Fluid Mechanics and Its Applications
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Lecture 14A
Further Examples of Exact Solutions

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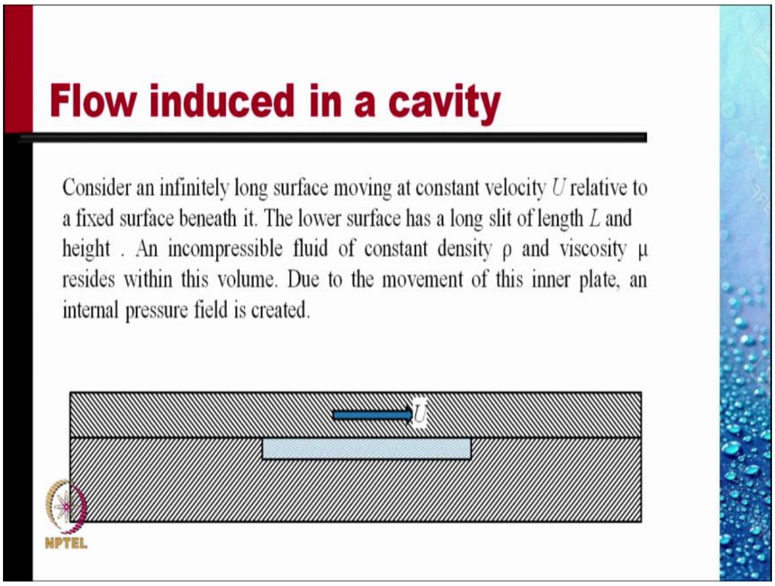

Let us do one more application: flow induced in a cavity. Consider a thin cavity in a 2-dimensional situation in a base plate with a capping plate moving at a velocity U towards the right as shown. The fluid is filling the cavity shown here. We want to find out what is the velocity that is introduced within the cavity, and the force *F* that is needed to keep this upper plate tightly enclosing the cavity.

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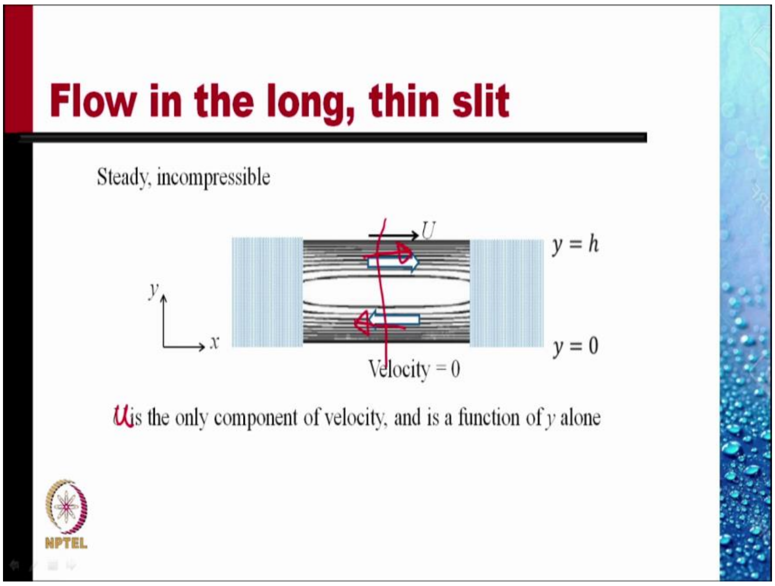

If you look at the cross section of the cavity, the thin long slit. The fluid would be moving like this. The fluid in the upper part would be moving towards the right, and fluid through the lower part for the moving towards the left. Across any section, it should be easy to see that the net volume flow rate should be 0. This appears to be quite complex flow, but we assume a long thin slit, and then we can ignore the ends, and then it looks like a fully-developed flow through a slit. Now, *u* is the only component of velocity and this is a function of *y* alone. I am talking of the lowercase *u* the velocity in the *x* direction.

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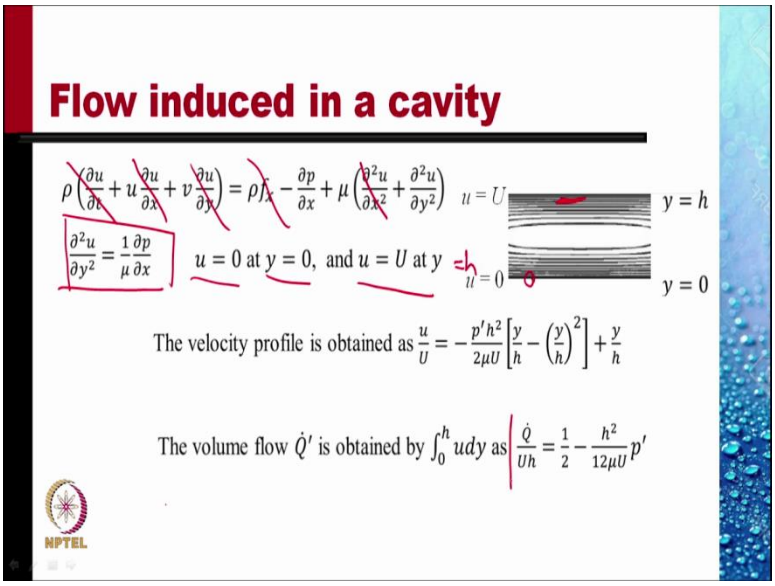

The *x*-direction component of the Navier-Stokes equation is this. In this term again, steady flow, *U* does not change in the x direction. Velocity *v* is zero, no body force in the x direction, $\partial p/\partial x$, the pressure gradient in the x direction, it will remain. Del squared u by Del x squared would be 0. And so, the equation would be this, to be solved with boundary conditions that $u = 0$ at $y = 0$ at the lower plate, the no slip condition. And $u = U$ at $y = h$, the no slip condition at the upper plate.

The velocity profile is obtained by integrating this, as we had done for the Couette flow. This is nothing but Couette flow. It is $\frac{u}{U} = -\frac{p'h^2}{2\mu U}\left[\frac{y}{h} - \left(\frac{y}{h}\right)^2\right] + \frac{y}{h}$. This is the velocity profile. The volume flow rate $\dot{Q}'$ is obtained by integrating udy from 0 to h, and this is obtained as $\frac{\dot{Q}}{Uh} = \frac{1}{2} - \frac{h^2}{12\mu U}p'$. This is 0.

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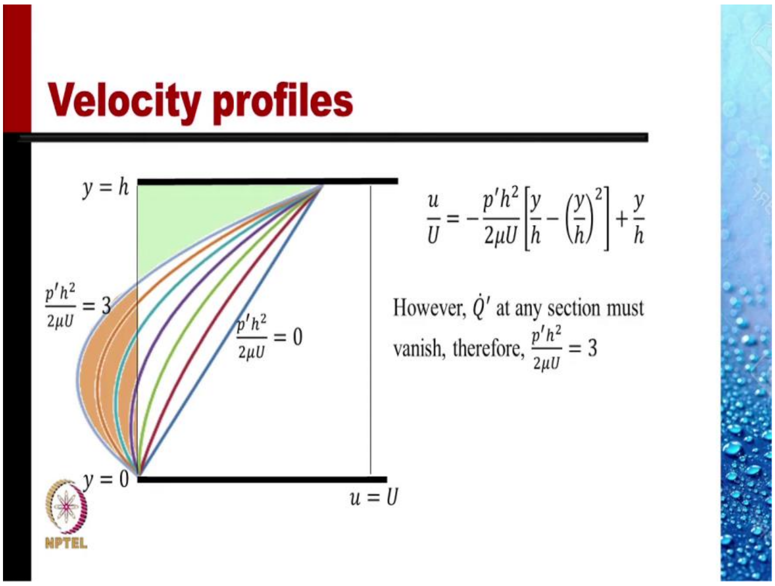

And so, we can evaluate p', the value of *u/U*, capital *U*, as a function of *p'* variable, is this, varies like this. For $\frac{p'h^2}{2\mu U} = 0$ here for this line, and the value is 3 for this line. And this 3 is where $\dot{Q}' = 0$, the volume flow rate positive and the volume flow rate negative, they cancel out, so $\dot{Q}'$ at any section must vanish, and this vanishes only when the value of that parameter is 3. the green area is exactly equal to the brown area, and the net flow is 0.

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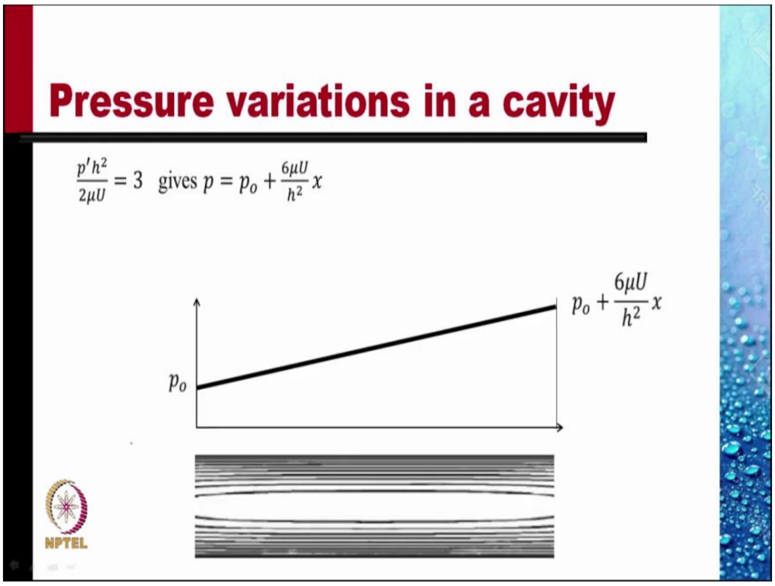

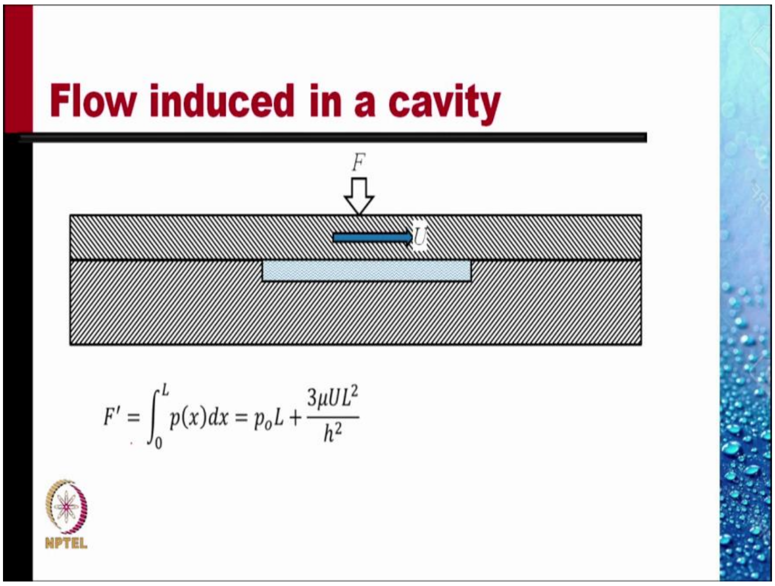

This gives the pressure varies as $p_o + \frac{6\mu U}{h^2}x$. This is the pressure variation, linearly along the slit, and the total force it can support is obtained by integrating the pressure $p(x)dx$ over length from 0 to L.

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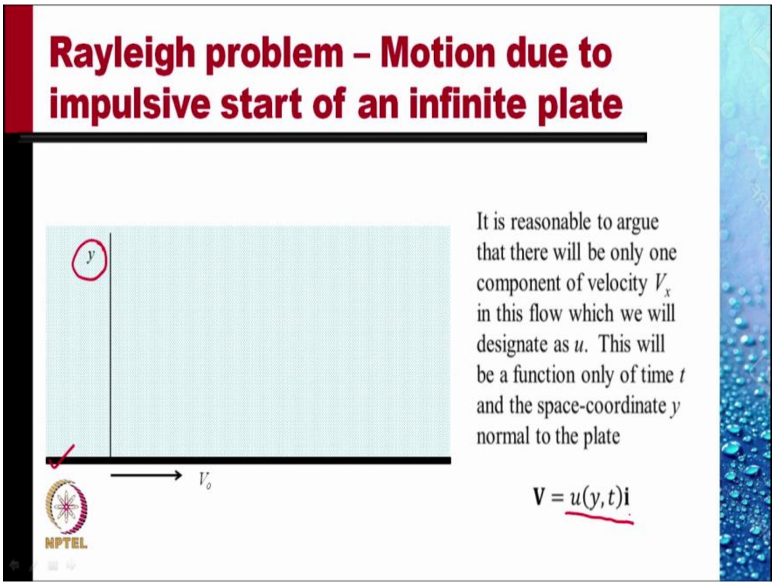

Let us do one more problem. The problem is known as Rayleigh problem, and this is an unsteady flow problem. It is the motion due to impulsive start of an infinite plate. Let this infinite plate be sitting in the fluid and the fluid is stationary, the plate is stationary. At time $t = 0$, this plate suddenly starts moving with a velocity V_o. What happens to the fluid on top?

It is reasonable to argue that there will be only one component of velocity V_x in this flow which we will designate as u. This will be function only of time and one space coordinate y, normal to the plate. So, the velocity vector of the fluid, the velocity field is simply the horizontal velocity u, which is now a function of y and t.

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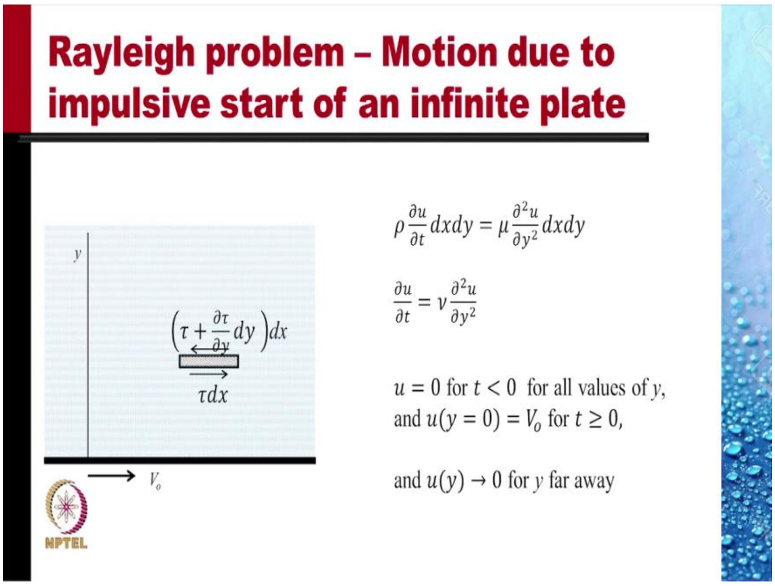

As before, we consider a small control volume, so the only horizontal forces are this shear stress and top and bottom surfaces. There are no pressure forces, or if there are pressure forces, they are equal and opposite on the two faces. So, they cancel out, and we do not write them. But unlike before, this fluid is accelerating. So, the mass of the fluid is $\rho dxdy$ acceleration is $\partial u/\partial t$. So, mass times acceleration is equal to the net force, and the net force is $-\frac{\partial \tau}{\partial y} dy\, dx$.

And if I put $\tau = \mu \partial u/\partial y$ then I get the net shear force as $\mu \frac{\partial^2 u}{\partial y^2} dxdy$, and the equation simplifies to this equation. It is an interesting equation. We can solve this equation with a little bit of trick. Let us look at the boundary conditions and the initial conditions. For this flow, we need initial condition as well because it is an unsteady flow. $u = 0$ for all times less than 0, and for all values of y. Everywhere, the fluid was at rest till time t is equal to 0.

At time t greater than 0, the velocity at the lower plate is V_o. So, $u(y = 0) = V_o$ at $t \geq 0$. And far away, the velocity would still be 0. That is, assuming that it takes infinite time for the effect of the motion of the plate to penetrate far away from the plate. So, we can take $u(y)$ tends to 0 for y far way.

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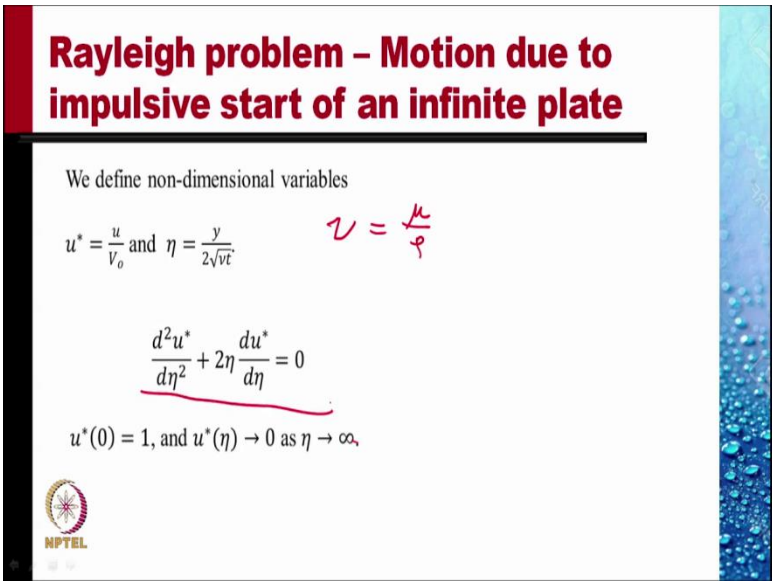

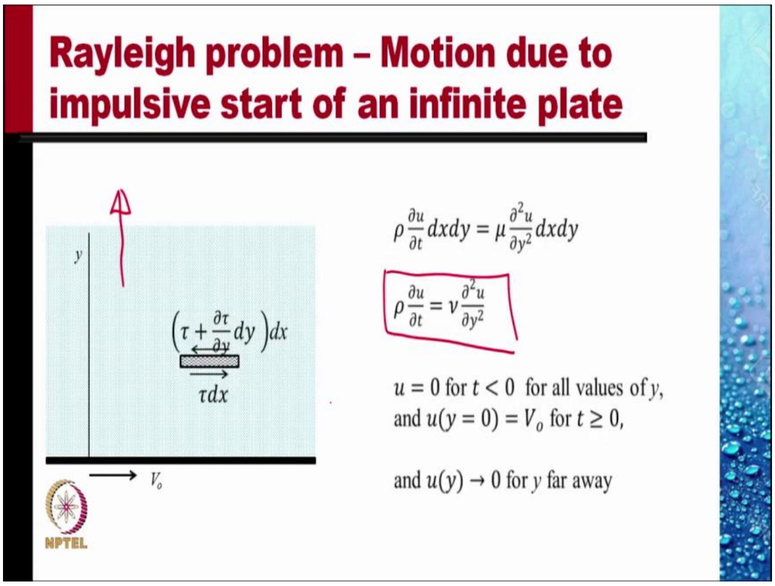

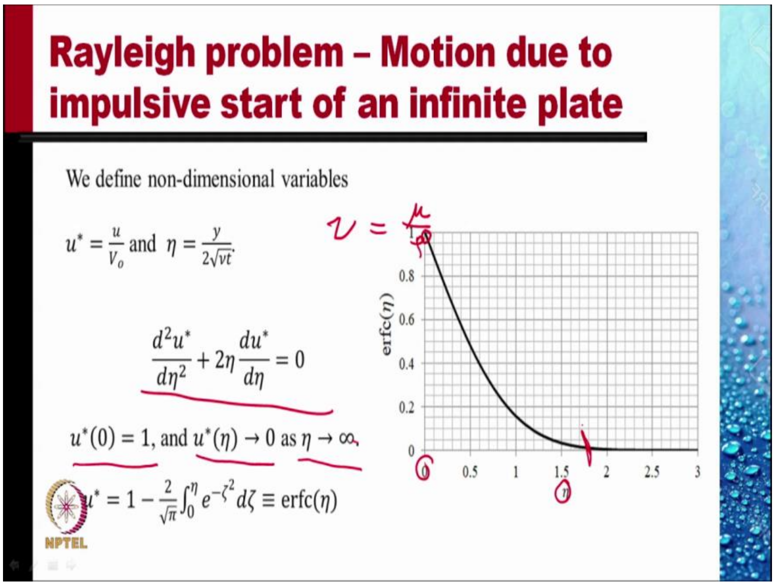

To solve this, we use a trick. We define two non-dimensional variables. We define u^*, star denotes that this is non-dimensional, $u^* = u/V_o$. Let me introduce a variable $\eta = \frac{y}{2\sqrt{\nu t}}$. Why this peculiar combination? We will explain this a little later, not now. For the moment, you take it as given we define $\eta = \frac{y}{2\sqrt{\nu t}}$, where ν is the kinematic viscosity of the fluid which is the dynamic viscosity μ divided by ρ, the density of the fluid. ν is equal to μ/ρ.

If we do this, and do a bit of algebra, we get this equation. Now, it is interesting. From a partial differential equation here, we have reached an ordinary differential equation here. The initial and the boundary conditions also change. We have $u^*(0) = 1$, and $u^*(\eta)$ tends to 0 as η tends to infinity. This equation has a standard solution, and the standard solution is what is called the complimentary error function. So, the final solution by direct integration is obtained is $1-\frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\zeta^2} d\zeta \equiv \operatorname{erfc}(\eta)$.

And this is the function defined as the complimentary error function which goes like this. The complementary error function $\operatorname{erfc}(\eta=0)=1$, and it decreases very sharply into a value about 1.8. It has reached a value less than 1 percent less than 0.01. $\operatorname{erfc}(\eta)$ represents u^*, the non-dimensional velocity.

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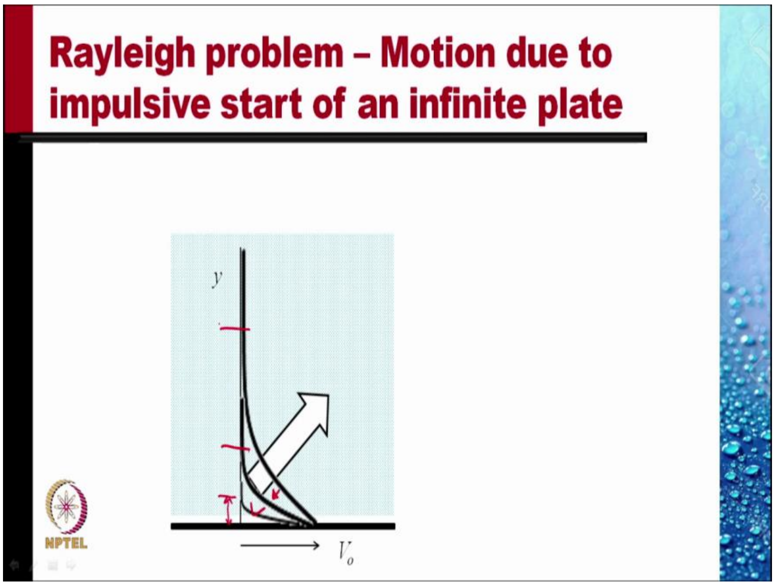

We plot this as the velocity profiles on the plate in the regular coordinates. These would be the velocity profiles for increasing times. For very small time, the velocity profile is like this. The velocity reaches 0 in a very small distance. At a little later time, this is the velocity profile, and now, the effect or the motion of the plate has penetrated to above this height. Above this, the fluid can be taken to be at rest.

At a little later time, the effect has penetrated to, say, up to here. This is what is called the penetration effect or viscosity. It is because of viscosity, the effect of the motion of the plate impulsively started, is penetrating into the bulk of the fluid.

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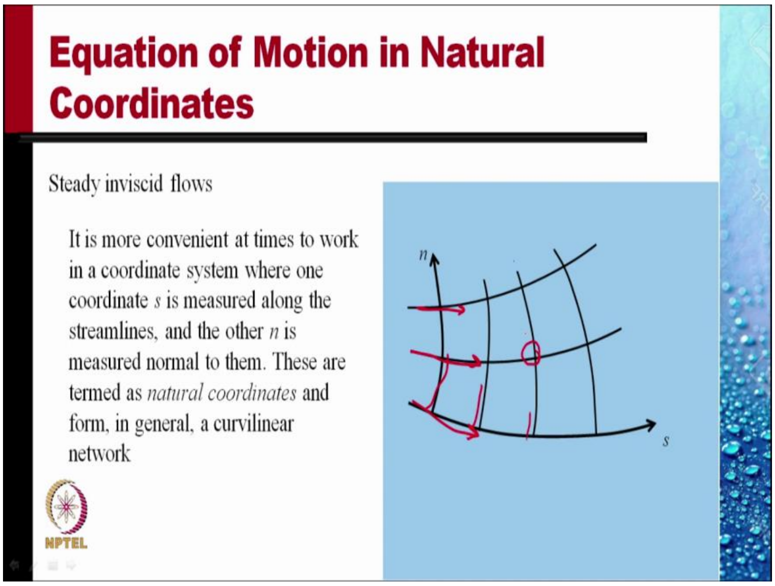

Let us now do the equation of motion in natural coordinates. We will deal with steady inviscid flows. It is more convenient at times to work in a coordinate system, in which one coordinate s is measured along the streamlines, and the other n is measured normal to them. These are termed as natural coordinates and form, in general, a curvilinear network. So, measure s along the stream lines, and measure n normal to the streamlines. Everywhere, these lines would be orthogonal. This is called the natural coordinate system.

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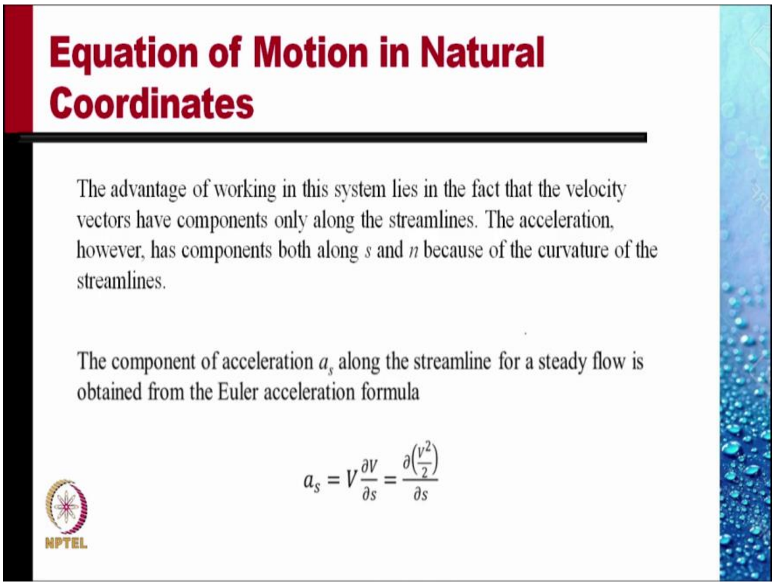

The advantage of working in this system lies in the fact that the velocity vectors have components only along the streamlines. There is no velocity vector normal to the streamlines. So, V_n is 0. The acceleration, however, has components both along the s and the n direction, because of the curvature of the streamlines. Since the streamlines the curving, so, the velocity vector is changing direction, and that would require a normal component of acceleration. The local acceleration is 0, because the flow is steady.

So, acceleration along this stream-wise direction is $V\frac{\partial V}{\partial s}$, which can be written as $\frac{\partial\left(\frac{V^2}{2}\right)}{\partial s}$.

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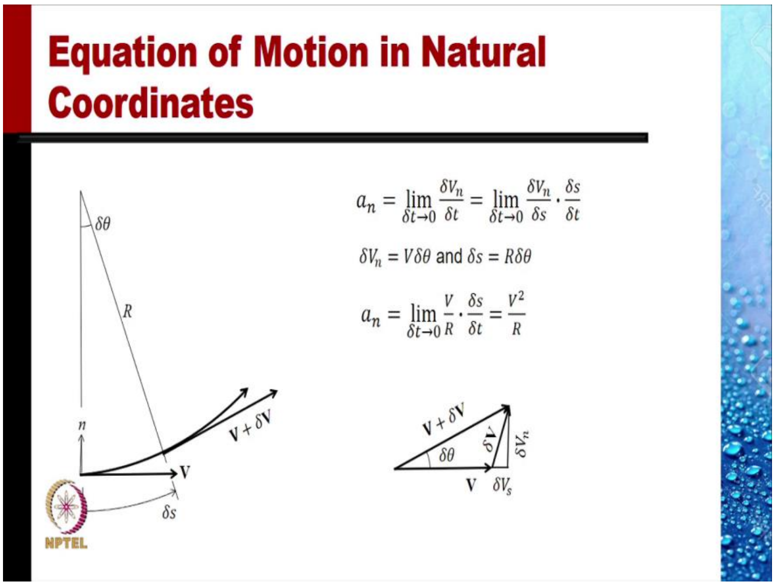

The normal component of acceleration is a little more complicated. Consider a fluid particle moving along this curve. This is a streamline. At the initial time the velocity is tangent at this point and the velocity is V. After a time δt the particle has executed a motion, and for small times, we can consider it to be arc of a circle of radius R, and let us say it has revolved through this δθ. The velocity vector now would be tangent to this arc at this point and let this magnitude be V plus delta V.

This was the original vector $\mathbf{V}$. After time delta t, it turned through an angle theta, and gets the magnitude of $\mathbf{V}+\delta\mathbf{V}$. So, change in velocity in this time δt is this $\delta\mathbf{V}$, which has two components, δV_s and δV_n. Acceleration in the normal direction is $a_n = \lim_{\delta t\to 0}\frac{\delta V_n}{\delta t} = \lim_{\delta t\to 0}\frac{\delta V_n}{\delta s}\cdot\frac{\delta s}{\delta t}$. And with $\delta V_n = V\delta\theta$ and $\delta s = R\delta\theta$, we get normal acceleration as $\frac{V^2}{R}$.

This is centripetal acceleration that you have done in your high school dynamics. So, the normal acceleration, acceleration in the normal direction, is like $\frac{V^2}{R}$, the centripetal acceleration.

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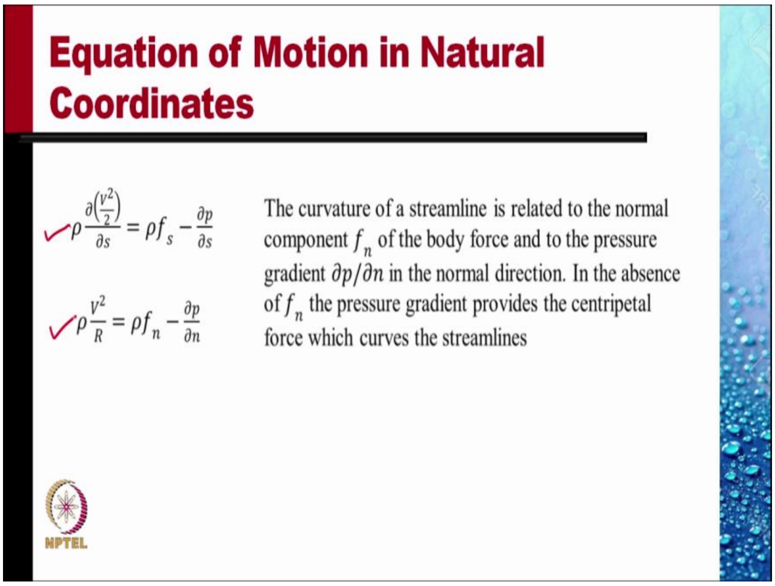

Now, the two equations can be written like this. This is the equation along the streamline: ρ times the acceleration along the steam wise direction is equal to the body force in the steam wise direction minus the pressure gradient in the steam wise direction, and rho times the normal acceleration is equal to the body force in the normal direction minus $-\frac{\partial p}{\partial n}$, the pressure gradient in the normal direction.

These are the equations of motion in the natural coordinates. The curvature of a streamline is related to the normal component f_n of the body force, and to the pressure gradient $\frac{\partial p}{\partial n}$ in the normal direction. In the absence of f_n, the body force, the pressure gradient provides a centripetal force which curves the streamlines.

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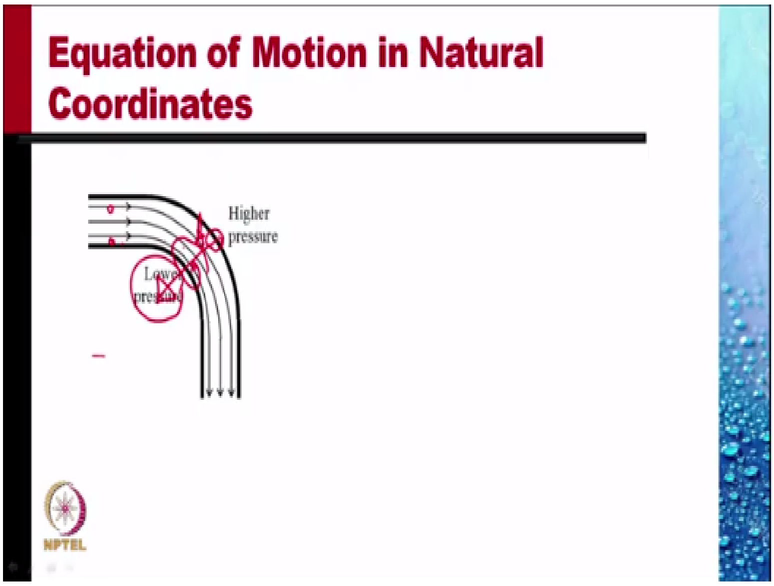

So, consider the flow through this elbow. So, clearly there must be a pressure gradient in that direction, or there must be the higher pressure here and the lower pressure there, or the pressure gradient is in the opposite direction. Only then, they would provide the centripetal force for curving of the streamlines.

In the stream wise-direction, the Bernoulli equation would show that the velocity here should be smaller than the velocity here. If the velocity was same here, the velocity at this point is smaller, then the velocity, because the pressure here is larger and the velocity at this point would be larger than this. So, the streamlines would be closer near the bend inside of the bend and would be farther apart on the outside of the bend.

So the streamlines shown make sense. They are farther apart on the outside of the bend and closer together at the inside of the bend.

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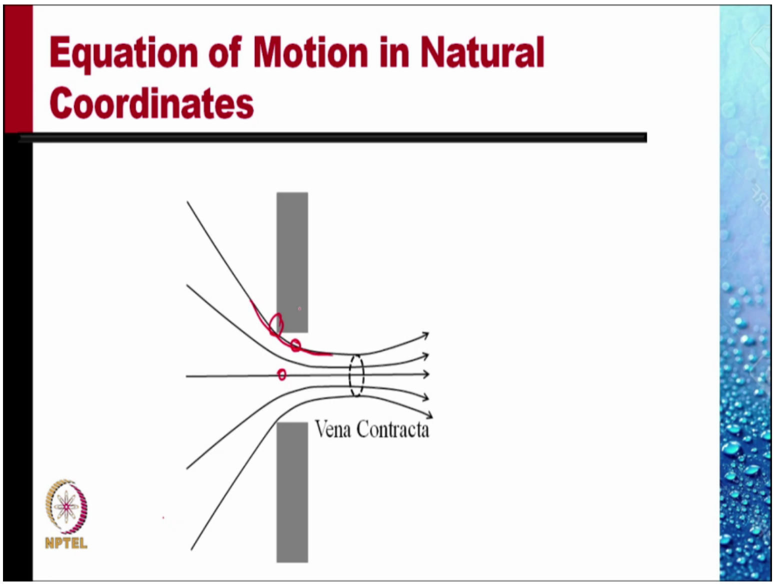

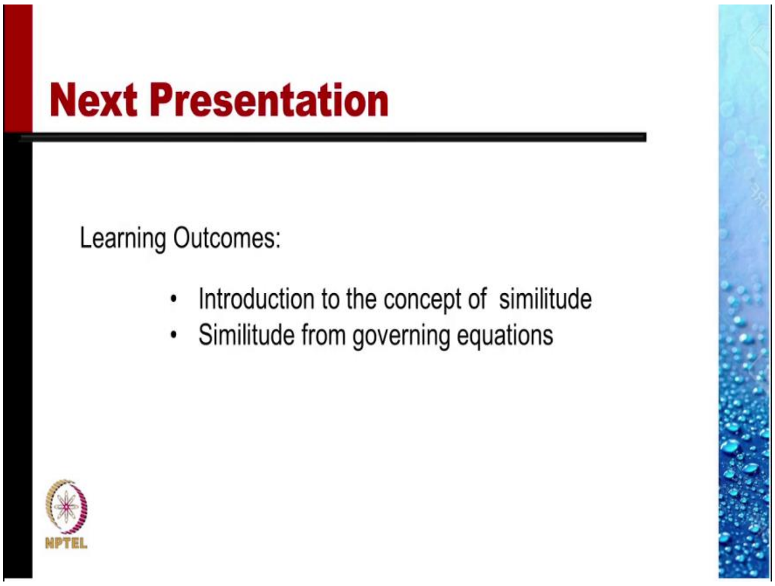

The flow through an orifice: the flow cannot turn sharply at a bend. So, this streamline would have a curvature, and this streamline has this curvature. That means the pressure here must be more than the pressure there. There would be a pressure gradient in the normal direction. Pressure on the outside is atmosphere. So, the pressure at the center of the jet would be more than the atmosphere.

It is only later, when the flow has straightens out at a location, which we call Vena Contracta, where the flow has straightened out. And when the flow has straightened out, then the pressure should be equal all across.

Thank you.