Fluid Mechanics and Its Applications
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Lecture 13A
A Simple Solution of Navier-Stokes Equation

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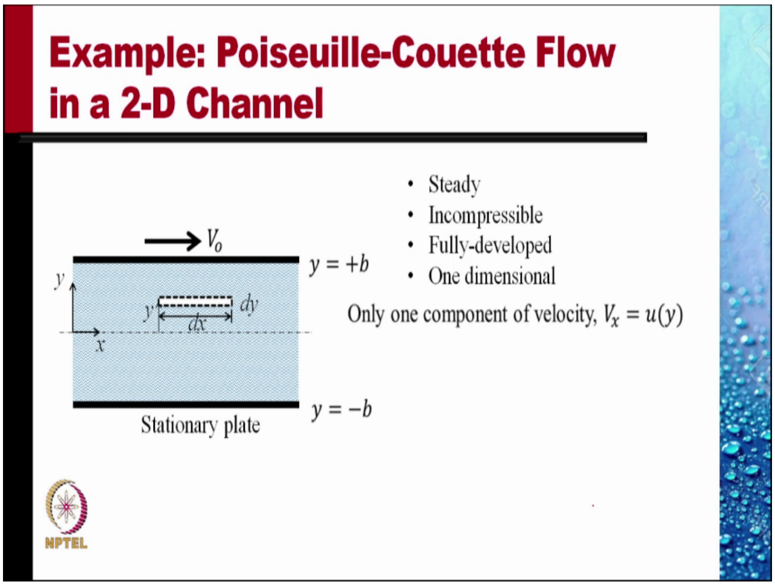

Let us now apply these equations to solve an example. We consider a channel made of two parallel plates with a pressure gradient applied along the axis. This flow is known as a Poiseuille-Couette flow. The plates are infinitely length, $2b$ distance apart. We take the origin in the middle of this channel, so that the upper plate is at $y = +b$ and the lower plate is $y = -b$. Let us assume that the lower plate is stationary while the upper plate is moving with a constant velocity V_o.

We are interested in a steady flow, so this plate has been moving for a long time, infinitely long plate; we assume the flow to be steady, incompressible, fully-developed, no changes in the x direction. There is an interesting argument about this, since this channel is infinitely long so there is no natural origin within this channel, no natural origin in the horizontal direction. If I close my eyes for one second and somebody moves the channel in the time being, and I reopen my eyes I will not be able to tell that the channel is moved.

This implies that the flow pattern that we see within the channel should not depend upon the location of the origin. It should be same everywhere. It cannot be a function of x. The flow is one-dimensional. There is only one component of velocity and that is in the x direction, V_x and we label it as u. And this is a function of only one space variable y. So, u a function of y is the only component of velocity.

Now let us consider a small element or a control volume within this of length dx and height dy.

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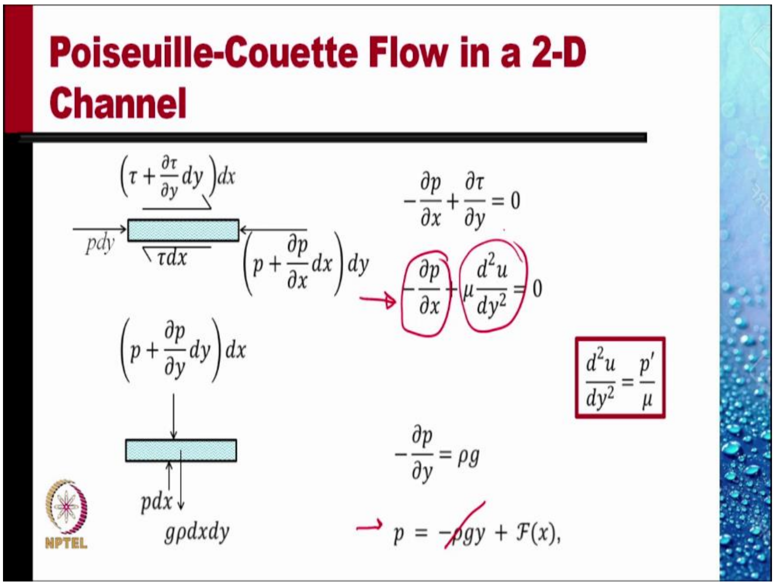

Let us take this out. There is a pressure force pdy acting on this face. By Taylor's expansion, the force on this face is $\left(p+\frac{\partial p}{\partial x}dx\right)dy$. The shear stress τdx on this side. The shear stress by Taylor's expression $\left(\tau+\frac{\partial\tau}{\partial y}dy\right)dx$ on this side. These are all the external forces applied on this, and so the force balance gives you $-\frac{\partial p}{\partial x}+\frac{\partial\tau}{\partial y}=0$. Since there is no acceleration all the forces must sum out to 0.

Now for τ, the shear stress, we plug in $\mu\frac{\partial u}{\partial y}$, by Newton's law viscosity, and so the expression is now this. If I consider the motion in y direction or the equation in y direction, obviously there is no acceleration in the y direction, there is no motion in the y direction.

The forces in the y direction are pdx on the lower face, $\left(p+\frac{\partial p}{\partial y}dy\right)dx$ on the upper face, and the weight of the element: $g\rho dxdy$. And sum of these forces should be 0, and that gives you $-\frac{\partial p}{\partial y}=\rho g$. Simple integration of this would give you $p=-\rho gy$ plus some function of x, because it is only the partial derivative with respect to y is ρg, so when we integrate a function of x or a constant can be added.

Now, let us go back to this equation. If I take the x derivative of this, this will drop, the first term will drop, so $\partial p/\partial x$ would be function of x alone. Now look in this equation. This is a function of x alone and this is a function of y alone, because u is a function only of y. So one term is a function of x alone, and the other term is a function of y alone. Then each of this term must not be a function of x or y, but should be a constant. And so we can equate the two, $\frac{\partial p}{\partial x}$ is replaced by p', and we get $\frac{d^2u}{dy^2} = \frac{p'}{\mu}$, and which is a constant. $\frac{p'}{\mu}$ is a constant, that means p' is a constant. dp/dx is constant, or the pressure gradient in the streamwise direction is constant. The pressure is varying linearly. This equation which governs the flow through this channel can be integrated rather easily.

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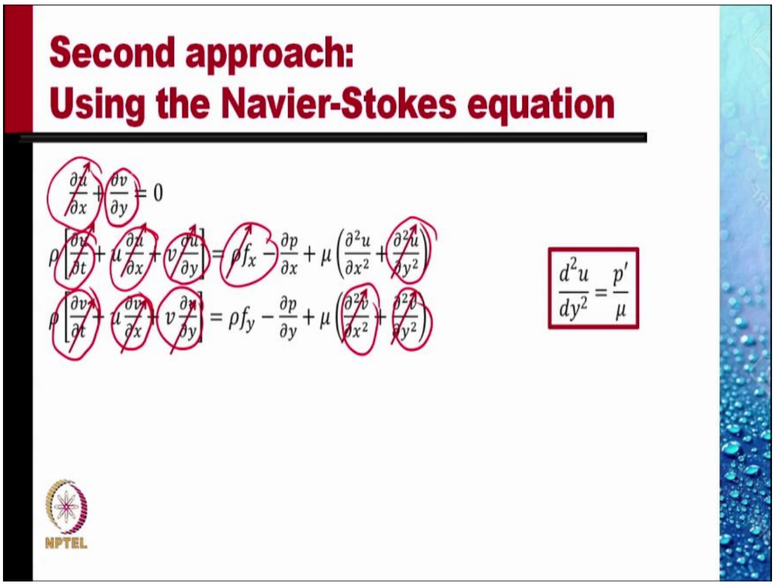

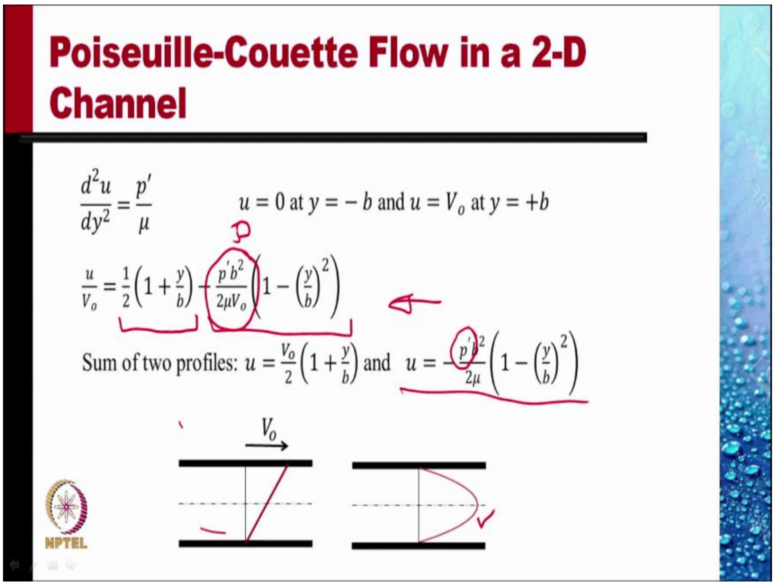

There is another approach that we can take using the Navier-Stokes equation that we had obtained earlier. When we have two dimension, x and y, as in this case, it is a two dimensional case, there are lot of terms that disappear. This term disappears because u is not a function of x. This term disappears because the flow is steady. This term disappears because u is not a function of x. This term disappears because v is 0, there is no vertical component of velocity. This term disappears because there is no body force in the x direction. There is body force only in the y direction. This force disappears because u does not vary in the y direction.

This term disappears because of the steady flow, or rather there is no v component of velocity. These terms also because there is no component of velocity. Because this $\partial u/\partial x = 0$, so $\partial v/\partial y = 0$. That means v is not a function of y. v at the lower plate is 0, v at the upper plate is 0, and v is not a function of y, so that means v must be 0 everywhere. So, after dropping all these terms, we recover the same equations as we had in the last slide, $\frac{d^2u}{dy^2} = \frac{p'}{\mu}$, same equation as in the last slide.

This equation is to be solved with the boundary conditions $u = 0$ at $y = -b$, that is, the lower plate is stationary, and because of no-slip, the velocity of the fluid at that plate is 0. And $u = V_o$ at $y = +b$, the upper plate is moving with the velocity V_o, and because of no-slip the velocity of fluid near the upper plate must be equal to the velocity of the plate V_o.

If we use this boundary condition to solve this simple linear equation, we get this solution. This solution has two parts, one, that does not depend upon the pressure gradient, and the other, that depends on the pressure gradient. The first part is a linear velocity profile, because of the motion of the upper plate that we have discussed earlier. The velocity varies from 0 at the bottom plate, linearly to the velocity V_o at the upper plate. The second part is the velocity profile between two parallel plates, because of a pressure gradient p'.

To obtain this flow, p' should be negative, so $-p'$would give you a positive velocity. This you can see is a parabolic velocity profile with the maximum at the middle. The velocity profile here is the combination of the two. Depending upon the various values of $p'$and V_o, we can have different profiles. This parameter here, which we denote as $\mathcal{P}$ determines the relative importance of the linear profile and the parabolic profile. The profile with the linear flow is called the Couette profile, and the parabolic profile is known as the Poiseuille profile because of the pressure gradient.

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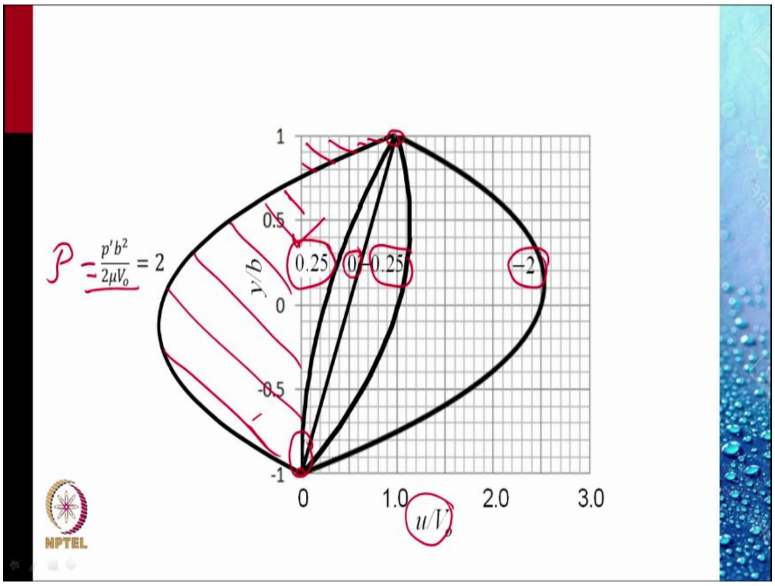

For different values of $\mathcal{P}$, we can plot the various curves. The straight line curve in the middle is for $\mathcal{P}$ is equal to 0, linear. The velocity u/V_o varying from 0 to 1. For the value of this parameter, $\mathcal{P}$ negative. That means pressure decreasing in the downward direction, for the value minus 0.25, we get this profile, and for minus 2 we get this profile, skewed parabolic profiles. For positive value p', that is, for the pressure increasing in the downward direction.

The profiles are shown on the left. This is for $\mathcal{P} = 0.25$; this is significant because at this value of the pressure gradient parameter, the slope of the velocity profile at the lower plate is 0. That means at this pressure gradient, the lower plate does not experience any shear stress. If the velocity gradient is 0 that means the shear stress is 0. So there is no shear stress on the lower plate.

For $\mathcal{P}$ greater than 0.25 we have a region of negative velocity. That means, the pressure drives the flow towards the left in the lower part of the channel, while in the upper part, because of the motion of the upper plate, the flow is in the positive y direction.

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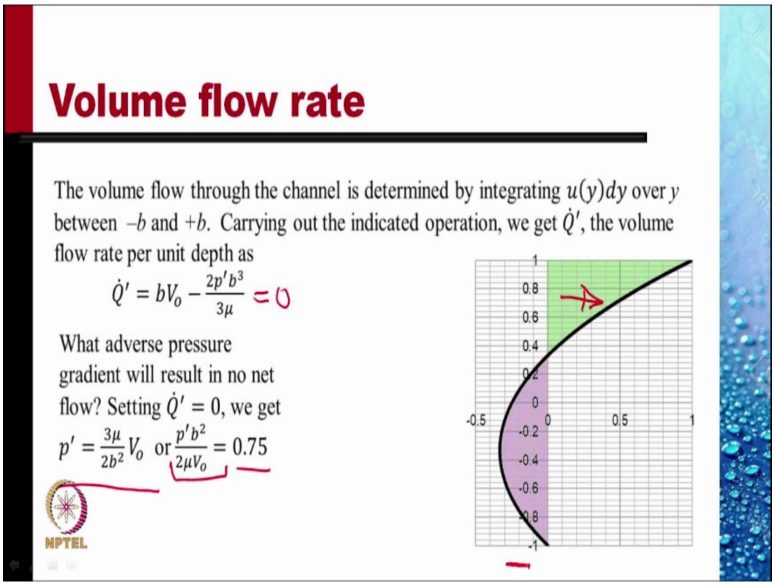

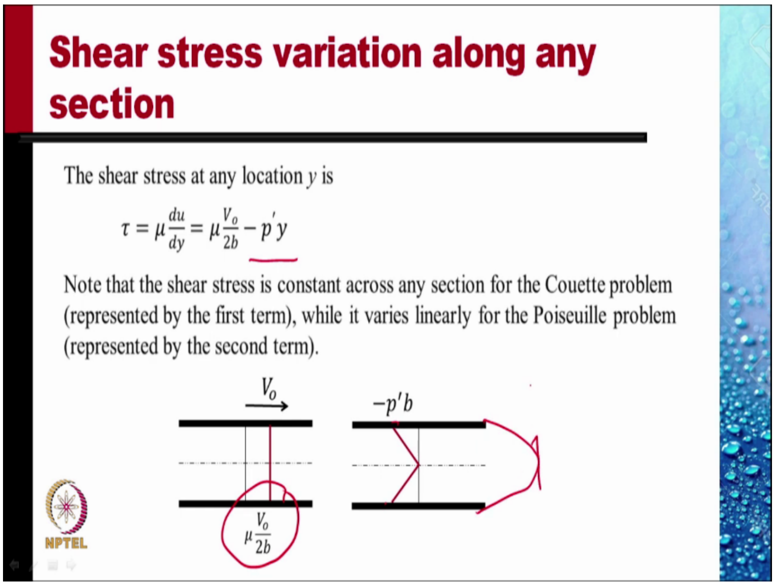

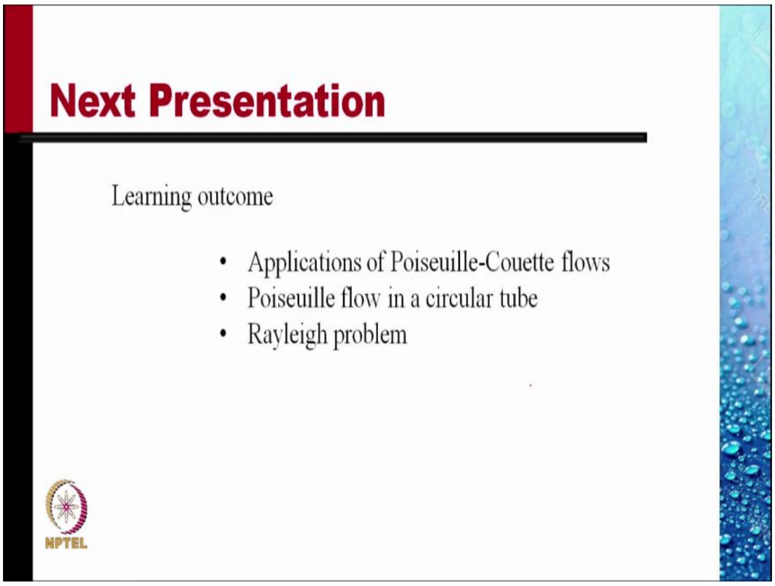

The volume flow rate through this channel is determined by integrating udy over y between $-$b and +b. Carrying out the indicated operation, we get $\dot{Q}'$, that is the volume flow rate per unit depth as $bV_o - \frac{2p'b^3}{3\mu}$. If there was no pressure gradient, then the volume flow rate would be bV_o, and if the upper plate was not moving, the volume flow rate would be $-\frac{2p'b^3}{3\mu}$.

Obviously, this kind of expression raises the possibility that there exists a value of V_o and p' such that there is no net flow across the channel. These can be equated to 0, and then we can determine the value of p'as $\frac{3\mu}{2b^2}V_o$ for which the net volume flow rate would be 0.

A positive pressure gradient pushing the flow backwards, and this gives you the pressure parameter defined earlier as $\mathcal{P} = \frac{p'b^2}{2\mu V_o} = 0.75$. We draw the profile. this is what we have. The flow to the right is exactly equal to the flow to the left. The shear stress variation along any section can be calculated by tau equal to mu du by dy, and that is obtained for the profile that we obtained as mu V naught by twice b minus p prime y.

Note that the shear stress is constant across any section for the Couette problem, that is no pressure gradient but only the upper plate moving with the velocity V naught. Then we have constant shear stress $\tau = \mu\frac{du}{dy} = \mu\frac{V_o}{2b}$. But the shear stress due to pressure gradient varies linearly. It is $-p'b$ at the top and bottom surface, and 0 in the middle. Of course, shear stress should be 0 at the middle, because the parabolic velocity profile is symmetric, so $\frac{du}{dy} = 0$ at the middle.

Thank you.