thus then, unsteady forces of the solid can be neglected if $\rho_s D^2/\mu t_c$is much less than 1, or t is greater than $\rho_s D^2/\mu_c$.

We can neglect it if *t* is greater than this. So, for time less than this, we need to consider the unsteady forces. For a steel sphere of diameter 0.1 mm settling in water, this is 0.028 s, a very small time. Following the same logic, the unsteady or the inertial effects of fluids are negligible for time greater than $\rho_f D^2/\mu$. This time for the same sphere is 0.011 s, less than half of the earlier time.

So, for time less than 0.011 s, we cannot neglect the inertial effects within the fluid. But, between 0.011 s and 0.028 s, we can neglect the inertial effects in the fluid, but not the acceleration of the steel sphere. But beyond 0.028 s, we could neglect the inertial effects completely, and the steel sphere would be traveling with terminal velocity.

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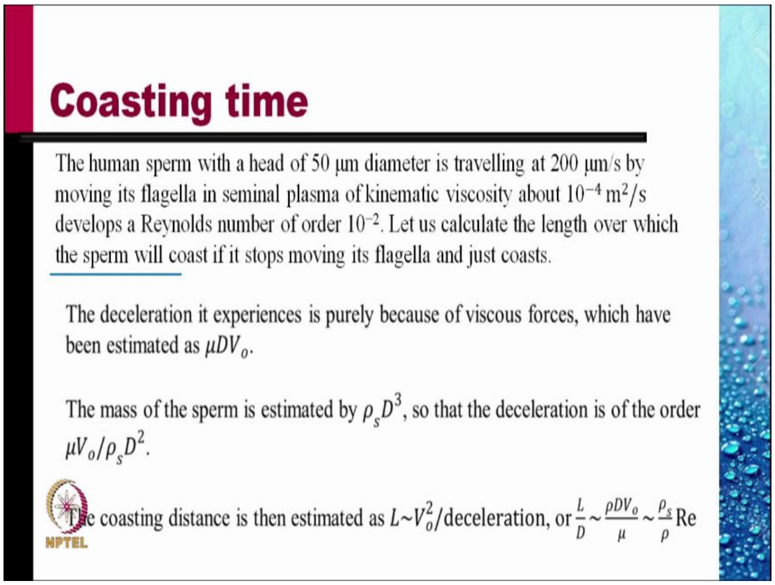

Let us do another example. The human sperm with a head of 50 μm diameter is traveling at 200 μm/s by moving its flagella in seminal plasma of kinematic viscosity about 10^{-4} m^2/s as discussed before. It develops a Reynolds number of the order of 10^{-2}.

So, it is a low Reynolds number flow indeed. Let us calculate the length over which the sperm will coast if it stops moving is flagella and just coasts. The deceleration it experience is purely because of viscous forces, which have been estimated as $\mu D V_o$. the mass of the sperm is estimated as $\rho_s D^3$. So, that the deceleration is of order $\mu V_o/\rho_s D^2$, and the coasting distance is then estimated as $V_o^2/$deceleration, or $\frac{L}{D}$, which is like $\frac{\rho D V_o}{\mu}$, is like $\frac{\rho_s}{\rho}$Re.

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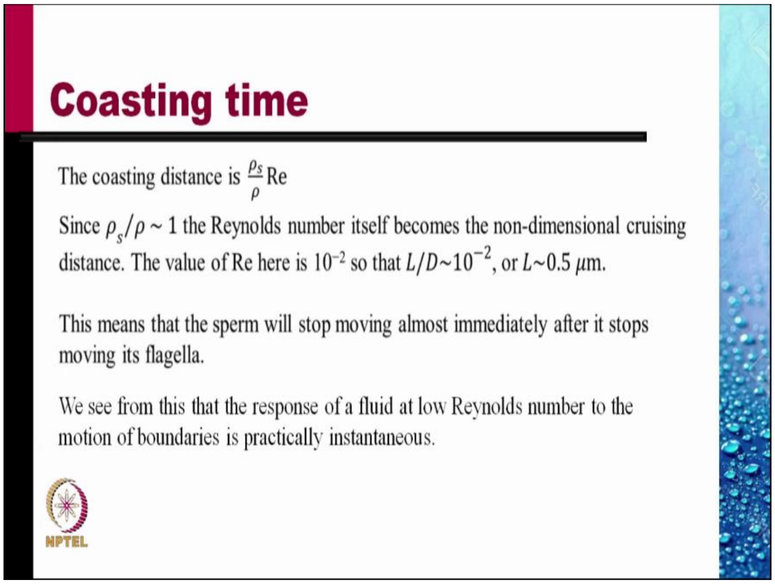

Since $\frac{\rho_s}{\rho}$ is like 1, the Reynolds number itself becomes the non-dimensional cruising distance. The value of the Reynolds number here is 10^{-2}. So, the value of L/D, the cruising distance, coasting distance divided by the diameter of the sperm is 10^{-2}, or the coasting distance is about 0.5 μm. This means that the sperm will stop moving almost immediately after it stops moving its flagella.

We see from this that the response of a fluid at low Reynolds number to the motion of boundaries is practically instantaneous.