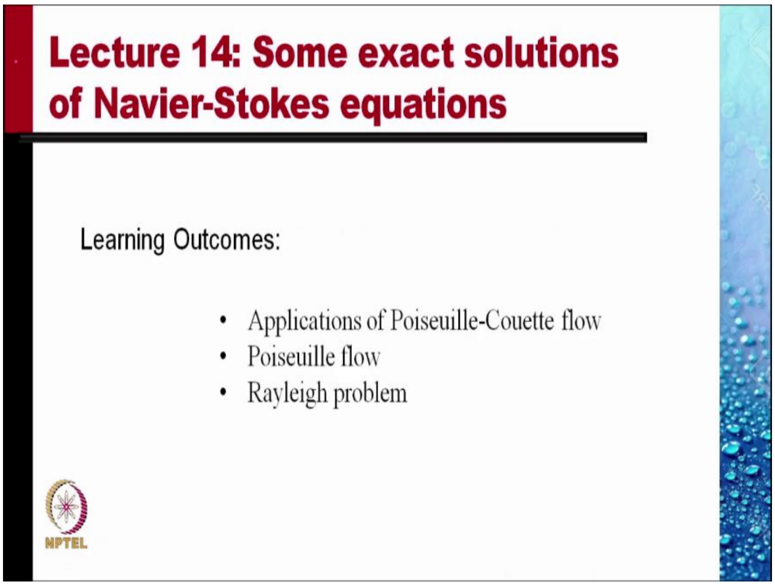
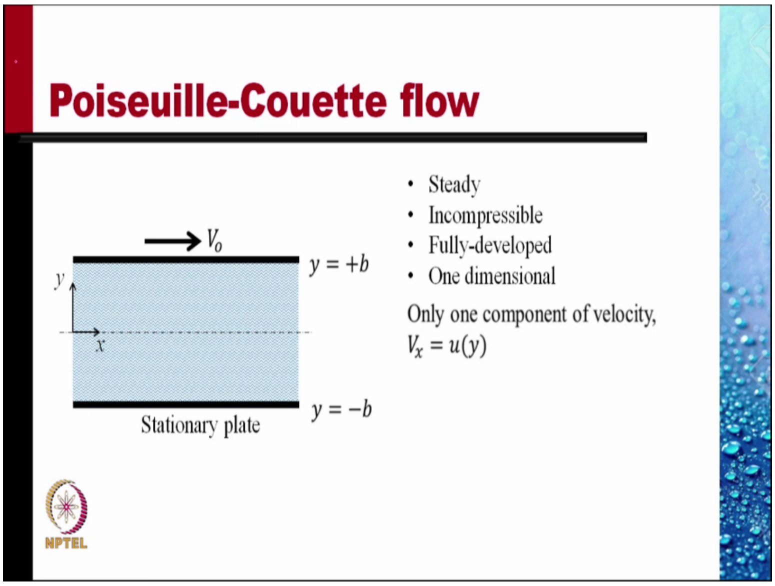
Fluid Mechanics and Its Applications
Professor. Vijay Gupta
Sharda University
Honorary Professor
Indian Institute of Technology, Delhi
Lecture 14
Some exact solutions of Navier-Stokes equations

(Refer Slide Time: 0:20)

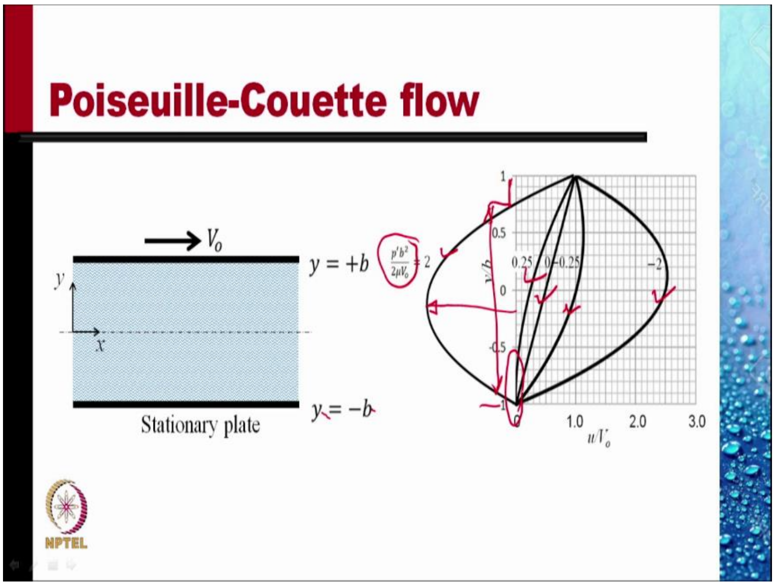




Welcome back. We continue with some exact solutions of Navier-Stokes equations. We have done the Poiseuille-Couette flows in the last presentation. The Poiseuille-Couette flow is the flow of liquid between two parallel plates, one plate being stationary, the other plate moving with a constant velocity V_o, and an applied pressure gradient along the length of the tube.

We assumed the flow to be steady, incompressible, fully-developed and one-dimensional. Under these assumptions, we showed that there is only one component of velocity V_x, which we denote with u, and it is only a function of y, since the flow is fully-developed there is nothing that depends upon x.

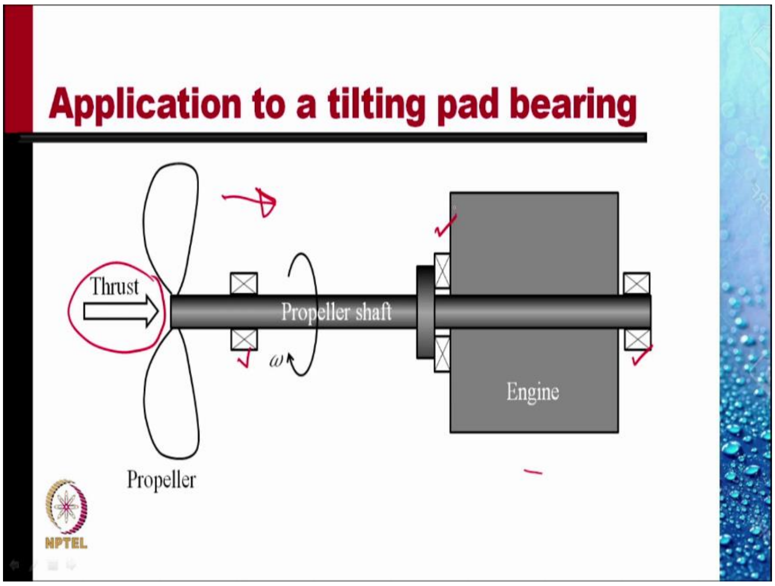
(Refer Slide Time: 1:21)



The velocity profiles were shown to be like this for the various values of the pressure coefficient, a non-dimensional number. When there was no pressure gradient along the tube, then the profile was a straight line profile. For negative pressure gradients, the velocity profiles are like this, and this. For positive pressure gradients, that is, the pressure increasing downstream, the velocity profiles are pushed back, this, and this.

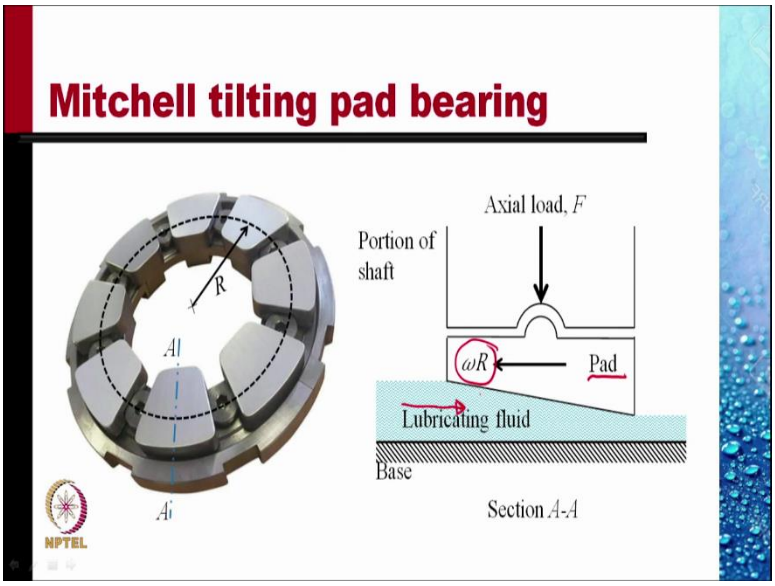
When the pressure coefficient has a value of 2, we notice here, that there is large depth of the flow across which the velocity is negative. It is only in this portion near the upper plate that the velocity is positive. The profile with a pressure coefficient of $+0.25$ has a special value, and that is, the shear stress at the lower plate is 0. $\frac{du}{dy} = 0$ at the lower plate. Let us apply this flow to a problem of thrust bearings.

(Refer Slide Time: 3:10)



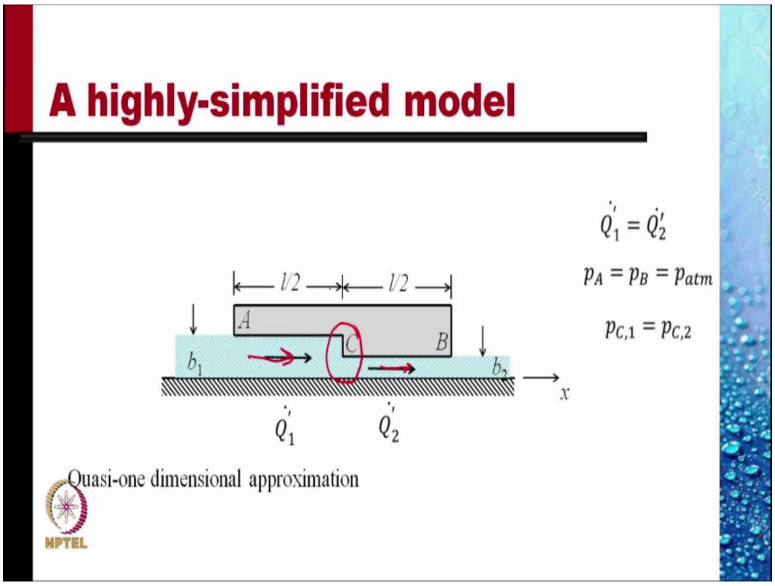
If we had a propeller shaft that was rotating, a propeller of a boat, we would need bearings so that the propeller shaft could rotate. But this propeller shaft would produce a thrust. Because of this thrust, the propeller shaft would be pushed back, and this would need a restraining device while the propeller is running, and this is provided by what is called a thrust bearing. This is the thrust bearing which permits the shaft to rotate easily with very low friction while resisting the axial force.

(Refer Slide Time: 4:20)



One of the common design of a thrust bearing is a Mitchell tilting pad bearing. This works on the principle of Couette flow. It consists of a number of pads like this, which are pivoted in such a manner that when the axial load comes on this and they are rotating, the pad is rotating in this direction with an angular velocity ω, so that the linear velocity is ωR, if R is the radius at which the pad is attached. Then it adjusts itself in such an orientation that the lower surface is tilting, and now, the lubricating fluid is pushed across this, in this direction. It forms a channel, which is a converging channel for the flow. Let us analyze this.

(Refer Slide Time: 5:44)

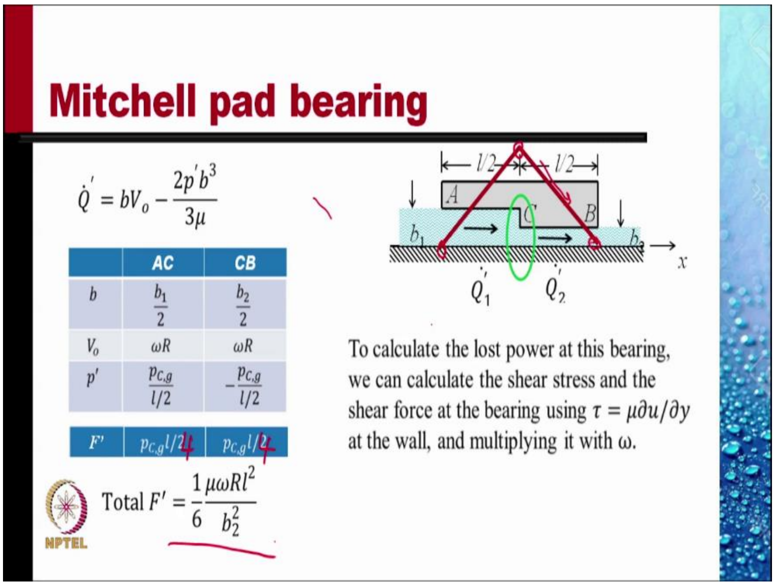


For analysis, tilting is a little more complex. So we convert this into a stepped channel. So, the tilt is replaced by two flat areas AC and CB. So that the first part is a channel of width b1, and the second part is a channel of width b2. Let both of them have equal length $l/2$ each. This is a quasi-one dimensional approximation. We assume the flow through each channel to be one- dimensional. Obviously, at the center, the assumption will break down in this will be quite complex.

If the length l is much larger than the width b, then this approximation could work. Now, we assume that a pressure gradient is created in this channel in such a manner that the flow through each part of the channel is the same. In a steady state, the flow through part 1 of the channel should be the same as the flow through part 2 of the channel. $\dot{Q}_1'$ should be equal to $\dot{Q}_2'$. The prime indicates we are talking of per unit depth.

Now, clearly $\dot{Q}_1' = \dot{Q}_2'$. The pressure point A and pressure point B are atmospheric, and so equal. And the pressure at point C should be continuous, that is, pressure at point C in the first part, just before point C, should be equal to pressure at point C in the second part, just after the step.

(Refer Slide Time: 8:11)



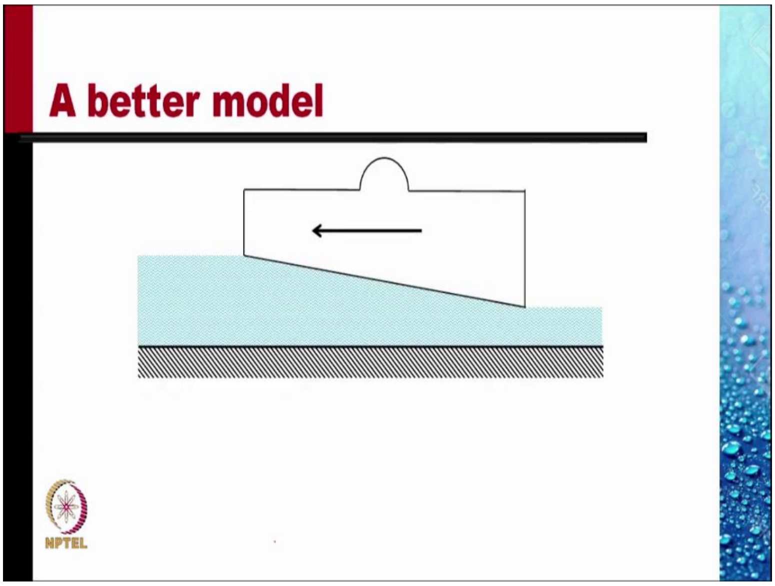
With these conditions now, we have derived in the last presentation, the volume flow rate per unit depth as $\dot{Q}' = bV_o - \frac{2p'b^3}{3\mu}$, where b is half-width of the channel, and p' is the pressure gradient along the length of the channel, μ is the viscosity of the fluid, and V_o is the speed of the upper plate. Now, for the two parts of the channels, AC and CB, we can write the values of b, V_o and p' to be used in there. b the half width for the first channel is $\frac{b_1}{2}$. For the part CB it is $\frac{b_2}{2}$. $V_o$in both cases ωR, p prime the pressure gradient in the first part is $\frac{p_{C,g}}{l/2}$. The pressure changes from 0 to $p_{C,g}$in the length $l/2$. For the conditions of Couette flow, this pressure gradient is independent of the x variable. So we can simply write p' as $\frac{p_{C,g}}{l/2}$. For the second part of the channel, that is part CB, the pressure gradient would be same, but with the opposite sign, $-\frac{p_{C,g}}{l/2}$.

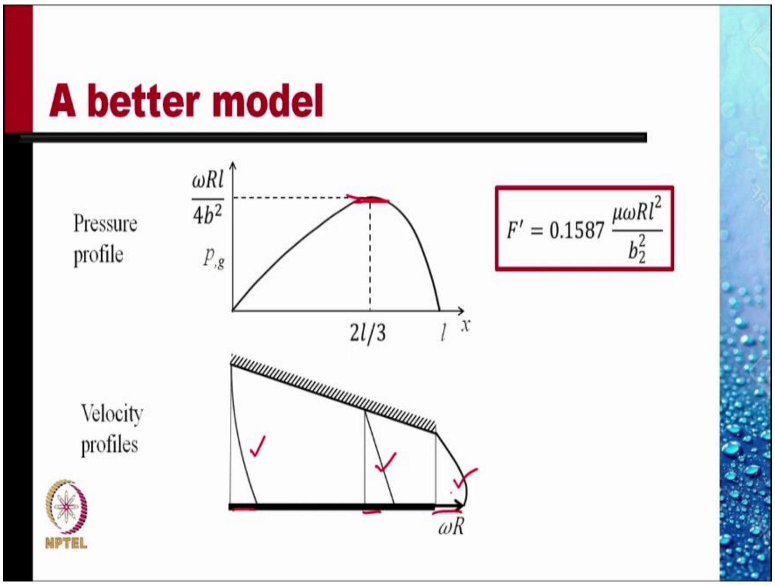
We can find out the values of the flow rate $\dot{Q}_1'$, and we can find out the value of $\dot{Q}_2'$ for the second part, and equate the two. All variables have the known value except for $p_{C,g}$. So, this way we can evaluate the value of $p_{C,g}$. Once we evaluate the value of $p_{C,g}$, we can evaluate the vertical force that it can support. The pressure in the first part AC is rising from 0 to $p_{C,g}$ and the second part the pressure is decreasing from $p_{C,g}$ back to 0.

So, pressure acting on the part AC varies from 0 to $p_{C,g}$ the average pressure is $p_{C,g}/4$, and so this is the force per unit length that we found.. $\frac{p_{C,g}}{2}/2$. The total force that we evaluate comes out to be this. This is the total thrust that this bearing can support when gap b_2 is one half of gap b_1.

To calculate the lost power by this bearing we can calculate the shear stress and the shear force at the bearing using $\tau = \mu \partial u/\partial y$ at the wall. And once we know the shear force, we can find out the torque by multiplying it with R, and then the power by multiplying it with ω. So, after finding the shear stress, we find out the shear force, then the torque, and then the power.

(Refer Slide Time: 12:34)



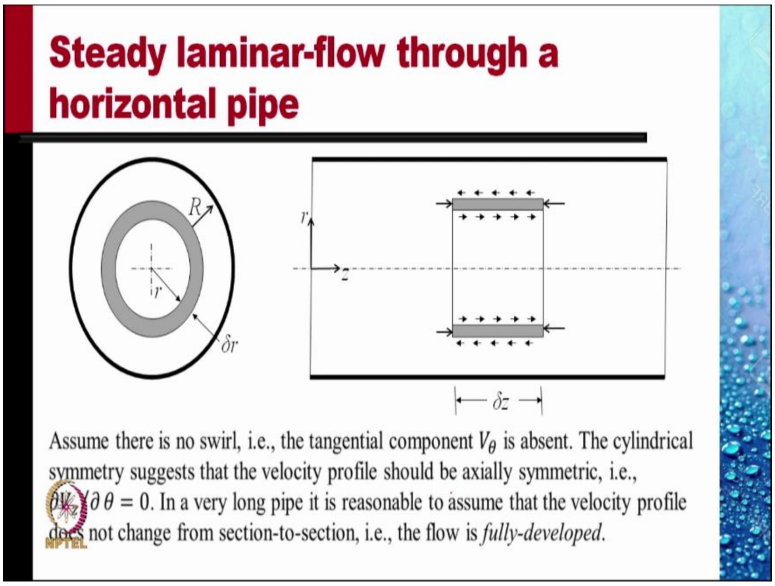


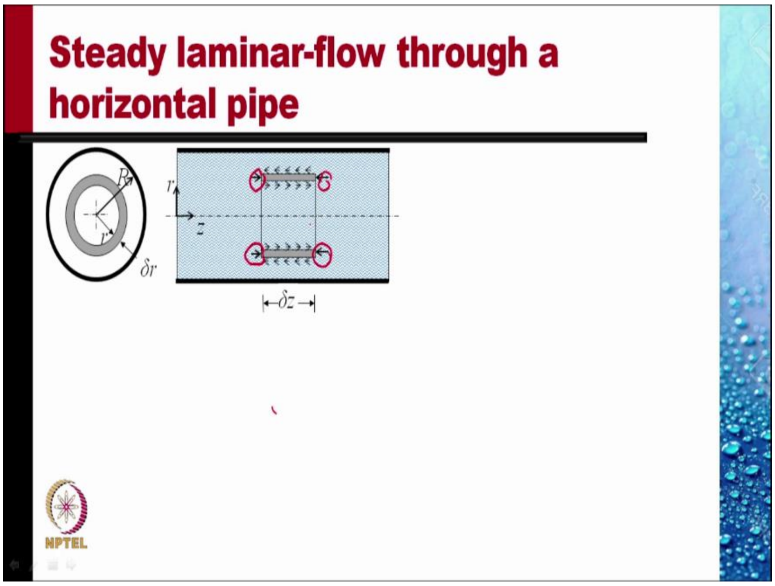
This was a very drastic model that we used in which we used one step. We could improve the model by considering a very large number of steps. We can take each step of length dx and calculate by equating the flow rate through each of those parallel-walled channels. And if we do this, we get a pressure profile like this. In this the maximum pressure is equal to $\frac{\omega R l}{4b_2^2}$ and occurs at two-thirds of the length from the left end. b is the average depth of the channel.

The velocity profiles through the channels look like this. In the first end, where we have a positive pressure gradient, we have a velocity profile that looks like this. Compare it with the pressure with the velocity profile that we are drawn earlier for positive pressure gradients. At x is equal $2l/3$, the pressure gradient is 0, and so we have a straight line profile. In the later third of the channel, the pressure gradient is negative, and we have a velocity profile which is typical of the negative pressure gradients.

The force that this bearing can support, the thrust, is $0.1587\ \frac{\mu\omega R l^2}{b_2^2}$. This is not very different from the estimate we made by a very crude model of the bearing. Of course, we notice that F' increases as μ increases, as the viscosity of the fluid, the lubricating fluid in the bearing increases, the load that the bearing can support would increase. But there would be a cost. This larger μ would result in larger shear stresses, larger torques, and larger power lost in rotation.

(Refer Slide Time: 15:39)



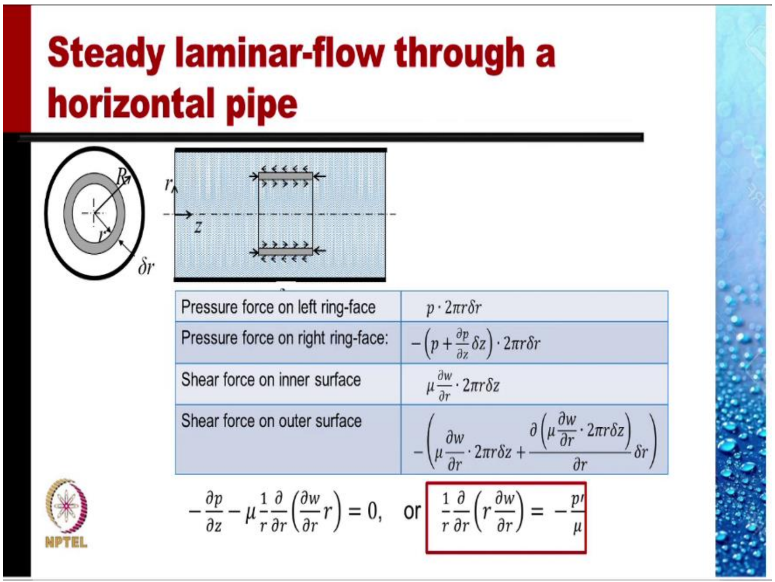


Let us now do one more example, steady laminar flow through a horizontal pipe. Consider a long, a very long pipe of radius R. Of course, the cylindrical polar coordinates would be preferred coordinates here. So, we take z coordinate along the axis of the pipe, and we measure the r radial distance from the center line. We assume there is no swirl, that is, the tangential component of V_θ is absent. The cylindrical symmetry suggests that the velocity profile should be axially symmetric, that is $\partial V_z/\partial \theta = 0$.

In a very long pipe, it is reasonable to assume that the velocity profile does not change from section to section, and the flow is fully-developed. We are discussed this in an earlier lecture too. To find out the velocity profile, we consider a small ring-shaped control volume as shown. A ring of radius r and thickness δr and of length δz. And we draw the forces acting on this. There would be pressure forces on the flat faces of the rings at the two ends, and the shear forces on the lateral surfaces of the ring.

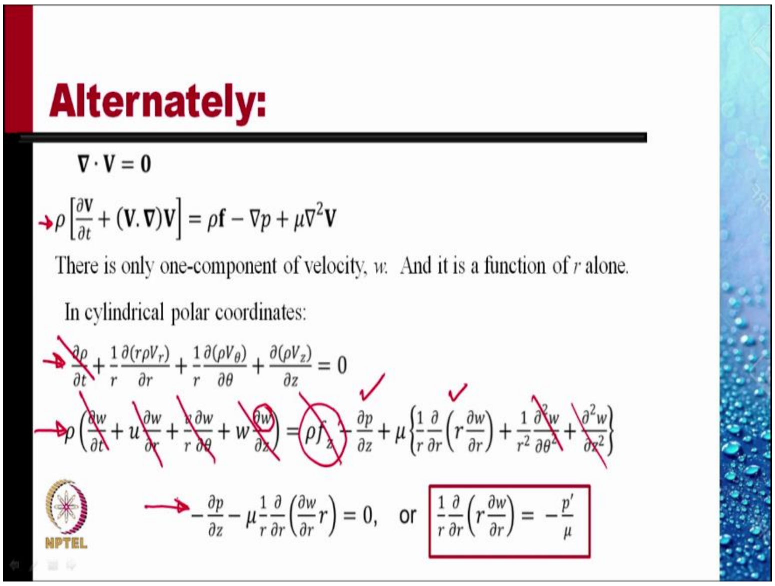
On the inner surfaces, the shear stress would be towards the right, because fluid inside this ring would be moving faster than this fluid. So, the shear stresses would tend to pull it towards the right. But the shear stresses on the outside would be to the left, trying to slow it down.

(Refer Slide Time: 18:19)



So, we will write the forces. The pressure force on the left ring face would be the pressure at that location, p times the area of that ring which is $2\pi r\delta r$. The pressure force on the right ring face is negative, and is obtained using the Taylor series expansion of the above result in the z direction. So it is $-\left(p+\frac{\partial p}{\partial z}\delta z\right)$, multiplied by $2\pi r\delta r$, the area of the ring face. Shear force on the inner surface is $\mu\frac{\partial w}{\partial r}\cdot 2\pi r\delta z$.. Shear force on the outer surface is obtained by Taylor's expansion of this alone. Now, notice that r also varies. So, we make a Taylor's expansion in which we take this whole expression. So here $\frac{\partial w}{\partial r}$ is changing as well as r is changing, w represents the velocity in the z direction. Now, since the velocity does not change in the fully-developed part of the flow through a pipe, there is no acceleration of the fluid. The velocities at any location z are the same. So, there is no acceleration, convective acceleration as well as the temporal acceleration $\partial w/\partial t$. So, the sum of all the forces should be 0. So if we take the sum of all these forces, we get this result. And this now is the equation that governs the flow through a pipe, a fully-developed steady flow of an incompressible fluid.

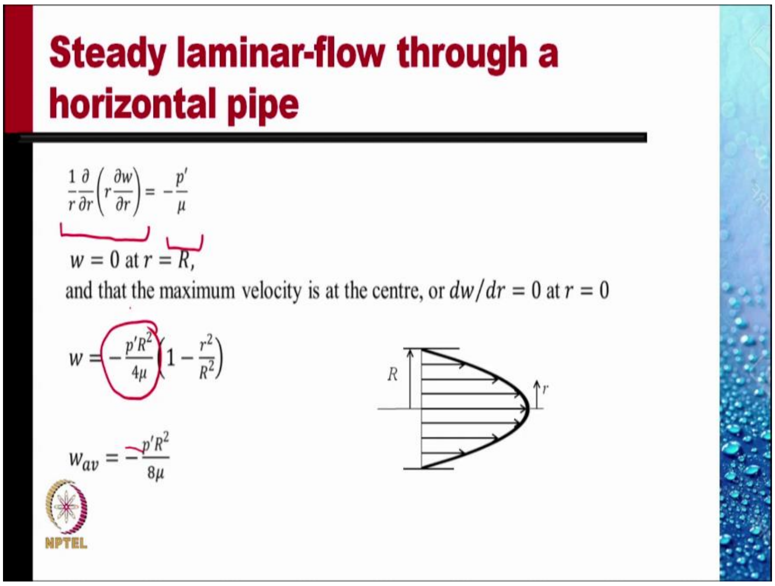
(Refer Slide Time: 21:32)



This could be obtained alternatively from the Navier-Stokes equation written in the cylindrical polar coordinates. The momentum equation in the cylindrical polar coordinates relevant here is only the z equation, because the only velocity component is w, that is, V_z, and it is a function of r alone. So the second equation here is the z component of the Navier Stokes equation in the cylindrical polar coordinates. This is the continuity equation, and with the fact that V_z is a function only of r, and there is no V_r and V_θ, and the flow is steady. This equation is automatically satisfied.

Now, here this term is 0 because the flow is steady. This term is 0 because there is no u, that is component in the r direction V_r. This is 0 because there is no v component, v is V_θ, the swirl component. This component is 0 because w velocity, though nonzero, is not varying with z. This is the body force term. And if we neglect this, we say the pipe is horizontal. Then this can be neglected. $\partial p/\partial z$ z remains. This term remains because w is a function of r. This term is 0 because w is not a function of θ. It just does not change in the circular direction, and it does not change in the z direction. So, this is the only part of the equation that remains. Simplifying we get in this.

(Refer Slide Time: 24:09)



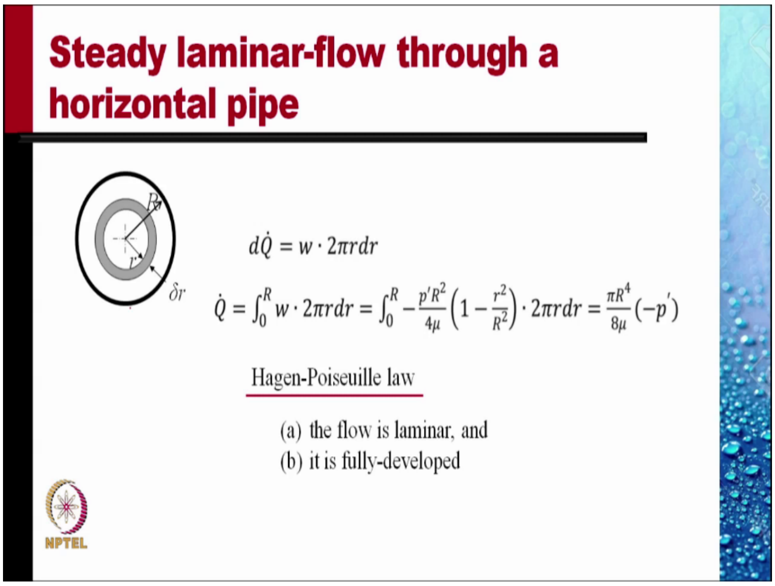
Now, this equation is handled in the same manner as the corresponding equation for the Couette flow. We know that this is a function only of r, w is a function of r. So, this would be a function of r. On the other hand, this p is a function only of z. p does not change with θ and r. So, this is only a function of z. One part is a function r, the other part is a function of z. The only possible way is if both of them are constant.

So, we take $-\frac{p\prime}{\mu}$ as constant, that is, we assume we take the pressure gradient along the z direction to be constant, or the pressure to be linearly decreasing in the z direction. And then, we can integrate this equation to get a variation of w, this w, the z component of velocity with r. This has to be solved with boundary condition $w = 0$, if $r = R$, the no slip condition. At the cylindrical wall the velocity should be 0, and that the maximum velocity at the center of the pipe, that is, $dw/dr = 0$ at $r = 0$.

With these two condition, if we solve this equation, we get the parabolic velocity profile, $w = -\frac{p\prime R^2}{4\mu}\left(1 - \frac{r^2}{R^2}\right)$, R is the radius of the pipe. The famous parabolic velocity profile. We have earlier determined the average velocity.

This is the maximum velocity w_{max} of V_{max}, as we noted. If V_{av} is one half of this, the average velocity in the z direction is $-\frac{p\prime R^2}{8\mu}$, the minus sign because p' has to be negative for the flow to take place towards the right.

(Refer Slide Time: 27:07)



We can find out the volume flow rate like we did before. $d\dot{Q} = w \cdot 2\pi r dr$. And integrating, we find out the volume flow rate is $\frac{\pi R^4}{8\mu}(-p')$. This is known as Hagen-Poiseuille law. We have assumed the flow to be laminar and fully-developed. Fully-developed is the assumption we made explicitly. Where have we made the assumption that the flow is laminar?

We made that assumption when we wrote that the shear stress is μ times the velocity gradient. That is valid only for laminar flows. In turbulent flows, the shear stress is much more complicated. So this Hagen-Poiseuille law is only for when the flow is laminar and the flow is fully-developed. Notice that we have said earlier, that laminar flows through pipes is a rarity. Most commercial pipe operations use turbulent flows.

Also when the flow is laminar, the developing length of the flow, the length through which the flow develops is quite a large value. It could be up to 100 times the diameter. So, the validity of Hagen-Poiseuille law is for a very limited range of applications. Flow through capillaries is one such. We would deal with the flow through pipes for laminar and turbulent flows later in much more details.