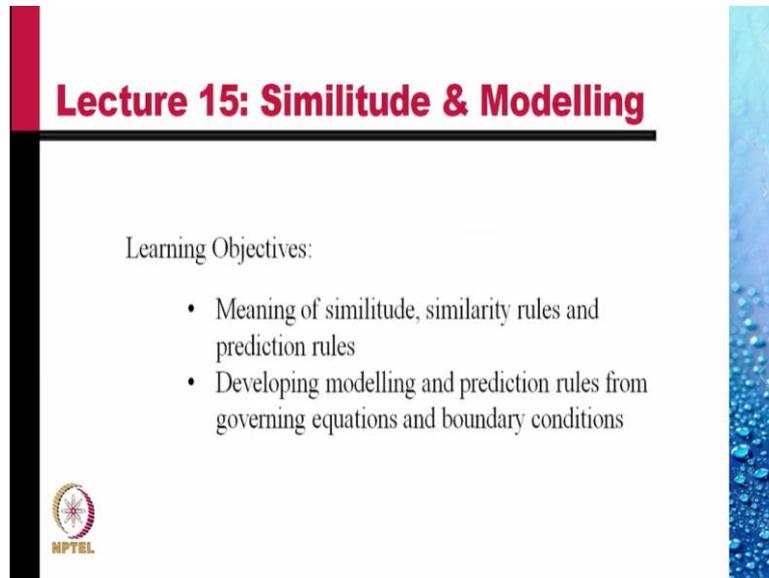**Fluid Mechanics and its Applications**
Professor Vijay Gupta
Indian Institute of Technology, Delhi
Lecture – 15
Similitude and Modelling
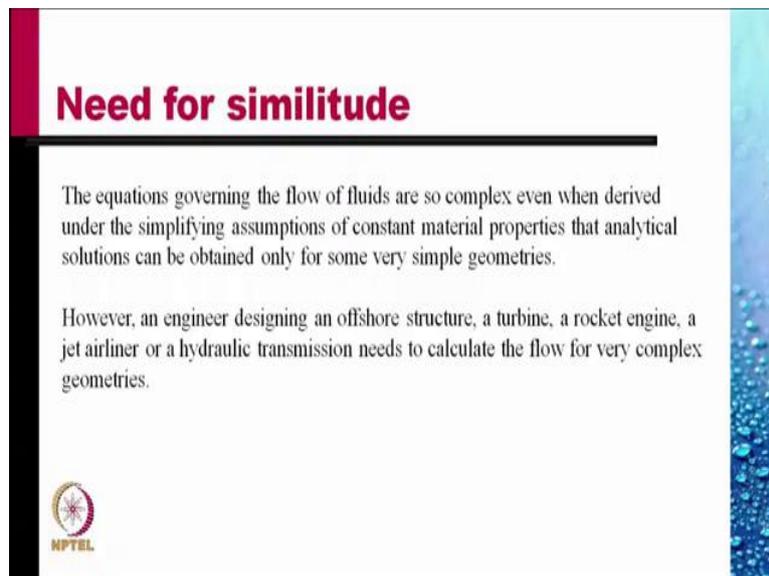
(Refer Slide Time: 00:19)

(Refer Slide Time: 00:25)

Welcome back.

The equations governing the flow of fluids are so complex that even when derived under the simplifying assumptions of constant material properties, the analytical solutions can be obtained only for some very simple geometries. However, an engineer designing an offshore structure, a turbine, a rocket engine, or a jet airliner needs to calculate the flows for very complex geometries.

So, what does he do in the process? The only way to do this is to conduct experiments. There two kinds of experiments that an engineer does. One is the actual physical experiment, and the other is the numerical experiment. He writes a computer code, runs it for certain conditions, and obtains a result.

For the experimental work, it is possible to do experiments on the actually designed structures, but there are problems. Sometimes, the structures are so big that if we make them in the first place without testing, we would not know whether they will work or not, and that calls for too much of investment. So, it is always advisable to test them in models. But how do we test them on models? What rules of model-making should be there and how to interpret the results of the model?

(Refer Slide Time: 02:49)
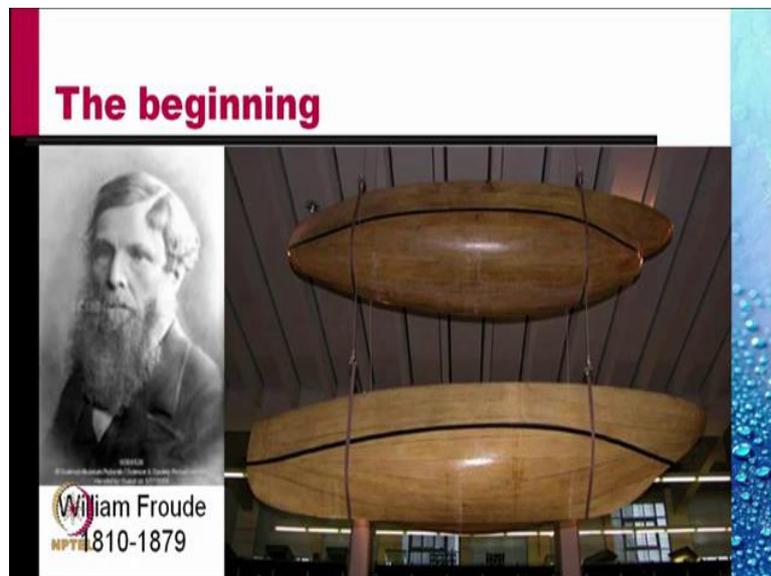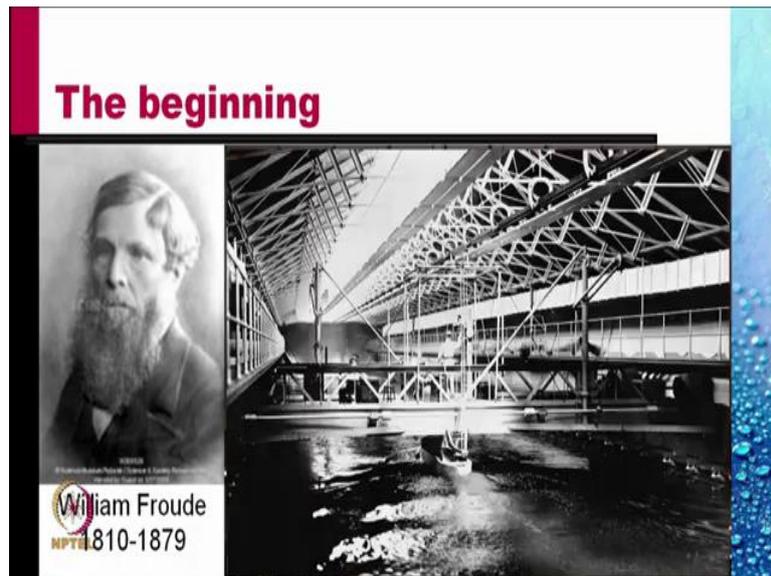
The beginning of this science is credited to the scientist William Froude who worked for the British admiralty and had been tasked to design ships. In the process of designing ships for low resistance, he came up with the idea of designing small models. The models that he made were various boats of the sizes of 3 feet, 6 feet and 12 feet. With these he found out the drag that these boats experienced.

And then he devised rules, which are known now as Froude formula to predict the drag on the full-sized boats. In fact, he was so successful that the British admiralty adopted that as a standard procedure for designing ships.

(Refer Slide Time: 03:45)

The next big name associated with model testing is the Wright Brothers. Frustrated by the failures of their earlier glider experiments, they chose to develop their own data in this wind

tunnel. They designed a structure in which a fan sucked air through a tube, a rectangular tube of cross-sectional area 1 foot squared in this case, and they tested the design of their wings in these.

They found that the data they obtained was very different from the data that had been reported earlier. And because of this they could design better aerofoils and better gliders. In fact, the propellers that they designed for their flyers were also designed using the data obtained in this small wind tunnel.

(Refer Slide Time: 04:56)
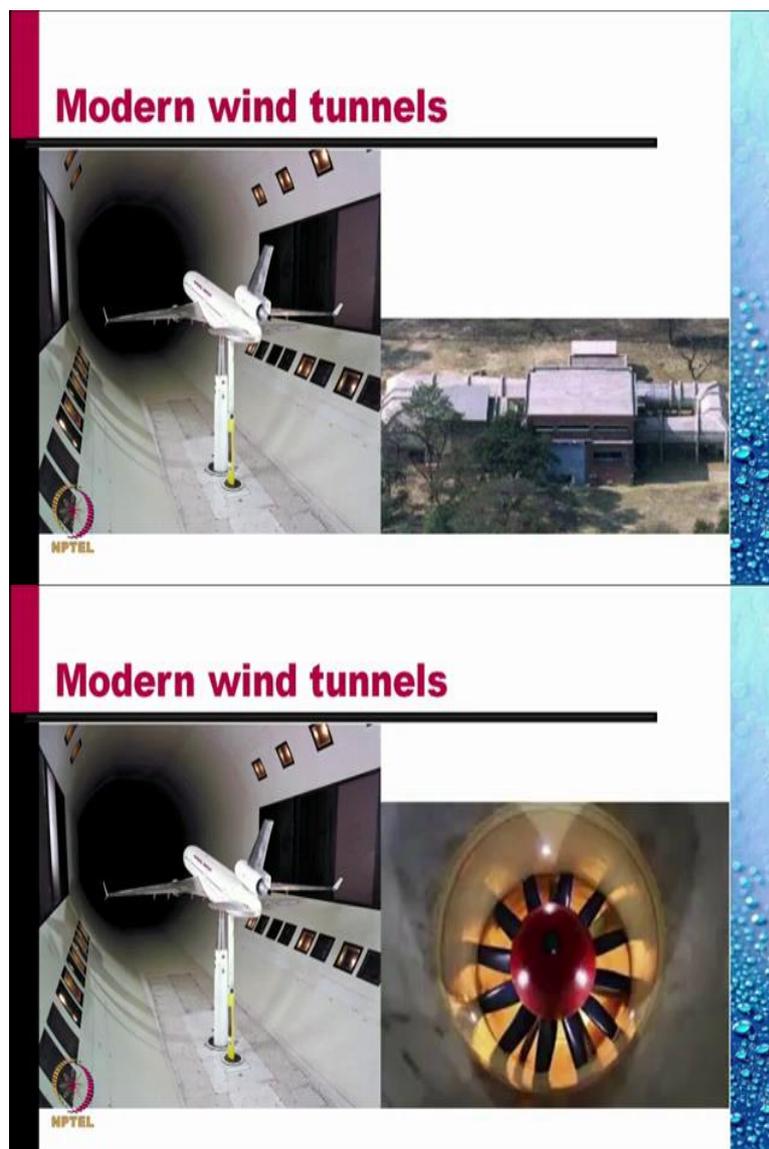
Wind tunnels today are very successful. There are wind tunnels in the world in which the whole aeroplane can go in. In this wind tunnel we see a model of an aircraft which has been instrumented to find out the forces that it will experience when flying in atmosphere. The

wind tunnels are small and big. This picture here is the wind tunnel at IIT Kanpur. It has a cross-sectional area of 8 feet by 10 feet and has been used extensively for testing the models. It uses a fan which consumes 1 MW of power to run it.

(Refer Slide Time: 05:52)
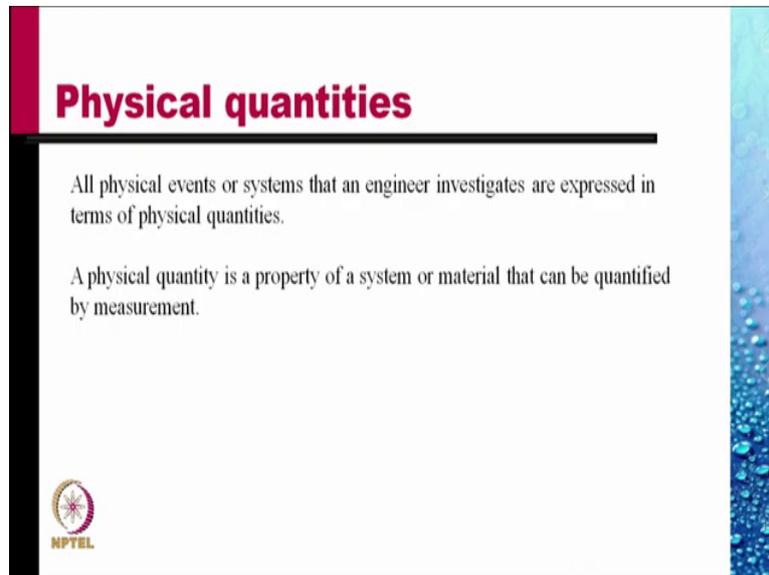
Before we develop the rules for modelling, let us understand a few things about physical quantities. All physical events or systems that an engineer investigates are expressed in terms of physical quantities. A physical quantity is the property of a system or material that can be quantified by measurement. It is usually expressed as a number and a unit.

A unit is a standard that has been established beforehand. For example, we measure lengths in terms of meters. How many meters does a length equal to? This meter to begin with was a metal rod which now sits in a museum in Paris which was used as a definition of a length.

(Refer Slide Time: 06:57)

There are four kind of quantities that are involved in any physical problems. And these quantities are the independent variables, the independent parameters, the dependent variables, and the dependent parameters.

(Refer Slide Time: 07:18)

Example: Pressure distribution on an airfoil

Independent variables:

- Location variables x, y
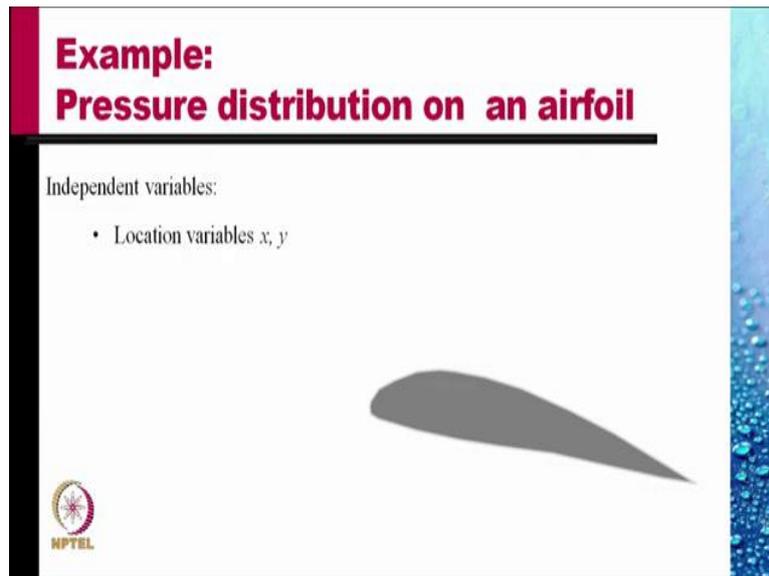
Let us take for example the problem of finding the pressure distribution on an airfoil for which we want to know what is the lift that will be experienced by this aerofoil when it is in flight in an atmosphere. We have independent parameters like geometry or the shape of the airfoil, the angle of attack, the angle the chord of this aerofoil makes with the oncoming flow, and the forward velocity. These are parameters that are needed to define what the problem is.

There are other independent parameters which include material properties such as air density, the viscosity of air, the pressure of atmosphere around the airfoil. These are all parameters for a given problem. Each one of them has one value. And all these parameters uniquely define a problem. Even if the value of one of these parameters changes, it is a new problem, a new experiment altogether.

It is for this reason they are termed as the uniquity parameters. They define the uniqueness of the problem. A problem is not completely defined till all the values of these parameters are specified. And once they are specified, the definition is complete. Anyone else can repeat the experiments and obtain the results. And he should obtain the same results as you obtained.

They are independent variables that vary within the problem. The parameters do not vary within a given experiment or a given problem. The location variables x and y are independent variables within the problems. We can have different locations which would have different values x and y.

Dependent variables: dependent variables are the results, variables which are the results. The velocity vector **V** and its component u, v and w are all dependent variables. Pressures are dependent variables. They are the results of the uniquity parameter, and they vary in the flow field and that is why they are variables. They are dependent because they are the results of a given experiment.

Then there are dependent parameters, the unique values, the total lift. It does not vary from point to point. For a given experiment there is one value of total lift, there is one value of total drag, or a pitching moment. The moment that tends to pitch the nose of the airfoil up, and that also is a parameter, a dependent parameter, a result of the experiment, a result of calculations.

The engineering problem consists of determining a dependent variable as a function of independent variables and of the uniquity parameters. Either determine a dependent variable or determine a dependent parameter, but a dependent parameter would be a function only of the uniquity parameters or the independent parameters, and not of independent variables.

For example, an engineering problem could be to determine pressure p as a function of the independent variables, the location variables x and y, and the uniquity parameters, shape, velocity far away or velocity of the airfoil, alpha, the angle of attack, rho, mu, the properties of the atmosphere, and the atmospheric pressure itself. Or we could have the x component of velocity u which would be the functions of the same quantities, independent variables as well as independent parameters.

Or we could have a problem in which we want to determine lift force L. And now this would be function only of the uniquity parameters, the independent parameters and not of the independent variables.

(Refer Slide Time: 13:27)
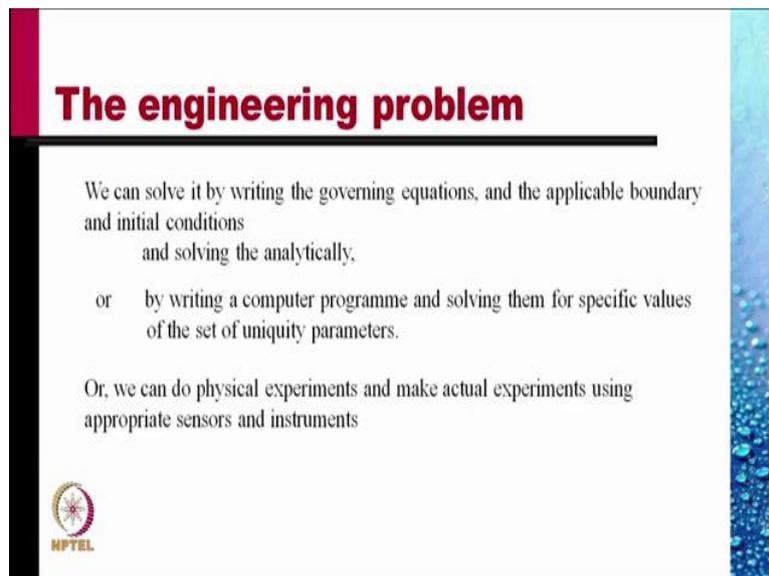
The engineering problem

We can solve it by writing the governing equations, and the applicable boundary and initial conditions
    and solving the analytically,

or   by writing a computer programme and solving them for specific values
      of the set of uniquity parameters.

Or, we can do physical experiments and make actual experiments using appropriate sensors and instruments
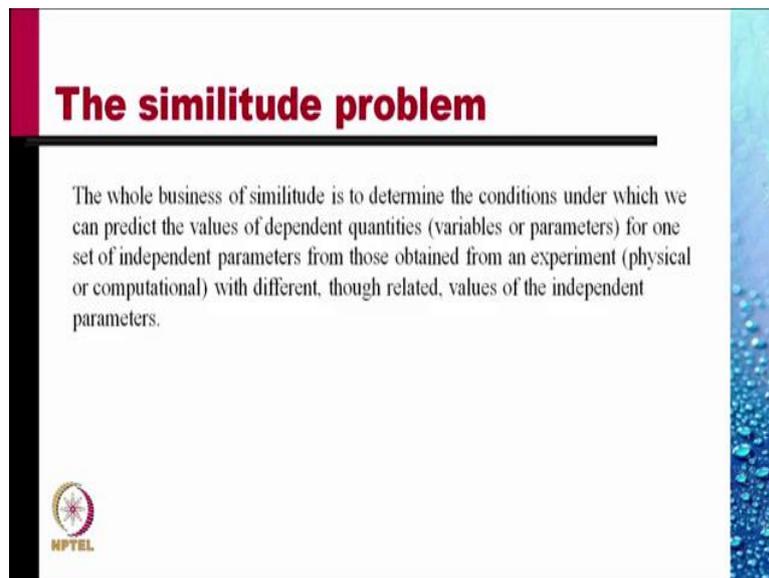
To solve this engineering problem, we could either write the governing equations and applicable boundary conditions, and solve this analytically using mathematics, or we could write a computer program for solving them, and then solve for a specific set of uniquity parameters.

When we are solving these equations analytically, we do not have to bother initially about the values of the uniquity parameters, but if we are solving the problems on a computer, then we

will have to give the values of the uniquity parameters. Then only the computation would proceed because the computation is playing around with numbers.

Or we can do physical experiments and make actual experiments using appropriate sensors and instruments. There also we will have to work with specific values of the set of uniquity parameters. We will have to set the velocity, we have to set the angle of attack, we have to set the pressure, we have to choose the fluid with a specific density and viscosity. So, a physical experiment cannot be done without an actual set of uniquity parameters.

(Refer Slide Time: 15:13)

The similitude problem

The whole business of similitude is to determine the conditions under which we can predict the values of dependent quantities (variables or parameters) for one set of independent parameters from those obtained from an experiment (physical or computational) with different, though related, values of the independent parameters.

The whole business of similitude is to determine the condition under which we can predict the values of dependent quantities, that is, of dependent variables or dependent parameters for one set of independent parameters from those obtained from an experiment, physical or computational, with different, though related, values of the independent parameters.

Thus, for example, we find the lift of an aerofoil with chord length of 10 centimetres, and then we were able to predict the lift on a full-sized aerofoil of chord length 1 meter. Then this is the business of similitude. This is the whole business of similitude.
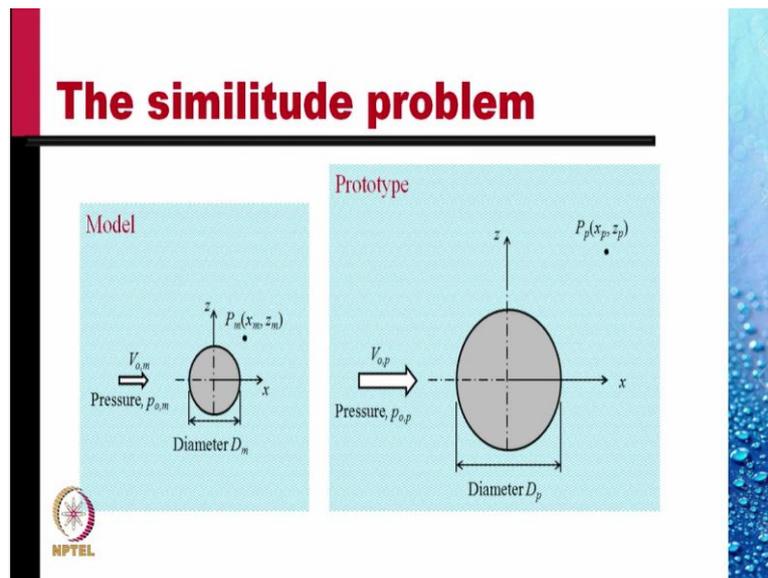
So, if we have two experiments, one we call a model experiment, the other we call a prototype experiment, then we do our experiment with the model for a given value of velocity $V_{o,m}$ for model, atmospheric pressure p, p_m, m for model, diameter D_m for model. And then from this, predict what would be the results on the prototype.

We have not conducted the prototype experiments; we have conducted only the model experiments. But we are able to predict the results for the prototype. If we can do that, then we have handled the similitude. Why the word similitude? Because if we are able to do this, then it means the flow of the prototype is similar in some fashion with the model.

The problem that we have on hand today, is to determine how to ensure that the flow is similar, and once we ensure the flow is similar, to determine the rules for predicting the results for the prototype, given the results on the model.

The various methods of analysing similitude, we will start with a standard method of non-dimensionalizing the governing equations. Let us consider a steady flow with constant density: incompressible steady flow. The equations governing these flows are the continuity equation and the Navier-Stokes equation that we have covered earlier. And these have to be solved subject to boundary conditions. We do not require initial conditions because the flow is steady. The boundary conditions could be $\mathbf{V}$, the vector velocity far away as as $x, z \to \pm \infty$ is $V_o \boldsymbol{i}$. The flow is only in the x direction with the velocity V_o and V is 0 on the boundary, the boundary $x^2 + z^2 = D^2/4$, where D is the diameter of the cylinder.

From, on this boundary the velocity is 0. This is the no slip boundary condition. And the pressure far away, as x tends to minus infinity far away, ahead of the cylinder at the centre line. In this problem the independent variables are x and y. Independent parameters include V_o, the velocity far upstream, p_o the pressure far upstream, the density ρ of the air, viscosity μ, g, the acceleration due to gravity, which determine the body forces, and the geometry of the boundary, here D represents the size of the body, and shape of the boundary which in this case is circular.
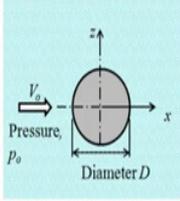
Dependent variables would be u and v as functions over x and y over the flow field, and p, the pressure at the various points x and y over the flow field. The dependent parameter could be the lift, the drag, the pitching moments, etcetera. Of course for a circular cylinder there will be no pitching moment. But this could be the list of dependent parameters. This is only as an example.

Non-dimensionalizing the Governing Equations

$V(\mathbf{x}) = \mathcal{V}(\mathbf{x}; V_o, p_o, \rho, \mu, g, D, \text{and the shape of the boundary})$

$p(\mathbf{x}) = \mathcal{p}(\mathbf{x}; V_o, p_o, \rho, \mu, g, D, \text{and the shape of the boundary})$

Introduce non-dimensional variables:

$x^* = x/D; \; z^* = z/D$ \qquad [or, $\mathbf{x}^* = \mathbf{x}/D$]

$u^* = u/V_o; \; w^* = w/V_o$ \qquad [or, $\mathbf{V}^* = \mathbf{V}/V_o$], and

$p^* = p/p_o$

Now, the velocity, dependent velocity V, that is, the component u and v as a function of the independent variable x and y can be written as function of **x**, the vector x, the location vector **x**, which consists of scalar quantity x and y, and the list of uniquity parameters, the independent parameters.

The two independent variables and the independent parameters are separated, typically, by a semicolon. Before the semicolon, the list of independent variables. After the semicolon, the list of independent parameters. Similarly, the pressure. Now we non-dimensionalize the quantities by introducing various non-dimensional variables. And how do we non-dimensionalize?

We take a quantity and divide this by its characteristic value. Suppose we say the diameter of the cylinder is 5 cm. 5 really has no meaning. We are comparing the diameter of the cylinder with a metal rod kept in a musty museum room in Paris. What does that rod have to do with this cylinder that we are working with?

So, we introduce what we call the natural unit. The cylinder, or any point on this or near this, could be characterized by the diameter of the cylinder or the radius of the cylinder, depending on whatever you wish. So that we may talk of a point x which is two diameters away from the cylinder in front of it, or one and the half diameters on top of it. This would be a natural measurement of lengths in this problem.

So, we define the location variable x star as the physical variable x in any units divided by the diameter in those units, x in meters, diameters in meter, x in inches, diameter in inches. Star

as a superscript here denotes that we non-dimensionalize the quantity x by D, or x^* would be a pure number. This is a measure of the location in terms of the diameter of the cylinder, or a length which is inherent to the problem.

The velocity we non-dimensionalize with respect to a characterising velocity, a velocity that characterizes the velocities in this problem. And what is a velocity that characterizes the velocities in this problem? The velocity far upstream, the undisturbed velocity of air flowing past this sphere, V_o. Similarly, pressure. We could non-dimensionalize by using atmospheric pressure as the characterizing pressure, so p by p_o.

(Refer Slide Time: 26:17)
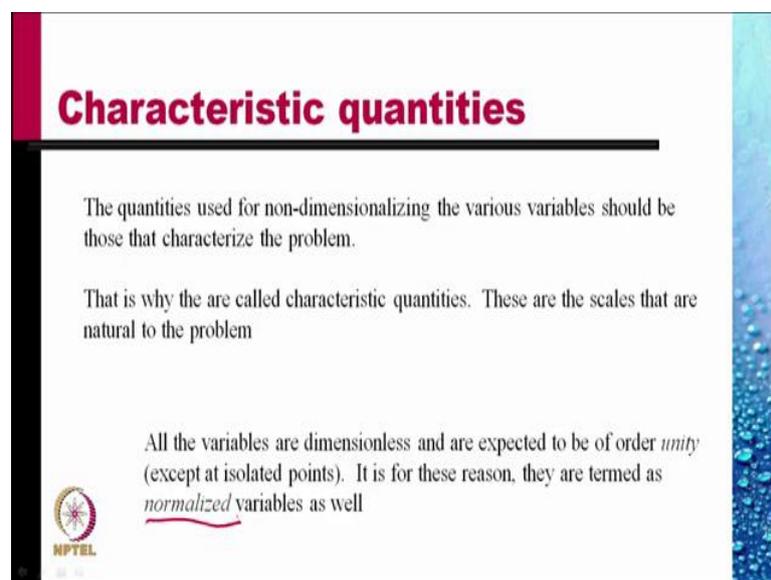
The quantities used for non-dimensionalizing the various variables should be those that characterize the problem. That is why these are called characteristic quantities. These are the scales that are natural to the problem as elaborated earlier. All these dimensional variables x^*, u^*, w^*, p^* now are dimensionless, and are expected order unity, accept at isolated points. They are neither too big nor too small. And it is for this reason that they are also termed as normalized variables. We will use this fact later on in this course.

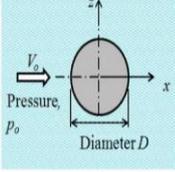## Non-dimensionalizing the Governing Equations

$$\nabla^* \cdot \mathbf{V}^* = 0$$

$$\mathbf{V}^* \cdot \nabla^* \mathbf{V}^* = -\left(\frac{p_0}{\rho V_o^2}\right)\nabla^* p^* - \left(\frac{gD}{V_o^2}\right)\mathbf{k} + \left(\frac{\mu}{\rho V_0 D}\right)\nabla^{*2}\mathbf{V}^*$$

with the boundary conditions

$$\mathbf{V}^* \to \mathbf{i} \qquad \text{as } x^*, z^* \to \pm\infty$$
$$\mathbf{V}^* = \mathbf{0} \qquad \text{on } x^{*2} + z^{*2} = 1/4$$
$$p^* \to 1 \qquad \text{on } z^* = 0 \text{ as } x^* \to \infty$$

where $\nabla^* = \mathbf{i}\frac{\partial}{\partial x^*} + \mathbf{j}\frac{\partial}{\partial y^*} + \mathbf{k}\frac{\partial}{\partial z^*}$

NPTEL

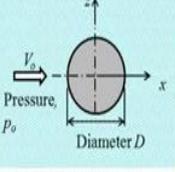## Non-dimensionalizing the Governing Equations

$$\mathbf{V}^*(\mathbf{x}^*) = \mathbf{V}^*\left(\mathbf{x}^*; \frac{p_0}{\rho V_o^2}, \frac{gD}{V_o^2}, \frac{\mu}{\rho V_0 D}, \text{geometry}^*\right),$$

and $$p^*(\mathbf{x}^*) = p^*\left(\mathbf{x}^*; \frac{p_0}{\rho V_o^2}, \frac{gD}{V_o^2}, \frac{\mu}{\rho V_0 D}, \text{geometry}^*\right)$$

Geometry* in the above list of parameters is the non-dimensional boundary curve $x^{*2} + z^{*2} = 1/4$.

$\frac{p_0}{\rho V_o^2}, \frac{gD}{V_o^2}, \frac{\mu}{\rho V_0 D}$ are termed as Pi numbers

NPTEL

So, now if we do those transformations, the non-dimensional governing equations look like these. Here, we have made the first term, the coefficient of first term, on the left as unity, and we have arranged the equation in this fashion. This term represents the inertial force, this is the pressure force, here this is the viscous force, and this is the gravity. These are the coefficients in front of these.
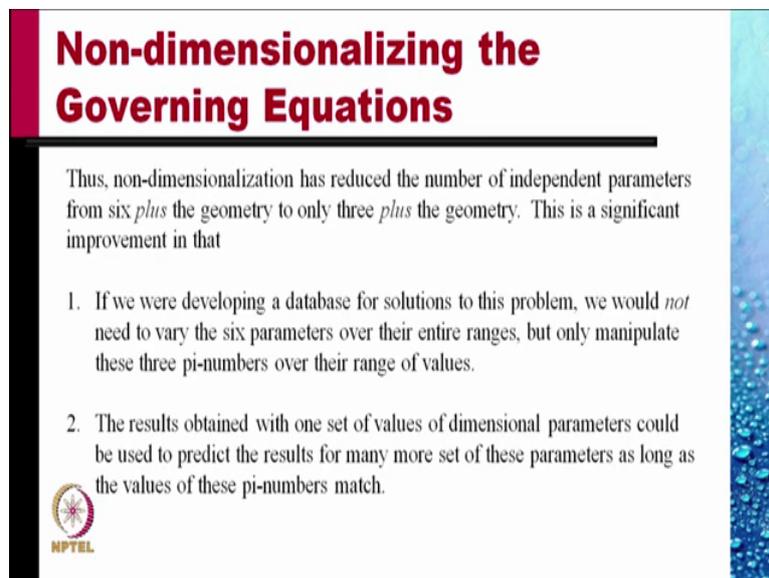
And similarly, we make the changes in the boundary conditions, and the boundary conditions simplify to these very simple things. $\mathbf{V}^* \to \mathbf{i}$, one unit in the x direction, as $x^*, z^* \to \pm\infty$, $\mathbf{V}^* = \mathbf{0}$ on the boundary non-dimensionalized $x^{*2} + z^{*2} = 1/4$ and the $p^* \to 1$, far upstream or downstream.

Now, you notice that there are only three groups of independent parameters, $\frac{p_0}{\rho V_o^2}$, $\frac{gD}{V_o^2}$, and $\frac{\mu}{\rho V_0 D}$. There are only these three parameters. So, if we were to write a problem, a problem would be like this. The non-dimensional velocity V^* as a function of x^* depends upon x^*, the location, and these three groups of independent parameters. The geometry in the above list of parameters is the non-dimensional boundary curve $x^{*2} + z^{*2} = 1/4$. The equation of the circular surface of the cylinder.

To me that is a big advancement: from six independent parameters plus geometry, we have now come down to three groups of independent parameters. So, there were six uniquity parameters, six parameters that whose values would change and would change the experiment.

Now, the value of p_o and V_o could change, but if they change in such a fashion $\frac{p_0}{\rho V_o^2}$ remains the constant, and these three groups remain the same, then the two experiments are the same, they will give the same non-dimensional solution V^* as a function of x^*. Similarly, for pressure. These three groups are termed Pi numbers.

(Refer Slide Time: 31:23)

Non-dimensionalizing the Governing Equations

3. The same holds for numerical solutions as well.

4. Since the variables and their various derivatives have all been normalized and are *expected* to be of order 1, the non-dimensional groups of parameters which now are rendered as coefficients of the various terms indicate the importance of the term in the equation. Thus, if the coefficient of any one term is much less than one, the term may be ignored as an approximation.

Thus, non-dimensionalization has reduced the number of independent parameters from six plus the geometry, to only three plus the geometry. This is a significant improvement in that. If we were developing a database of solutions to this problem we would not need to vary the six parameters over the entire ranges, but only manipulate these three parameters or these three Pi numbers over their range of values.

Another significance, the result obtained with one set of values or dimensional parameters could be used to predict the results of many more set of these parameters, as long as the values of these Pi numbers match.

The same holds for numerical solution as well. I will not have to run the program for various values of velocities and diameters and densities and viscosity. I can run the program against a constant value of Reynolds number. Since the variables and the various derivatives have all been normalized and are expected to be order 1, the non-dimensional groups of parameters, which now are rendered as coefficient of the various terms indicate the importance of the terms in the equation. Thus, if the coefficient of any one term is much less than the others, that term may be ignored as an approximation. This is an idea that we develop again a few lectures later.

These three Pi numbers that we obtained in a previous slide have been given names. In fact, we name all such Pi numbers after great scientists who have made contributions in that field. The first number $\frac{\rho V_0 D}{\mu}$ is named as Reynolds number after Osborne Reynolds, a British physicist who did the experiment showing that the fluid flowing through a tube remains laminar for values of this parameter below a certain threshold, and then becomes turbulent. We had discussed this experiment in an earlier lecture.

The second number, $\frac{\rho V_o^2}{p_0}$ is called the Euler number after Leonard Euler, a great mathematician. The third number, $\frac{V_o}{\sqrt{gD}}$, in fact this is square root of the parameter group that we obtained earlier, has been termed as Froude number after William Froude who first started experimenting in this area.

So, this is what we have now. We can obtain the result from one experiment V^* and if Euler number, Reynolds number and Froude number match, then the results for the prototype would have the same non-dimensional values of V^* and p^*.

So, if the two experiments have to be similar, if the two flows have to be similar, the Reynolds number should match, that is, $\frac{\rho V_0 D}{\mu}$ of the prototype should be the $\frac{\rho V_0 D}{\mu}$ of the model. Similarly, for the Euler number and Froude number.

The two flows, model and prototype are similar flows if the values of the non-dimensional Pi numbers formed with the uniquity parameters are identical in the two flows. In such a situation, the normalized dependent variables have the same values on all sets of homologous points.

This statement can be broken down in two parts The part one, as I said before, are the modelling rules. These are the requirements for similarity. Two flows are similar if the values of the Pi number formed with the independent parameters in the two flows are the same, or have the same values. And the second part, the prediction rules. If the two flows are similar,

the values or the normalized dependent variables in one flow are the same as in the other flow at homologous points.

(Refer Slide Time: 37:12)

To illustrate this let us consider the dependent variables, shear stress τ at any location within the flow field. We know from Newton-Stokes relation for stress in the flow that τ is μ times the rate of deformation, which in two dimensional flows would be $\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$.

On non-dimensionalization this becomes $t\tau = \frac{\mu V_o}{L}\left(\frac{\partial u^*}{\partial z^*} + \frac{\partial w^*}{\partial x^*}\right)$, or we can simplify this to say $\tau^* = \frac{\tau}{\rho V_o^2}$, this non-dimensionalized shear stress, is equal to $\frac{1}{Re}\left(\frac{\partial u^*}{\partial z^*} + \frac{\partial w^*}{\partial x^*}\right)$.

So, if we do the experiment with the same value of Reynolds number and Froude number and Euler number, and measure the τ in a model, then from this I can calculate τ^* by dividing this by the model value for ρV_o^2. The τ^* that I obtained in this fashion would have the same value for the prototype as well.

And now if I take this tau star which we measured from the model and multiply it with the value ρV_o^2 for the prototype, I will get the shear stress for the prototype. In similar flows, the non-dimensional RHS of the above would have the same values at homologous points, and therefore, $\tau_p^* = \tau_m^*$.