Fluid Mechanics and its Applications
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Indian Institute of Technology, Delhi
Lecture – 17
Similitude (continued)

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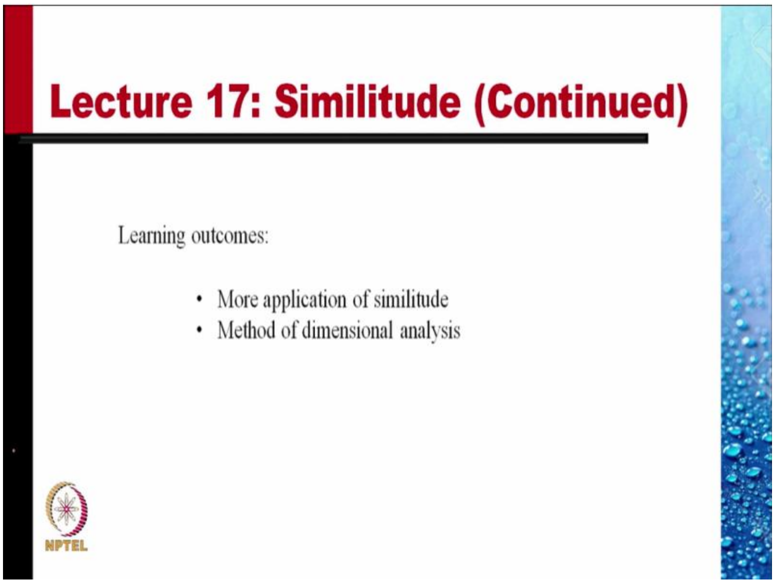

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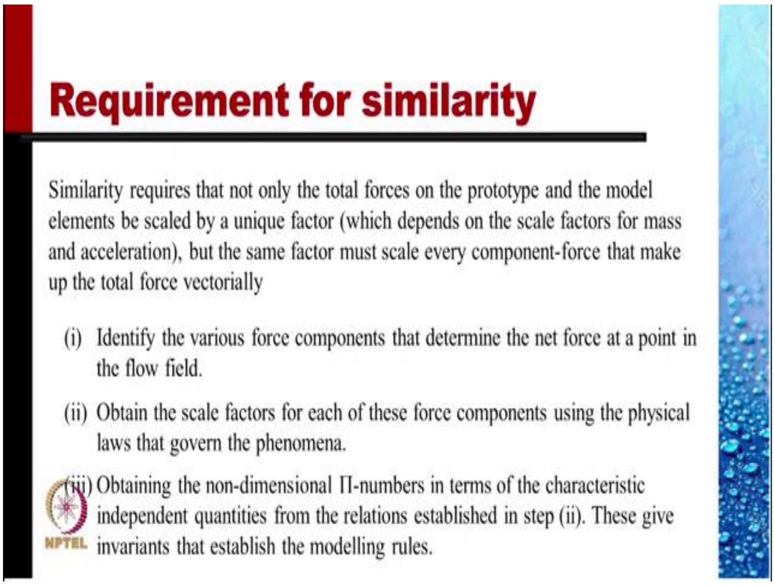

Welcome, back.

In the last couple of lectures we have discussed that similarity requires that not only the total forces on the prototype and the model elements need to be scaled by a unique factor, which depends on the scale factors for the mass and acceleration, but the same factor must scale every component force that make up the total force vectorially.

So, for this we identify the various force components that determine the net force at a point in the flow field. Obtain the scale factors for each of these force components using the physical laws that govern the phenomena. And obtain the non-dimensional pi numbers in terms of the characteristic independent quantities from the relations established in step above. These give the invariants that establish the modelling rules.

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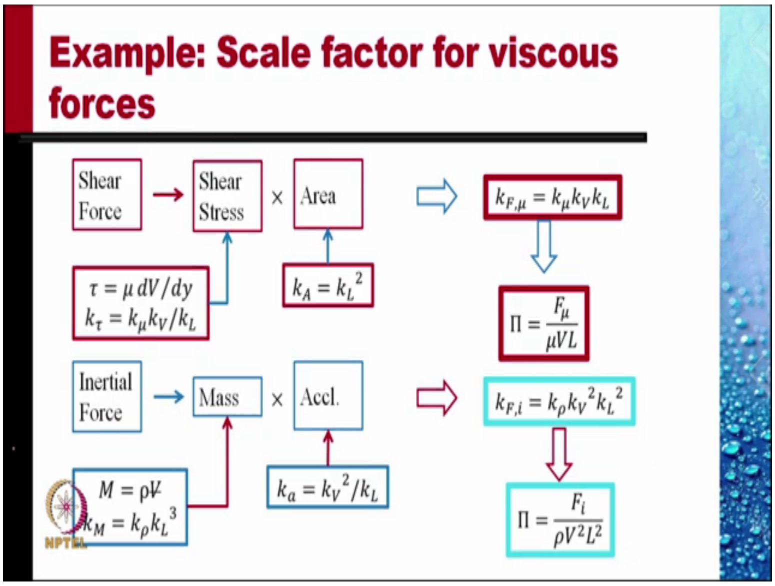

For example, let us look at the viscous forces. The shear force is shear stress into area, the shear stress $\tau = \mu\, dV/dy$, and so k_τ, where τ stands for shear stress is equal to $k_\mu k_V / k_L$. The area scale factor k_A is $k_L{}^2$, as was shown in the last class, and therefore, k for the viscous force is equal to $k_\mu k_V k_L$. This is the scale factor for the viscous forces.

Similarly, we do for the inertial force. Inertia force is mass times acceleration, the mass is density times the volume, so $k_M = k_\rho k_L{}^3$. Since the volume is like length cubed, and the acceleration, from the convective acceleration, we obtained $k_a = k_V{}^2 / k_L$. This was all done in the last class, from this we obtain $k_{F,i} = k_\rho k_V{}^2 k_L{}^2$. And from this I get the pi number $\frac{F_i}{\rho V^2 L^2}$.

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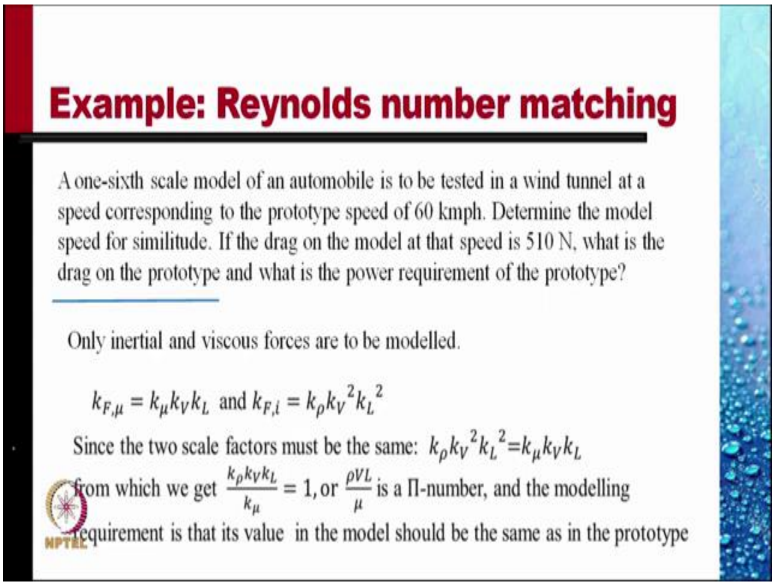

Now, the two scale factors must be equal. We will apply this to an example. A one-sixth scale model of an automobile is to be tested in a wind tunnel at a speed corresponding to the prototype speed of 60 kilometres per hour. Determine the model speed required for similitude. If the drag on the model at that speed is 510 N, what is the drag expected on the prototype, and what is the power requirement of the prototype?

In this case of an automobile moving on the road or as tested in a wind tunnel, the only forces of importance are the inertial and the viscous forces, and they need to be modelled. As you have just seen $k_{F,\mu}$, the viscous for scale factor is $k_\mu k_V k_L$ and the inertial force scale factor is $k_\rho {k_V}^2 {k_L}^2$, just obtained. The two scale factors must be same and from this we get $\frac{k_\rho k_V k_L}{k_\mu} = 1$, or from this we can obtain $\frac{\rho V L}{\mu}$ as a pi number. The modelling requirement is that the value of this pi number for the model should be the same as for the prototype.

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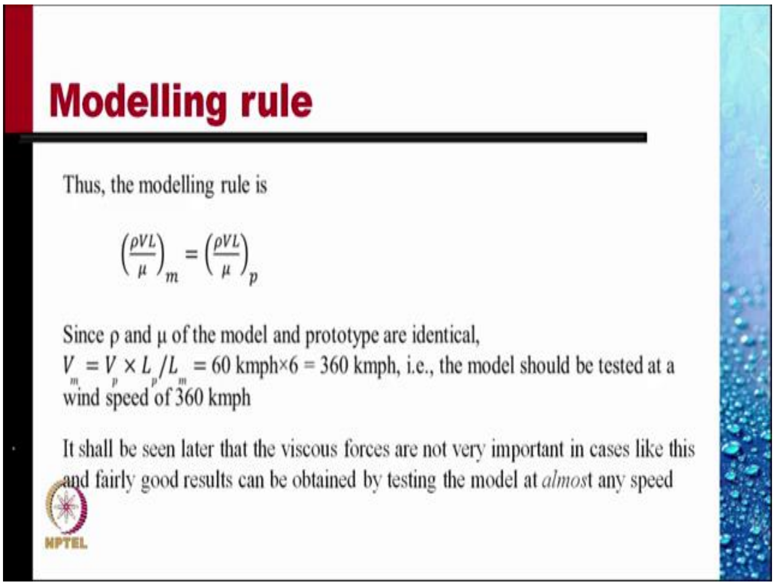

Thus, we equate the value of this pi number for the model and the prototype to obtain the velocity for the model. ρ and μ for the model and prototype are identical. And so, V for the model is V for the prototype times L of the prototype divided by Lm. L for the prototype divided by Lm is the length scale factor, which was given as 6. And since the prototype speed is 60 kmph, the model speed should be 360 kmph. So, model should be tested at a wind speed of 360 kilometres per hour, a rather high speed.

It shall be seen later that the viscous forces are not very important in these cases, when the Reynolds number are very high, and the body is blunt or bluff, as an automobile is typically. The viscous forces are not very important for drag modelling. And fairly good results can be obtained by testing the model at almost any speed. Anyway, we will come to that a little later.

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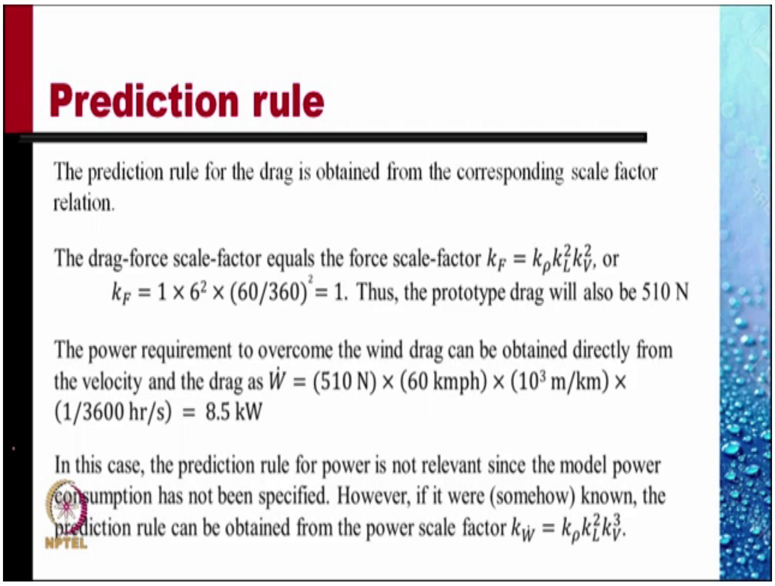

For the time being let us predict the drag. The prediction rule for the drag is obtained from the corresponding scale factor relation. The scale factor for drag would be same as scale factor of any other force, and as we have seen for the initial force the scale factor is $k_\rho {k_V}^2 {k_L}^2$, so the scale factor of force is $k_F = 1 \times 62 \times (60/360)^2 = 1$,. Thus, the prototype drag will also be 510 N, the drag on the model, because the scale factor of forces is 1.

The power requirement to overcome the wind drag can be obtained directly from the velocity and drag as power is the drag, 510 N, into velocity 60 kmph, and converting kmph to m/s, we get the power requirement as 8.5 kW. In this case, the prediction rule for power is not relevant, since the model power consumption has not been specified. However, if it were somehow known, the prediction rule can be obtained from the power scale factor $k_{\dot{W}} = k_\rho k_L^2 k_V^3$.

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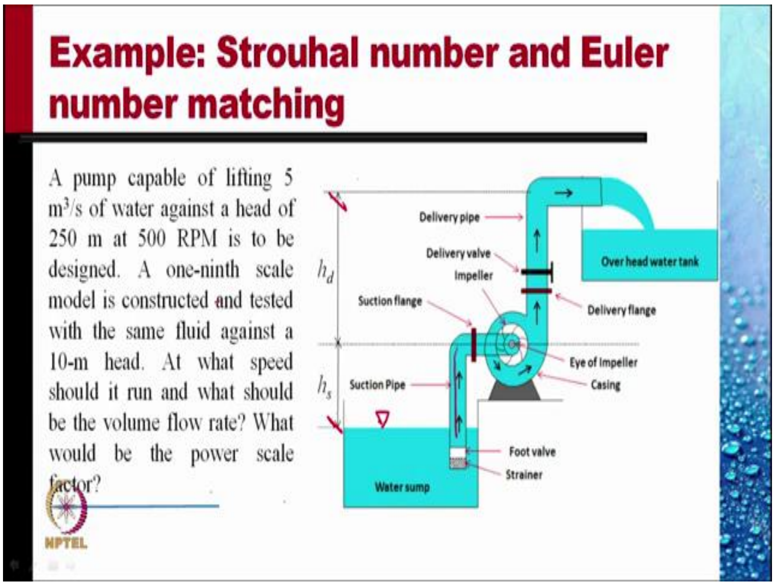

Let us do another example where the Strouhal number and Euler number matching is to be done. This is related to a pump, which is taking water from a water sump and pumping it to an overhead water tank. The pressure through this pipe is sub-atmospheric, because pressure here is atmospheric, so pressure through this pipe is sub-atmospheric. There is a jump in pressure at the pump, and then the pressure reduces further, and at the exit the pressure is atmospheric again.

A pump capable of lifting 55 m^3/s of water against the head of 250 m, that is this head, total head from here to here, is 250 m, and the pump is running at 500 rpm. Before designing this pump, we need to make a model and test it. A one-ninth scale model, k_L is equal to 9, length of the prototype divided by the length in the model is 9.

A one-ninth scale model is constructed and tested with the same fluid against a 10-m head, so it is a 250-m head, we deliver against a 10-m head. At what speed should it run, and what should be the volume flow rate, if there has to be similarity? What would be the power scale factor? These are the questions that need to be answered.

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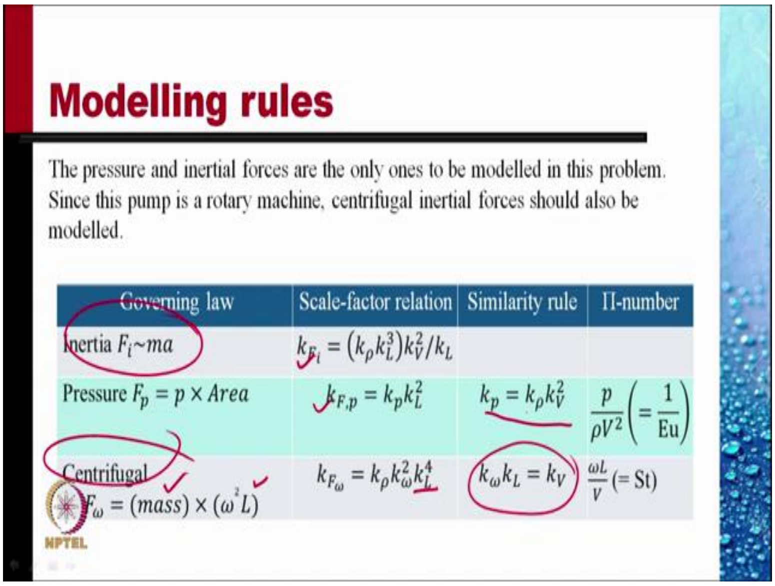

The pressure and the inertial forces are the only forces that one need to model in this problem. Since the pump is a rotary machine, centrifugal inertial forces should also be modelled. We organize our calculation in this manner. The governing law for inertia is inertial force is equal to *ma*, and we plug in the scale factors of mass and acceleration to obtain the scale factor of our initial force as $(k_\rho k_L^3)$ for mass, and for acceleration k_V^2/k_L. The pressure forces would have pressure times the area.

So, the scale factor of pressure force would be k_p, the scale factor of pressure, times that of area which is k_L^2. Matching these two, this should be equal to this, and that gives us $k_p = k_\rho k_V^2$. From this we get the pi number $\frac{p}{\rho V^2}$, which was shown to be 1 over Euler number $\frac{1}{\mathrm{Eu}}$ in the last class.

The centrifugal force $F_\omega = (mass) \times (\omega^2 r)$, and so the scale factor for the centrifugal force is $k_\rho k_\omega^2 k_L^4$, 3 L's from mass and 1 L here. So k_L is to power 4. From this, and this, we obtain similarity rule $k_\omega k_L = k_V$, which gives us $\frac{\omega L}{V}$, which is named Strouhal number, and denoted by St, Strouhal number.

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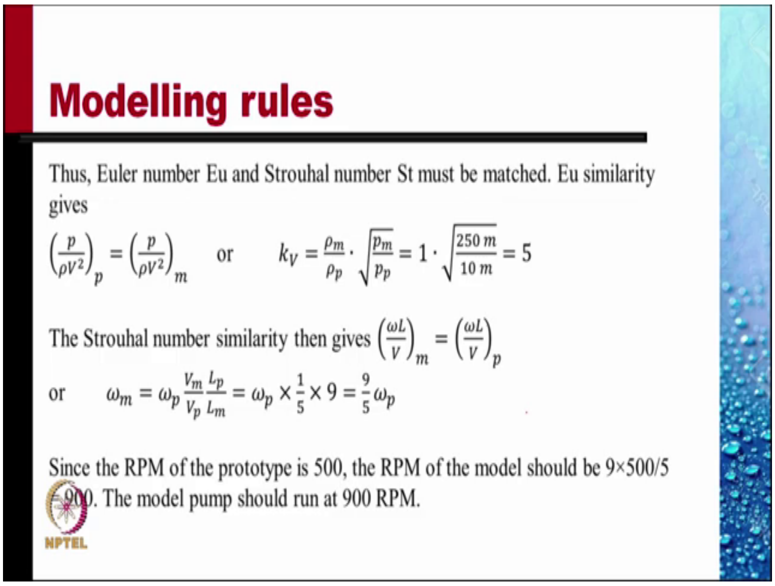

Thus, the similarity requires that the Euler number and Strouhal number must match. The Euler number similarity gives p divided by rho V squared for the prototype should be same as that for the model, and from this we obtain kV is equal to rho m by rho p into under root p m by p p. Rho m by rho p is 1, since we are using the same fluid, so the density is same. And the pressure developed in model and pressure developed in prototype are 250 meters divided by 10 meters. So, this gives you a scale factor for velocity to be 5.

Then we use Strouhal number similarity, and from this, we get $\frac{\omega L}{V}$ for the model is same as that for the prototype, and so, ω for the model is obtained as 9 by 5 of the value for the prototype. Since the rpm the prototype is 500, the rpm of the model should be 9 into 500 divided by 5, that is 900. The model pump should run at 900 rpm as against the prototype pump which is to run only at 500 rpm.

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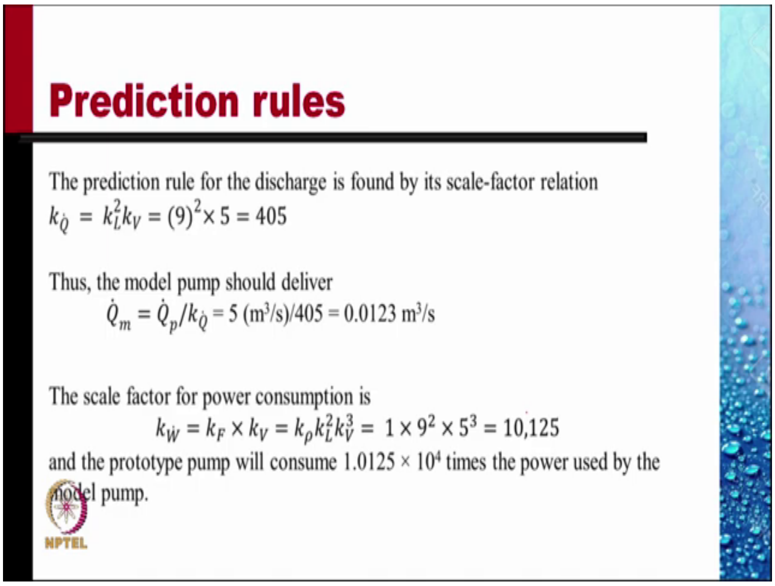

The prediction rule for the discharge is now obtained by the scale factor for discharge. The discharge is like velocity times the area of the pipe, and so $k_{\dot{Q}}$ would be k_L^2 for the area in k_V for the velocity of the fluid through the pipe, k_L is 9, and k_V has been determined to be 5. So, the scale factor for the discharge is 405. The prototype pump would discharge 405 times the discharge of the model pump, and since the prototype pump is to discharge at 5 m^3/s, the discharge in the model would be measly 0.0123 m^3/s.

The scale factor for power consumption is $k_{\dot{W}}$, the power is force times the velocity, so $k_{\dot{W}}$ is equal to $k_F \times k_V$, and we use $k_\rho k_L^2 k_V^2$ for k_F, the inertial force factor. All forces are same. So, this gives you a power scale factor of 10,125. So, the prototype will consume more than 10,000 times the power used by the model pump.

I hope you appreciate the ease with which you can make the prediction rules based on simple relations of scale factors.

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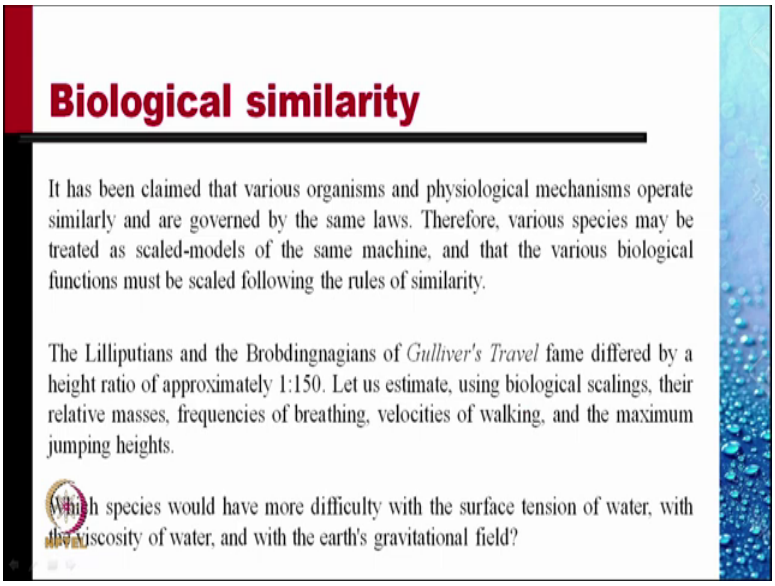

Let us do another example, an interesting example of biological similarity. It has been claimed that various organisms and physiological mechanisms operate similarly, and are governed by the same laws in different species. Therefore, various species may be treated as scale models of the same machine. And that the various biological functions must be scaled following the rules of similarity as developed here.

An example: Let us consider the Lilliputians and the Brobdingnagians of the Gulliver's Travel fame who differed in the height ratio by approximately 1 is to 150, actually the ratio was 1 is to 144, because it was mentioned in Gulliver's Travel that an inch of Gulliver was a foot for the Lilliputians, and a foot of Gulliver was an inch for Brobdingnagians.

Let us estimate using biological scalings, their relative masses, frequencies of breathing, velocity of walking, and the maximum jumping heights. Which species would have more difficulty with the surface tension of water, with the viscosity of water, and with the earth's gravitational field?

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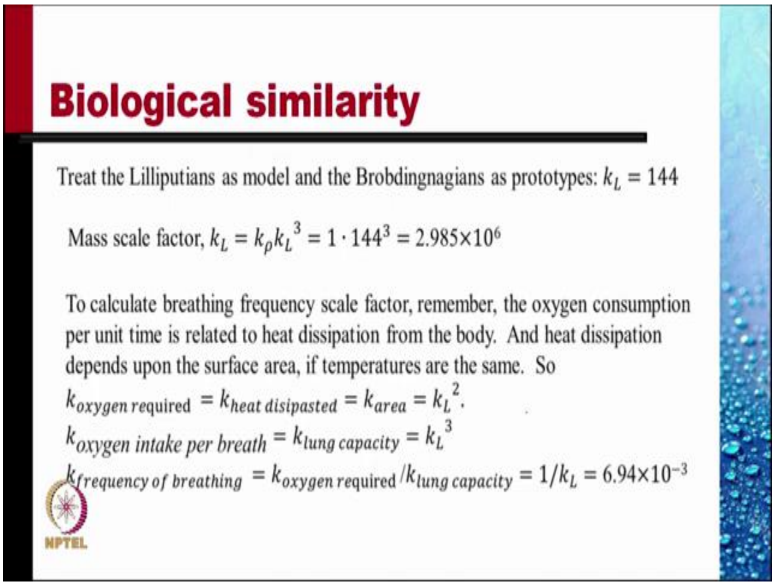

So, let us treat Lilliputians as the model, the smaller one, and Brobdingnagians as prototypes. The scale factor is taken as 144. The mass scale factor is simple $k_\rho {k_L}^3$, and so is $1 \cdot 144^3 =$ 2.985×10^6. So, giants would weigh about three million times the weight of the Lilliputians.

To calculate the breathing frequency, remember that the oxygen consumption per unit time is related to heat dissipation from the body and the heat dissipation from the body depends upon the surface area of the body, if the temperatures are the same.

So, the oxygen required, the scale factor of oxygen required, would be the same as the scale factor for the heat dissipated, which would be like the scale factor for area and so this will be ${k_L}^2$. The oxygen intake per breath would be the scale factor for lung capacity, and which would be ${k_L}^3$, the volume.

So, the frequency of breathing, the scale factor for frequency of breathing, would be the scale factor for the oxygen required divided by the scale factor for lung capacity, the volume breathed per breath. And so that turns out to $1/k_L$. So, it would be 6.94×10^{-3}. The giants would breathe much slower, order of magnitude, two orders of magnitude, three orders of magnitude slower, than the Lilliputians.

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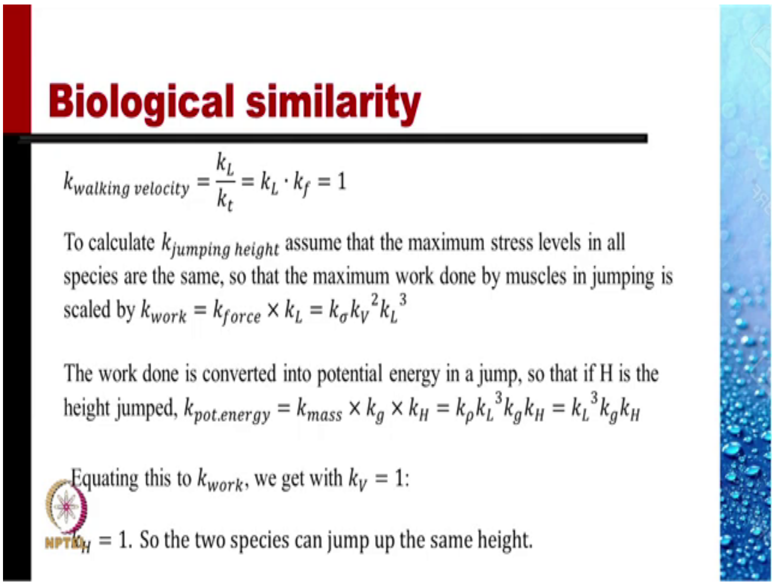

Walking velocity $\frac{k_L}{k_t}$, velocity is like length scale factor divided by time scale factor, and that is $k_L \cdot k_f$, and we are just seeing that the frequency scale factor is like $1/k_L$. And so, the walking scale factor would be 1. The two species would walk at about the same speed, one taking longer strides but with far lower frequencies.

To calculate the jumping height, we use a trick. We assume that the maximum stress level in all species are the same, maximum stress level in the muscles. These muscles do work, they exert force, and we assume that the maximum stress levels in all muscles, in all species, is the same. So, that the maximum work done by the muscles in jumping is scaled by k_{work} is $k_{force} \times k_L$ and k_{force} is like $k_\sigma {k_L}^2$, so that k_{work} is like $k_\sigma {k_L}^3$. k_σ is 1, stress level being same, so the work done scale factor is like k L cubed.

This work done is converted into a potential energy in a jump, and if capital H is the height jumped, $k_{pot.\,energy}$ is equal to $k_{mass} \times k_g \times k_H$.. And for k_{mass}, we put $k_\rho {k_L}^3$. k_g is equal to 1, this is ${k_L}^3 k_g k_H$. Equating this to k_{work}, we get k_H equal to 1, because k_g is also 1. So, k_H would be 1, that is the two species can jump up the same height.

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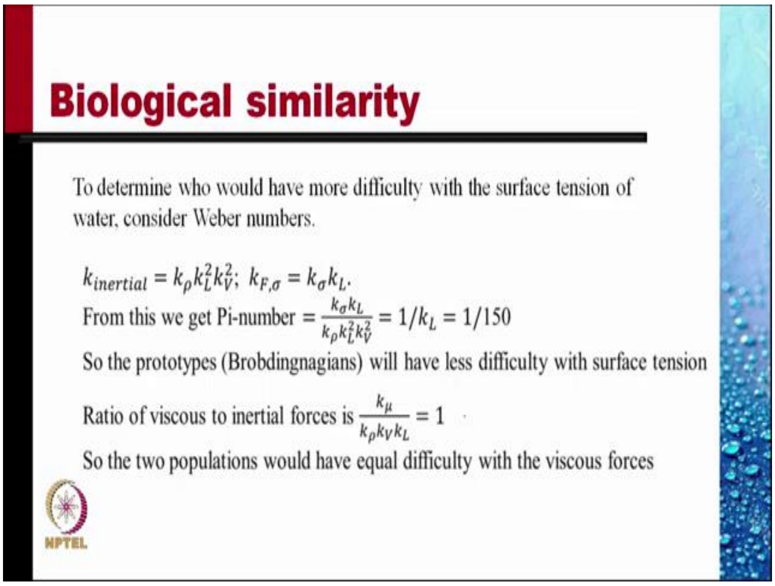

To determine who would have more difficulty with the surface tension of water, consider the Weber number. Weber number gives the relation between the inertial forces and the surface tension forces. $k_{inertial}$, we have established a number of times is equal to $k_\rho k_L^2 k_V^2$, and the surface tension force $k_{F,\sigma} = k_\sigma k_L$, because surface tension force is sigma times the length.

From this we get the pi number, $\frac{k_\sigma k_L}{k_\rho k_L^2 k_V^2}$ and which is $\frac{1}{k_L}$, k_σ being 1, same water, k_ρ being 1, so we have this scale factor 1/144. So, the prototypes, which is Brobdingnagians, will have less difficulty with surface tension than would the Lilliputians.

Similarly, the ratio of the viscous to inertial forces is like 1 over Reynolds number, and so $\frac{k_\mu}{k_\rho k_V k_L} = 1$ should be 1. So the two populations would have equal difficulty with the viscous forces.

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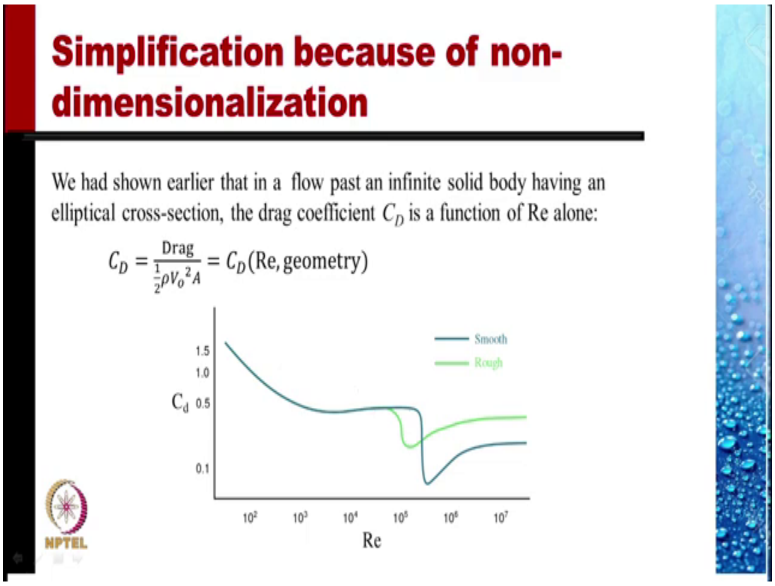

We had earlier shown that the flow past an infinite solid body having an elliptical cross section, the drag coefficient C_D is a function of Reynolds number alone. We define drag coefficient as $\frac{\text{Drag}}{\frac{1}{2}\rho {V_o}^2 A}$. And this was a function of Reynolds number and geometry. $\frac{1}{2}\rho {V_o}^2$ is like the characteristic pressure force times area would give you a force. So, drag divided by $\frac{1}{2}\rho {V_o}^2 A$. We have to select a characterizing area.

And this we make a plot for low Reynolds number, low means of the order of 10, up to about 10^4. The drag coefficient is a function of Reynolds number alone nothing else. And at high Reynolds number, there is a difference between the drag experienced by rough spheres and by smooth spheres. We will explain this difference in a later lecture.

In fact, this is the difference that is used in many of the ball games that we play. In cricket, in golf, but the point to note here is that we have simplified. So instead of expressing the drag as a function of a number of independent parameters, we now write a drag coefficient as a function of one single number Reynolds number, which is a group of parameters, a non-dimensional group of parameters, and the geometry.

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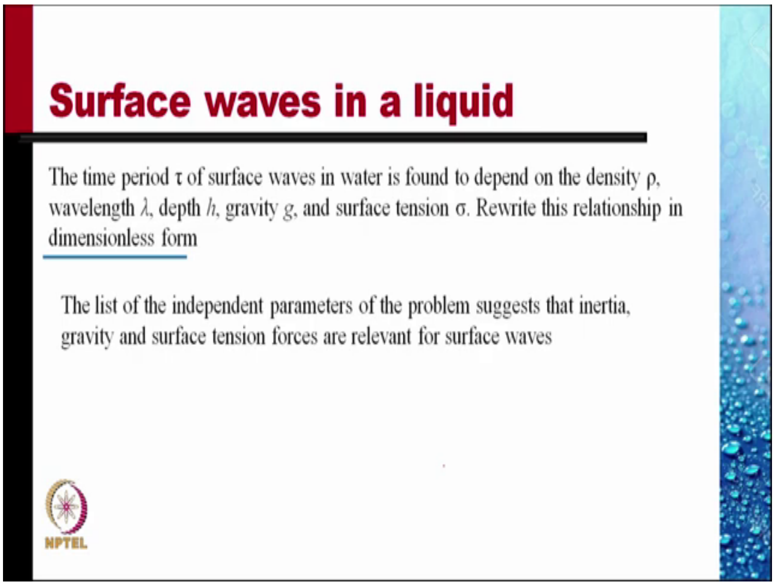

Can we do this for other situations? Let us apply it to the surface waves in a liquid. On a pond of water, in an ocean the waves travel. The time period τ of the surface waves in water is found to depend on the density ρ, the wavelength λ, the depth h, gravity g, and the surface tension σ. Rewrite the relationship in dimensionless form.

So, we see that the time period of the dependent parameter is a function of five independent parameters: ρ, λ, h, g and σ. Can we obtain a simpler relationship in a dimensionless form? The list of independent parameters of the problems suggest that inertia, gravity and surface tension forces are relevant for surface waves. Inertia must be there because density is important, gravity because g is important, surface tension, because σ is relevant.

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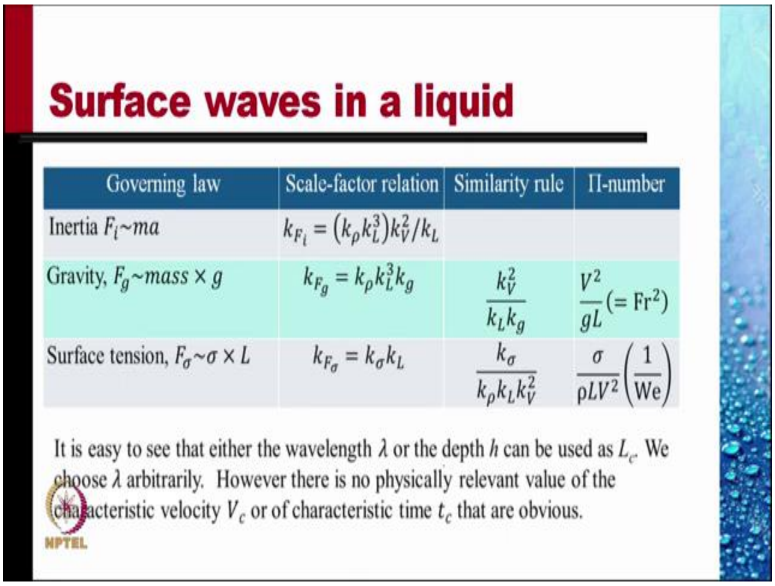

And we convert these forces into pi numbers. Three forces, so we get two pi numbers as ratios. Using the same process that we followed earlier we write the scale factor for inertial forces. We write scale factor for gravity forces, mass times g, so k_{F_g} would be $k_\rho k_L^3$ for mass and k_g. For surface tension, which is like σ times L, k_{F_σ} would be $k_\sigma k_L$.

From this we get two similarity rules: $\frac{k_V^2}{k_L k_g}$ should be 1. And $\frac{k_\sigma}{k_\rho k_L k_V^2}$ should be 1. And from this we get two pi numbers, $\frac{V^2}{gL}$ in the two flows, the model and the prototype, should have the same values, and $\frac{\sigma}{\rho L V^2}$ should have the same values in the two flows.

The first one is recognized as the square of Froude number, and the second one is the reciprocal of the quantity defined as Weber number. So, similarity requires that these two pi numbers must match. It is easy to see that either the wavelength λ or the depth h can be used as the characterizing length. We choose λ, the wave length of the waves, arbitrarily, as the characterizing length. However, there is no physically relevant value for the characteristic velocity V_c, or for the characteristic time t_c. So, we will use this a little later.

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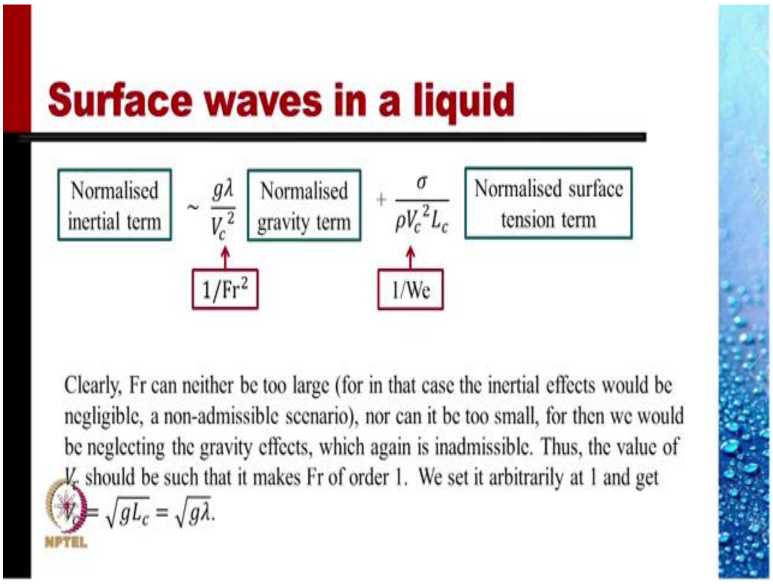

Here again, the normalized inertial force would, somehow, balance gravity force and the surface tension force. So, the equation that we write for the phenomenon would have a term inertial force on the left-hand side, normalized. Then on the right-hand side, we have normalized gravity force and the normalized surface tension force, the normalized gravity force would have $\frac{g\lambda}{V_c^2}$ as its coefficient, and the normalized surface tension force would have $\frac{\sigma}{\rho V_c^2 L_c}$as its coefficient.

The first is recognized as $1/\mathrm{Fr}^2$, and the second is recognized as 1/We. Clearly, Froude number can neither be too large, for in that case the initial force would be negligible, a non-admissible scenario, nor can it be too small, for then we would be neglecting the gravity effects altogether, and we know gravity effects cannot be neglected.

Thus, the value of the Froude number should be of the order 1. Since V_c is yet unknown, we assert that the value of V_c should be such that it makes the Froude number of order 1. We set it arbitrarily equal to 1, and get V_c as $\sqrt{gL_c}$. And since L_c has been taken as λ so the characteristic velocity is $\sqrt{g\lambda}$ a. If we use this as a characteristic velocity, then there would be only one parameter left, and that would be the Weber number.

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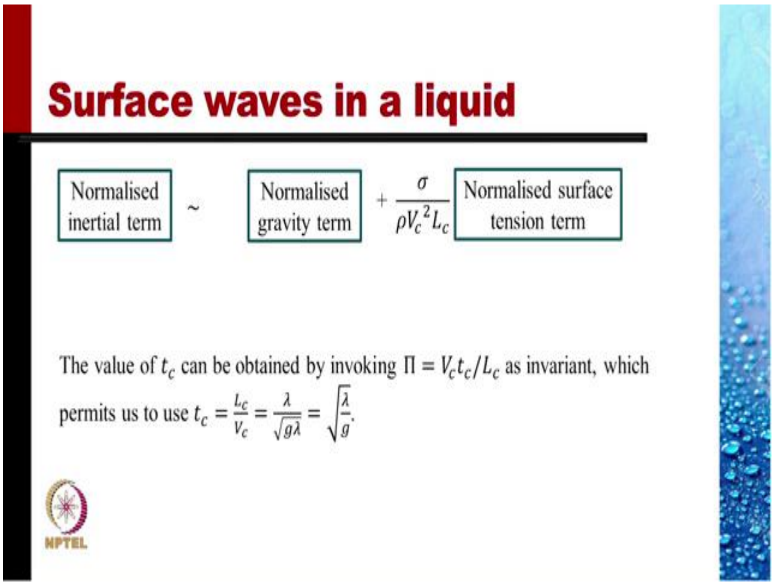

Once we know the value of V_c, we can obtain the value of the characteristic time invoking pi $\frac{V_c t_c}{L_c}$ as invariant, which permits us to use $t_c = \frac{L_c}{V_c} = \frac{\lambda}{\sqrt{g\lambda}} = \sqrt{\frac{\lambda}{g}}$. This is the characteristic time.

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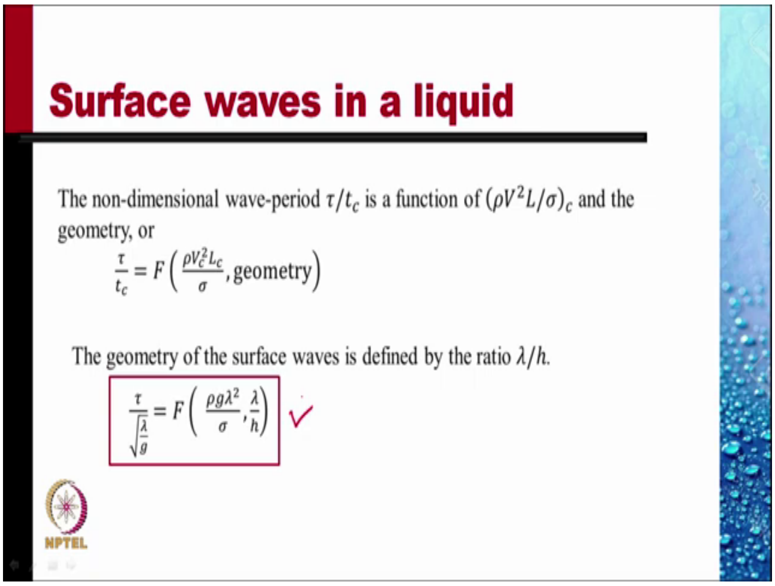

And so, the non-dimensional wave period which should be of the form τ divided by the characteristic time, is now a function of Weber number alone, and the geometry. τ/τ_c, the non-dimensional wave period is a function of the characteristic Weber number and the geometry.

The geometry of the surface wave is defined by the ratio *λ/h*, so this is the final equation that we get for non-dimensional wave period. A much simpler expression than we began with for deep sea waves. When h is very large, λ is neglected so, the time period of the waves simply depends upon $\rho g\lambda^2/\sigma$ which is 1 over Weber number.