Fluid Mechanics & its Application
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Lecture 21
Oscillating Boundary Problem

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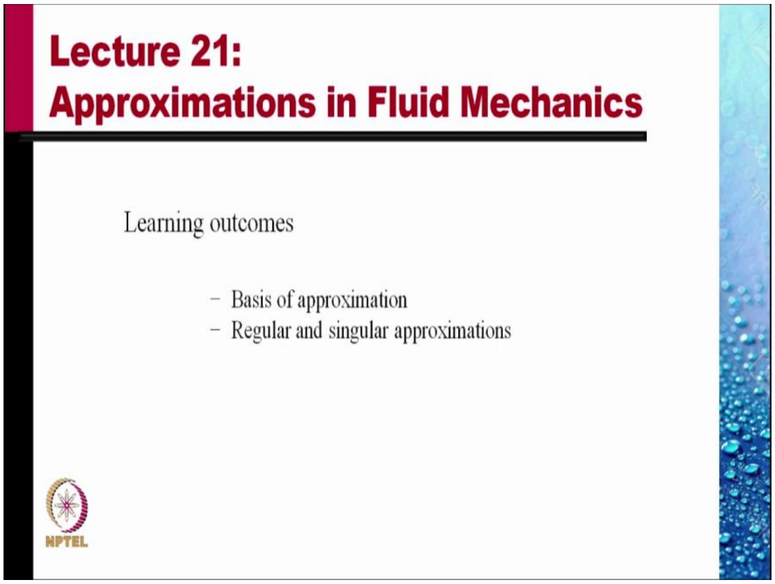

Welcome back.

In this lecture, we will cover the basis on which we make approximations in fluid mechanics.

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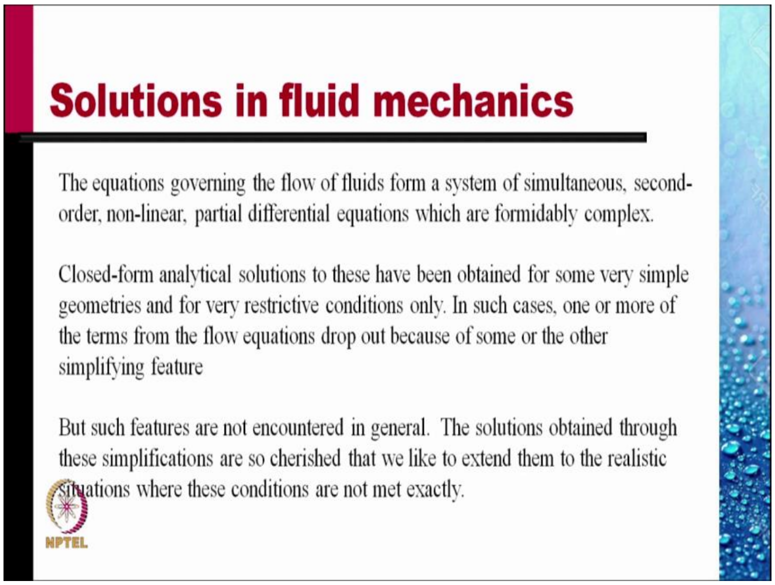

The equations governing the flow of fluids form a system of simultaneous second order non-linear partial differential equations which are formidably complex. Closed from analytical solutions to these have been obtained for some very simple geometries, and for very restrictive conditions.

In such cases, one or more of the terms from the flow equations drop out because of some or the other simplifying features. But such features are not encountered in general. The solutions obtained through these simplifications are so cherished, that we like to extend them to the realistic situations where these conditions are not met exactly, and so, we have to resort to approximations.

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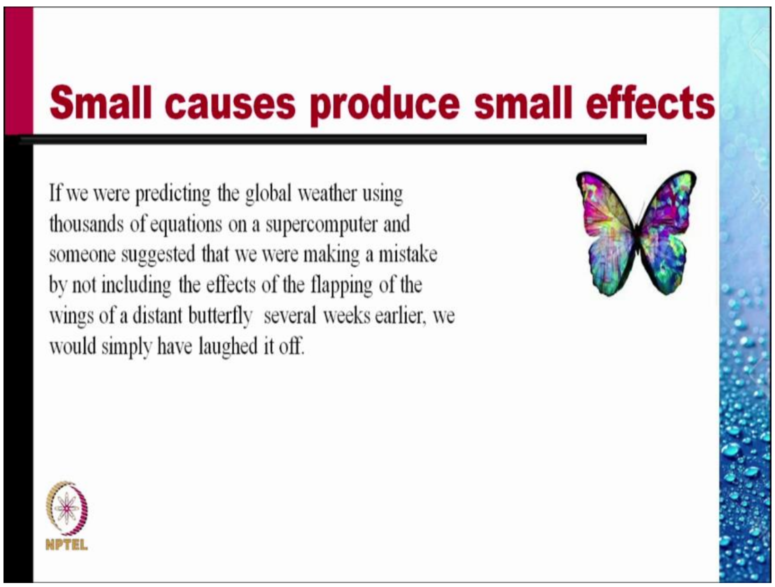

If we were predicting the global weather using thousands of equations on a supercomputer, and someone suggested that we are making a mistake by not including the effect of the flapping of the wings of a distant butterfly several weeks earlier, we would simply have laughed it off.

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But consider that in December 2019, a gentleman in a city of Wuhan took some bat soup and soon there were repercussions which are beyond imagination. The whole world got infected with COVID 19 virus. Up till now, 90 million people have been infected, 2 million have died. The medical facilities are under severe stress. The cities have been locked out, the schools and colleges shut down, and the economy is shattered.

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Could this butterfly flapping its wings result in a tornado a few months later?

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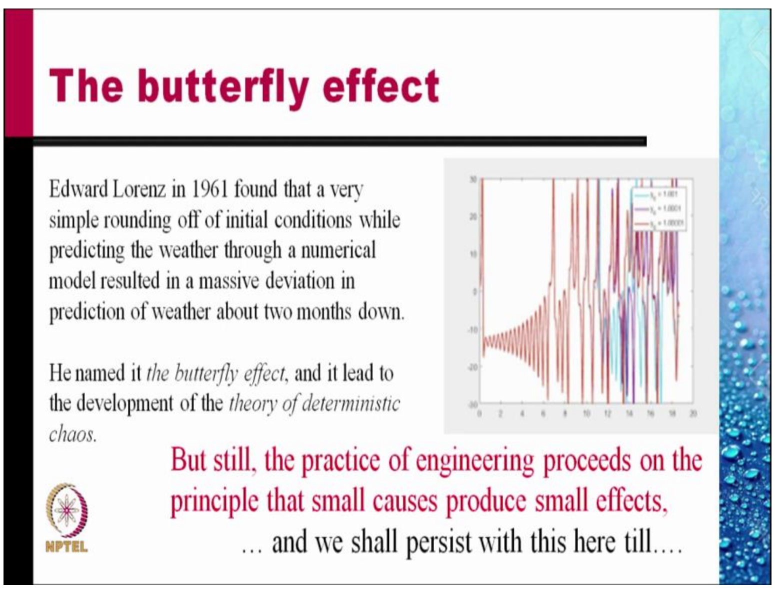

Edward Lorenz in 1961 found that a very simple rounding off of initial conditions while predicting the weather through a numerical model resulted in a massive deviation in prediction of weather about two months down. He named it the butterfly effect, and it led to the development of the theory of deterministic chaos. In this graph that we are showing, we started out with three solutions with boundary conditions slightly different, only slightly different. After a while the three solutions diverged.

But still, the practice of engineering proceeds on the principle that small causes produce small effects. So if we can show that a cause is small, the expected results are small. And so we can neglect that small cause, and we shall persist with this principle, till something goes wrong and we will have to reconsider.

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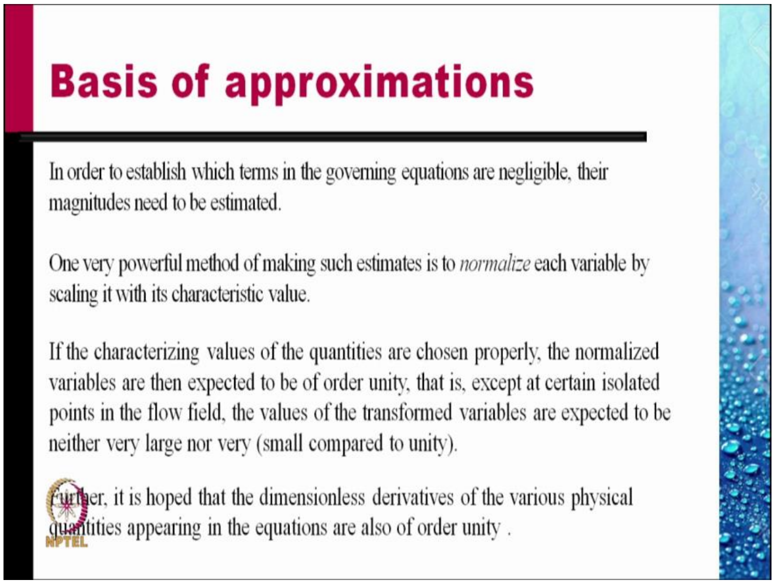

So, we need to determine which terms in the governing equations are negligible, and for this, we need to estimate the magnitudes of various terms, so that we can neglect terms which are small in magnitude compared to terms which are larger in magnitude. One very powerful method of making such estimate is to normalize each variable by scaling it with its characteristic value.

If the characterizing values of the quantities are chosen properly, the normalized variables are then expected to be of order unity, that is, except at certain isolated points in the flow field, the values of the transformed variables are expected to be neither very large nor very small compared to unity. Further, it is hoped that not only the variables, but the dimensionless derivatives of the various physical quantities appearing in the equations are also of order unity.

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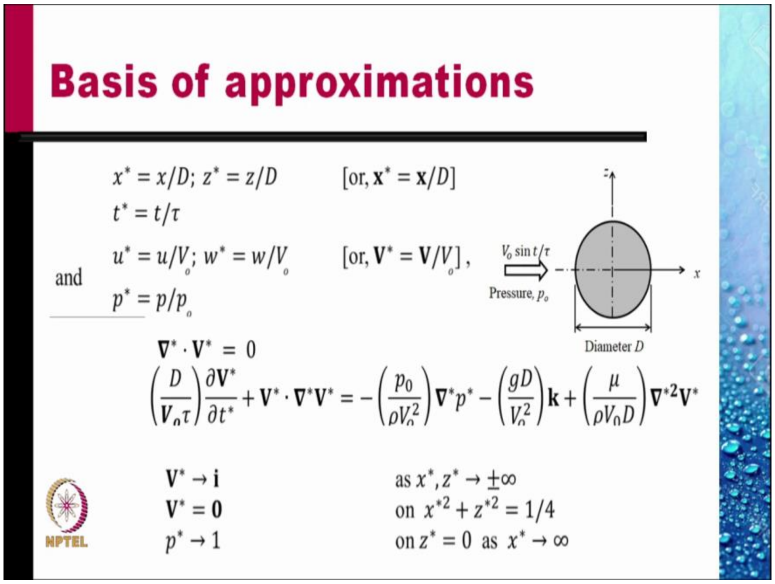

Let us consider a flow past a circular cylinder of diameter D. A fluid at the far away pressure p_o, and a far away velocity which is pulsating like $V_o \sin\frac{t}{\tau}$ flows past the cylinder. Following the procedure discussed earlier, we can normalize the various variables $x^* = x/D;\ z^* = z/D$, $t^* = t/t_c$, and the time period of pulsation tau could be a characteristic time. Also, $u^* = u/V_o;\ w^* = w/V_o$. We could non-dimensionalize pressures as $p^* = p/p_o$, the standard procedure. And if you do this, this is the form of the resulting continuity equation and the Navier Stokes equation, unsteady flow. And as before we see that the various parameters are now concentrated as non-dimensional coefficients of the various terms. And the boundary conditions are non-dimensionalized into these.

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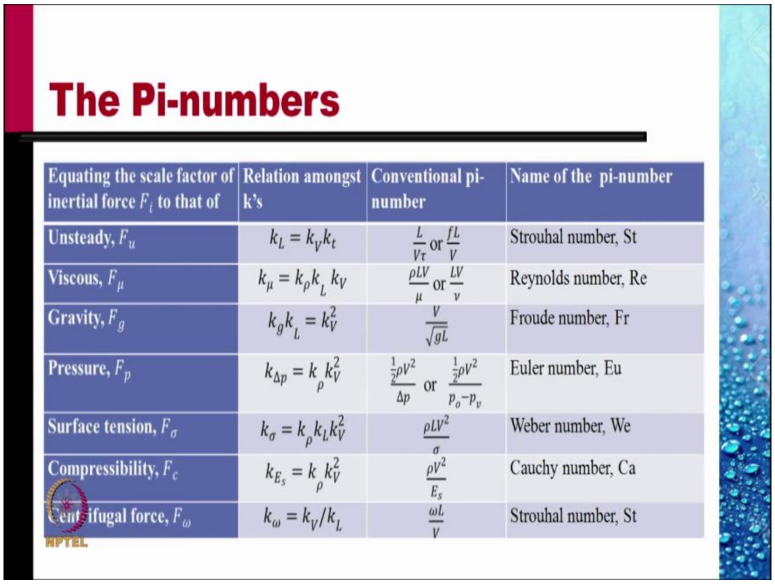

Equating the scale factor of inertial force F_i to that of	Relation amongst k's	Conventional pi-number	Name of the pi-number
Unsteady, F_u	$k_L = k_V k_t$	$\frac{L}{V\tau}$ or $\frac{fL}{V}$	Strouhal number, St
Viscous, F_μ	$k_\mu = k_\rho k_L k_V$	$\frac{\rho LV}{\mu}$ or $\frac{LV}{\nu}$	Reynolds number, Re
Gravity, F_g	$k_g k_L = k_V^2$	$\frac{V}{\sqrt{gL}}$	Froude number, Fr
Pressure, F_p	$k_{\Delta p} = k_\rho k_V^2$	$\frac{\frac{1}{2}\rho V^2}{\Delta p}$ or $\frac{\frac{1}{2}\rho V^2}{p_o - p_v}$	Euler number, Eu
Surface tension, F_σ	$k_\sigma = k_\rho k_L k_V^2$	$\frac{\rho L V^2}{\sigma}$	Weber number, We
Compressibility, F_c	$k_{E_s} = k_\rho k_V^2$	$\frac{\rho V^2}{E_s}$	Cauchy number, Ca
Centrifugal force, F_ω	$k_\omega = k_V / k_L$	$\frac{\omega L}{V}$	Strouhal number, St

The coefficient of the various terms which are nothing but non-dimensional groups of various parameters form the Pi numbers as we have discussed a few classes ago. The various Pi numbers like Strouhal number, Reynolds number, Froude number, Euler number, Weber number, Cauchy number, Strouhal number, again for centrifugal forces.

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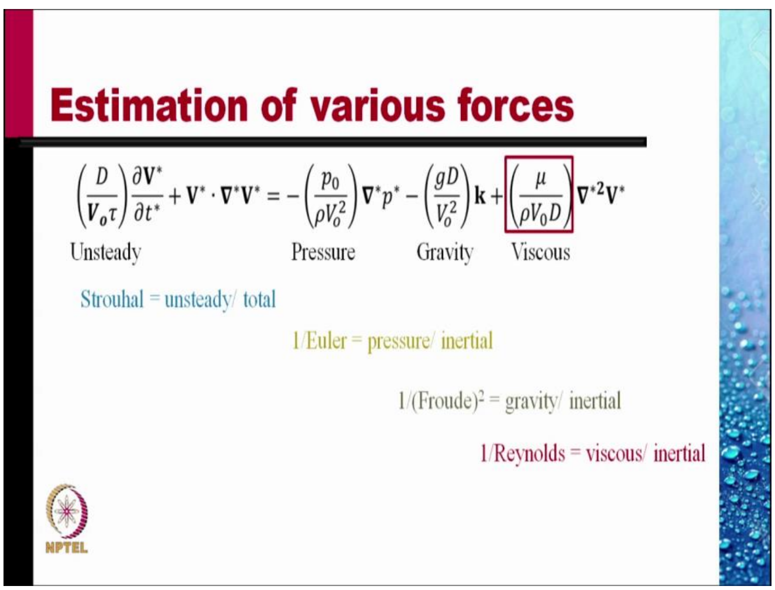

So, that this equation, the first term is the unsteady term, the second is the convective acceleration, and we have rendered the coefficient of this as unity. The first term on the right is the pressure term, the second the gravity, and the third the viscous term. Since the coefficient of the convective acceleration term is 1, the coefficients of the various other terms measure the relative magnitudes of those quantities with respect to the convective acceleration. We recognize $D/V_o\tau$ as the Strouhal number. In this we clearly see as the ratio unsteady forces to the total forces, the inertial forces, the convective inertial forces.

The second coefficient $p_o/\rho V_o^2$ is recognized as 1/Eu, and $p_o/\rho V_o^2$ would then be the ratio of the pressure forces to the inertial forces.

The third term is recognized as 1 over Froude number squared, because Froude number is $V_o/\sqrt{gD}$, and this is the ratio of gravity over the inertial forces.

The coefficient of the last term is 1 over Reynolds number and which is viscous over inertial forces, the ratio of the typical viscous over inertial forces. So, clearly, magnitudes of these coefficients tell us something about the importance of these terms in this equation. If any of these coefficient is much less than 1, then the term is negligible compared to the convective acceleration term.

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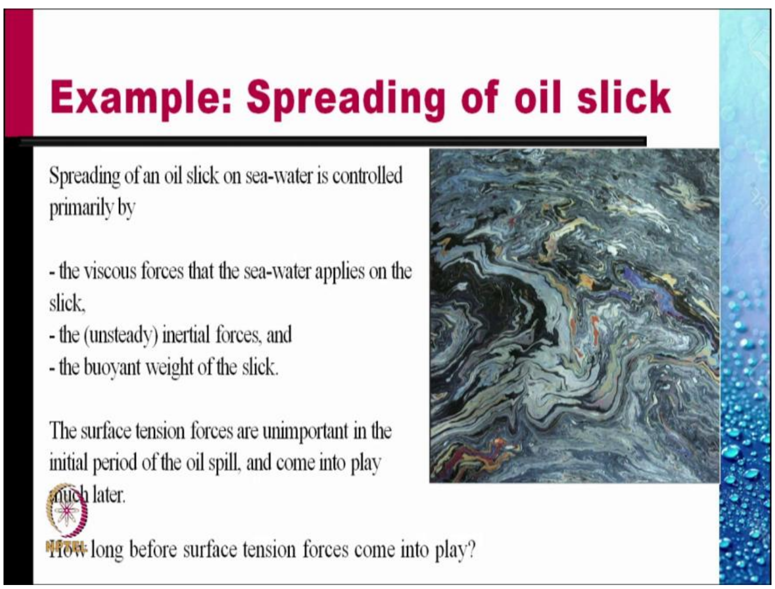

Let us, do an example of carrying out this analysis, and this is concerned with the spreading of oil slick, when an oil spills in oceans. The spreading of oil slick on seawater is controlled primarily by the viscous forces that the seawater applies on this slick. The unsteady inertial forces, and the buoyant weight of the slick, that is the weight of the slick minus the buoyancy force.

This surface tension forces are unimportant in the initial period of the oil spill, and come into play much later. Can we estimate the time that it takes for the surface tension forces to come into play? Clearly, the approach would be that we would estimate the surface tension forces, and see if it contains any time parameter.

Then as long as the coefficient of the surface tension force in the non-dimensional equation, that is, an estimate of surface tension forces with respect to the inertial forces, is much less than 1, then the surface tension forces are negligible. But they would come into play when that coefficient approaches the value 1.

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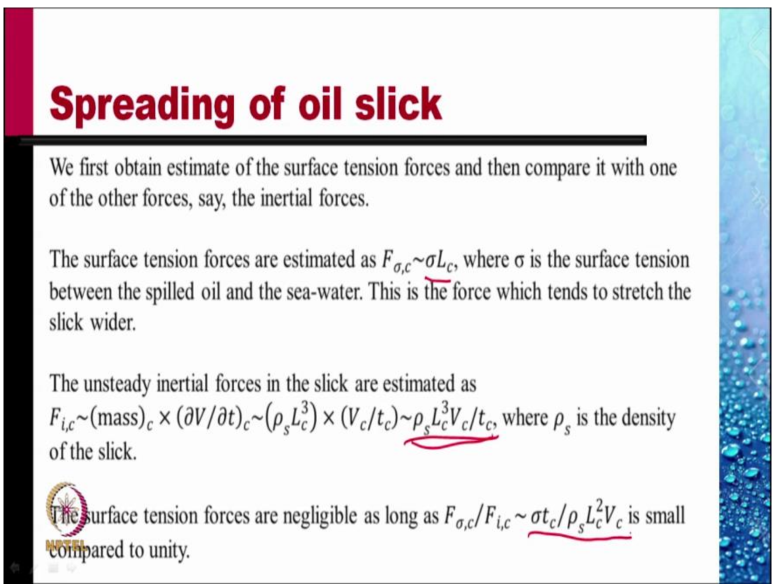

So, we estimate these forces. The surface tension forces are estimated as $F_{\sigma,c} \sim \sigma L_c$, where L_c is a characteristic length, and σ is the surface tension between the spilled oil and the sea water. This is a force which tends to stretch the slick wider. The unsteady inertial forces in the slick, which oppose that stretching, are estimated as, as before, $F_{i,c} \sim (\text{mass})_c \times (\partial V/\partial t)_c$.

And the mass is estimated as $(\rho_s L_c^3)$, where ρ_s is the density of the slick, the unsteady acceleration, which would be like (V_c/t_c) and so, this is an estimate of the inertial forces. So, surface tension forces are negligible as long as this over this, is small compared to 1, which is here. This contains t_c, and as the time increases, this parameter increases. Let us, estimate for what time would this become of order 1.

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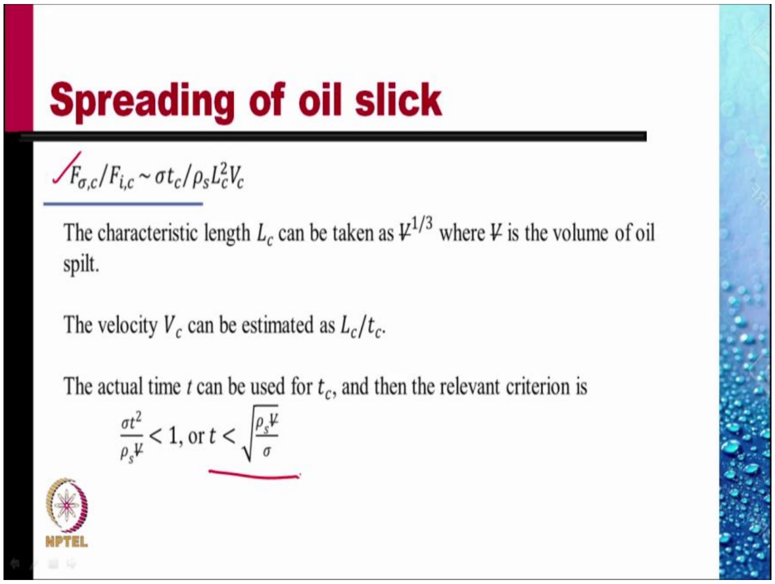

That is the ratio that we obtain. The characteristic length L_c can be taken as the cube root of the volume of the oil that is spilt. The velocity L_c can be estimated as L_c/t_c. The actual time t can be used for the characteristic time, and the relevant criteria then is, that this is negligible as long as $t < \sqrt{\frac{\rho_s V}{\sigma}}$.

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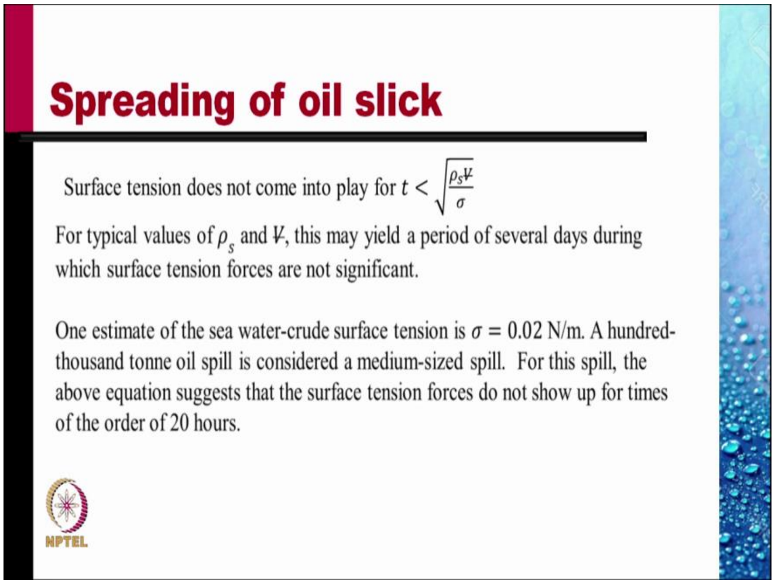

So, surface tension does not come into play for $t < \sqrt{\frac{\rho_S V}{\sigma}}$. For typical values of $\rho_S$and V, this may yield a period of several days during which surface tension force is not significant. One estimate of sea water crude surface tension is 0.02 N/m. A 100,000 tonne oil spill, which is considered a medium size spill, the above equation suggest that the surfaces tension forces do not show up for times of the order of 20 hours. So, for the initial 20 hours the surface tension forces can be neglected. It is only after that the surface tension spreads the oil on water.

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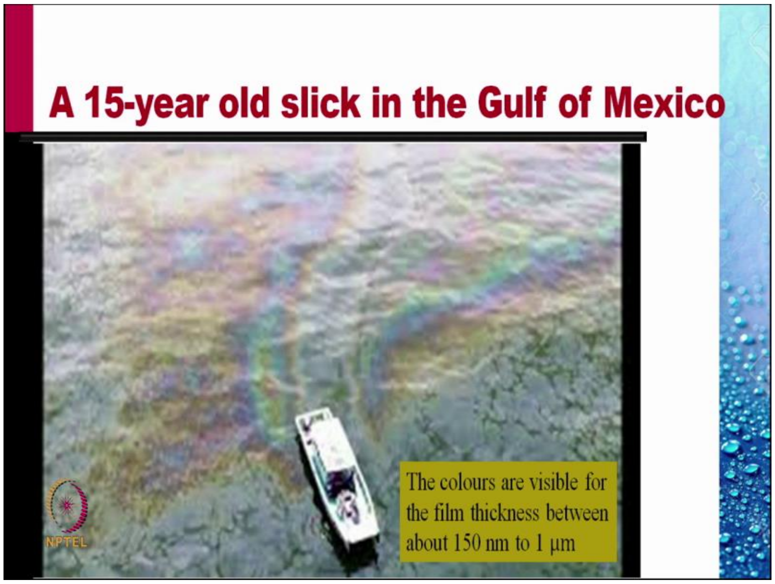

This picture shows a 15 year old slick in the Gulf of Mexico. The colours are visible for the film thickness between 150 nm and 1 μm. So, the oil has thinned out into a film so thin.

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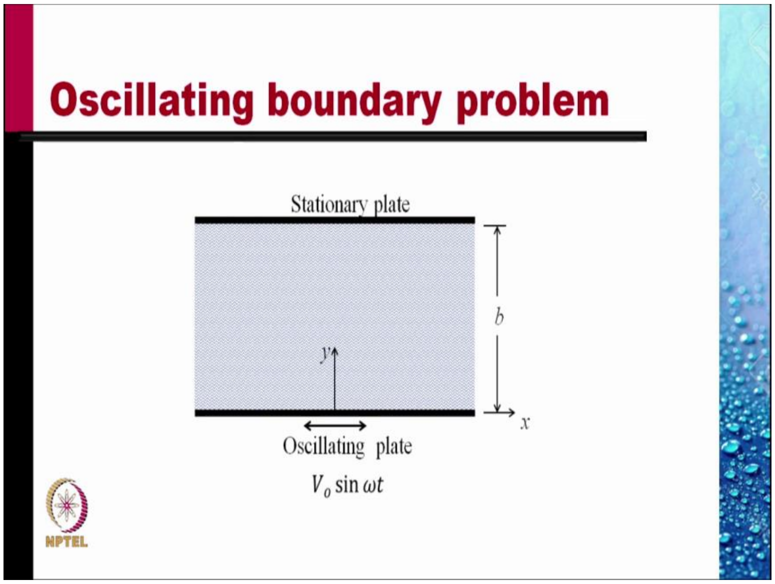

Let us, illustrate this concept further by taking a problem in which a fluid is confined between two plates. The upper plate is stationary, but the lower plate is oscillating at a velocity of $V_o \sin \omega t$. The gap between the plates is b.

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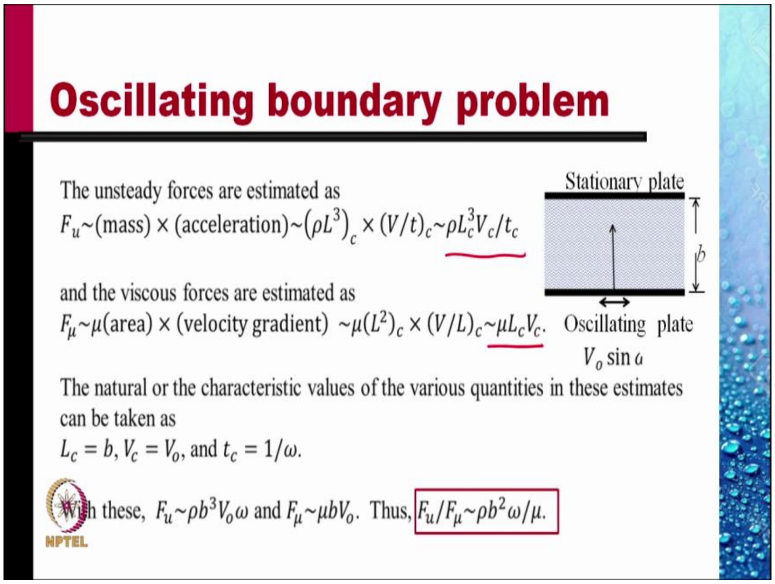

Let us estimate the forces. The unsteady forces, which are like mass times acceleration, would using a usual procedure result in an estimate of this with a subscript c denotes that these are the characteristic quantities. The viscous forces are estimated as mu times the area times the velocity gradient, and as before, it comes out to be $\mu L_c V_c$.

The natural or the characteristic values of the various quantities are: L_c can be taken equal to b, V_c can be taken is equal to V_o, the amplitude of the velocity fluctuations, and the characteristic time can be taken as 1/ω. If we do this, the ratio of the unsteady forces to the viscous forces is estimated as $\rho b^2 \omega/\mu$.

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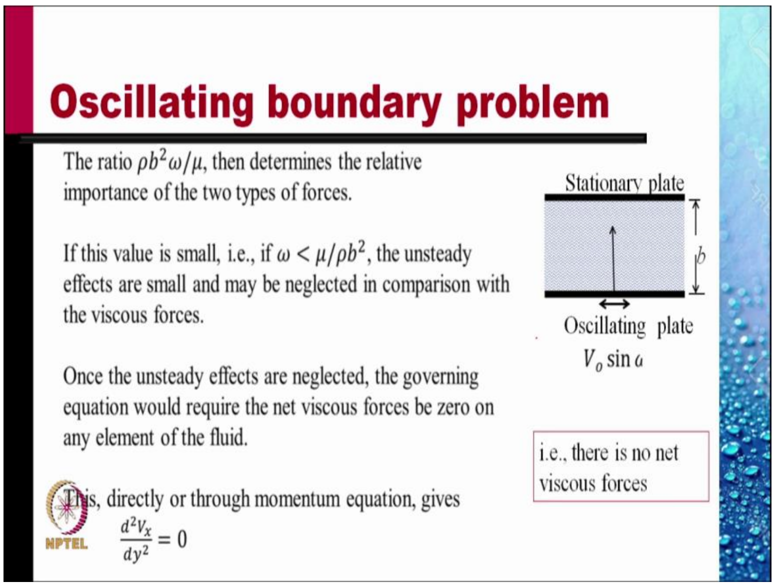

The ratio $\rho b^2 \omega/\mu$ then determines the relative importance of the two types of forces. If this value is small, that is, $\omega < \mu/\rho b^2$, the unsteady effects are small and maybe neglected in comparison with the viscous forces. Once the unsteady effects are neglected, the governing equation would require that the net viscous forces be zero on this element.

Because the governing equations is a balance of unsteady forces plus the inertial forces equal to the pressure forces plus the viscous forces. Because of the geometry, the pressure forces are 0, because the flow does not change in the x direction, the inertial forces are 0. So, the governing equation just balances the unsteady forces with the viscous forces.

And if the unsteady forces are 0 that means, we can neglect the left hand side, then the only term remaining the equation would be the viscous term is equal to 0, the net viscous force should be 0, $\frac{d^2 V_x}{dy^2} = 0$. This is the solution that we obtained for the Couette flow. There is no net viscous force.

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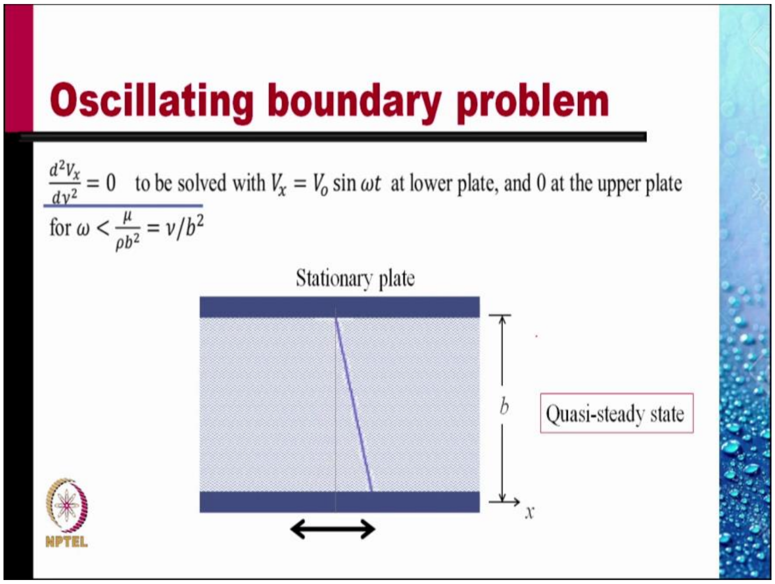

This equation is to be solved with V_x is equal to $V_o \sin \omega t$ at the lower plate, and 0 at the upper plate. For $\omega < \mu/\rho b^2$, which is also ν/b^2, where ν is the kinematic viscosity. The solution looks like this. The solution is a straight line, but the boundary condition at the plate changes with the time, $V_o \sin \omega t$, that is why it is a straight line profile which is swinging between two extremes, $-V_o$ to $+V_o$. At any instant it is a straight line. This is a quasi-steady state solution.

At any given moment, we can neglect the inertial forces within the fluid. We can treat the fluid to be steady at any moment, and solve for the instantaneous boundary conditions. But, those instantaneous boundary conditions are changing with the time. They are changing slowly enough for the setup. So that at any instant, the profile is as if it is a steady flow, with the instantaneous velocity of the lower plate as the velocity of the lower plate, and 0 as the velocity of the upper plate. A very interesting solution.

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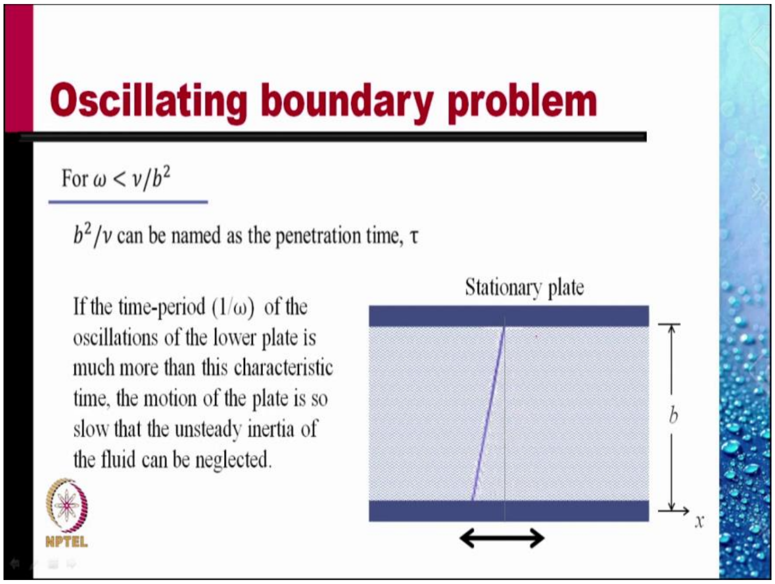

This is all for $\omega < \nu/b^2$. The rate of change of velocity is slow compared to ν/b^2. b^2/ν, the reciprocal of that, can be named as the penetration time τ. This is the time it takes for the effect of the motion of the lower plate, or change in velocity of the lower plate to penetrate up till the upper plate. More the viscosity, lesser is the penetration time. In a very viscous flow the penetration is almost instantaneous.

Now, if omega is much less 1 over this penetration time, then at any instant the effect of the change of velocity of the lower plate penetrates almost instantaneously through the upper plate, and that is why we get a quasi-steady flow. If the time period 1/ω of the oscillations of the lower plate is much more than the characteristic time, the motion of the plate is so slow that the unsteady inertia of the fluid can be neglected. This is what we showed in our discussion.

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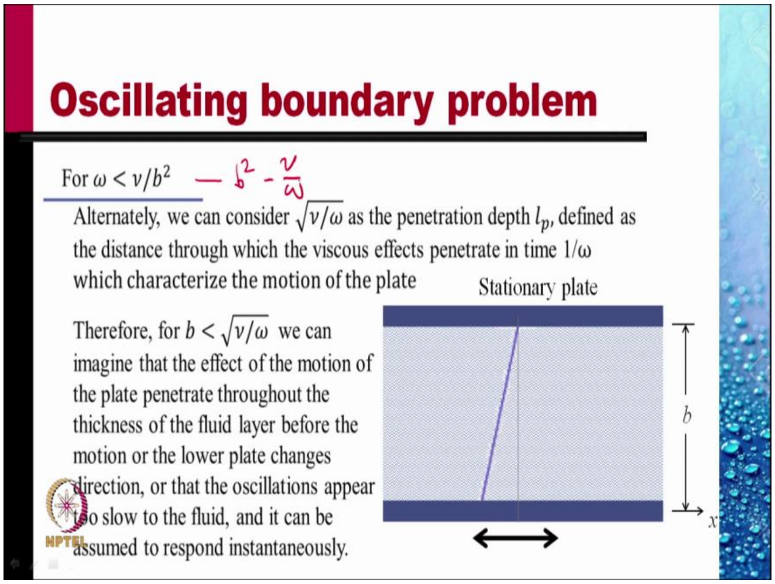

Alternately, we can consider ν/ω. From this we get $b^2 = \nu/\omega$, or $b = \sqrt{\nu/\omega}$. We can consider $\sqrt{\nu/\omega}$ as the penetration depth L_p, p for penetration, defined as a distance through which the viscous effects penetrate in time 1/ω. 1/ω is the time period of the fluctuations, or related to the time period of fluctuations which characterize the motion of the lower plate.

Therefore, for b much less $\sqrt{\nu/\omega}$ or the gap between the plate less than the penetration depth, we can imagine that the effect of the motion of the plate penetrate throughout the thickness of the fluid layer before the motion of the lower plate changes direction, or that the oscillation appears too slow to the fluid, and it can be assumed to respond instantaneously. Same thing we said in the last slide, repeated in different words.