Fluid Mechanics and Its Applications
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Lecture 13
Momentum Equation

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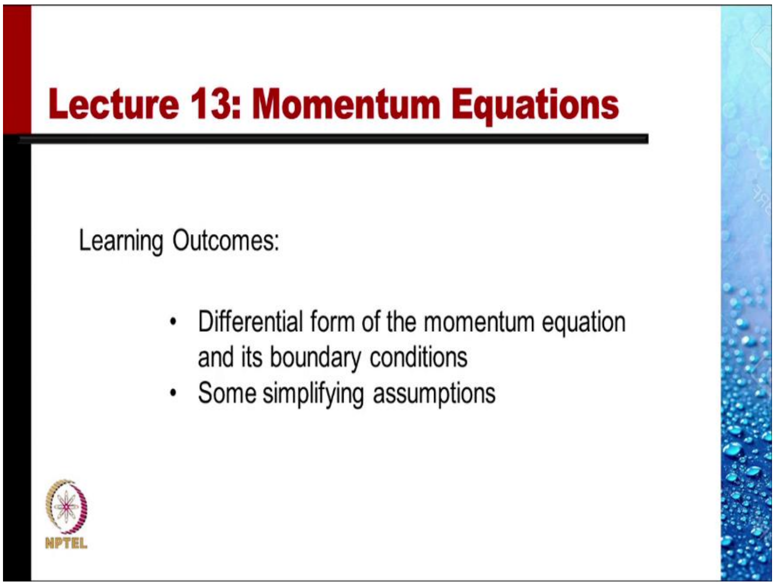

Welcome back. In today's presentation, I would be dealing with the differential form of the momentum equation and its boundary conditions, and then cover some simplifying assumptions.

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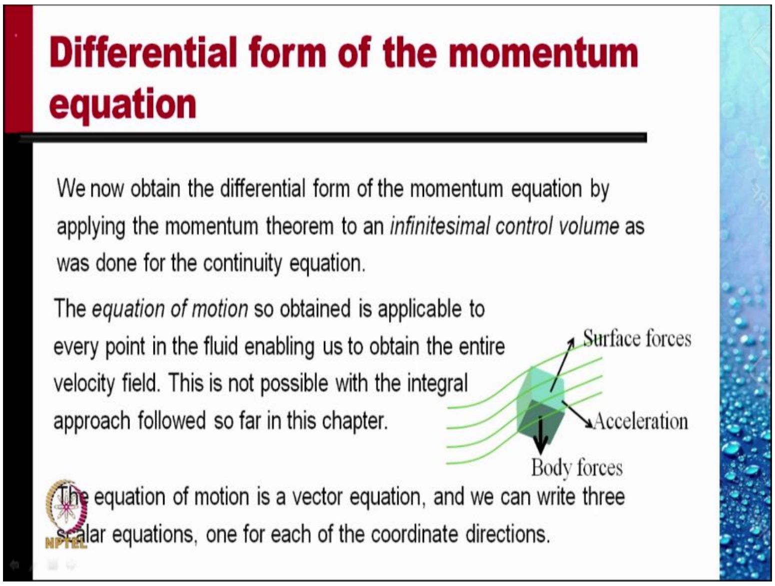

We have so far obtained the momentum equation for a finite control volume. That is a very useful theorem, but it cannot give us the velocity field at every point of the flow. We now obtain the differential form of the momentum equation by applying the momentum theorem to an infinitesimal control volume, as was done for the continuity equation.

The equation of motion so obtained is applicable to every point in the fluid field enabling us to obtain the entire velocity field. This is not possible with the integral approach followed so far. This equation of motion is a vector equation, and we can write three scalar equations, one for each of the coordinate directions.

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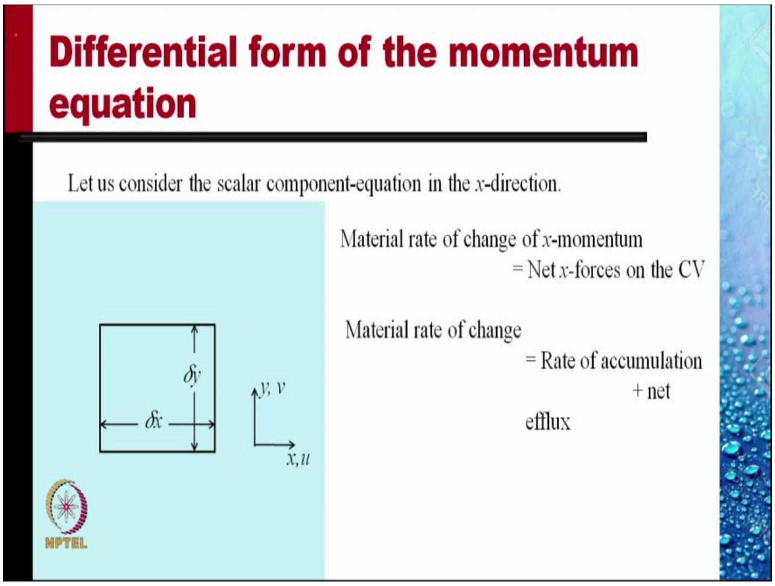

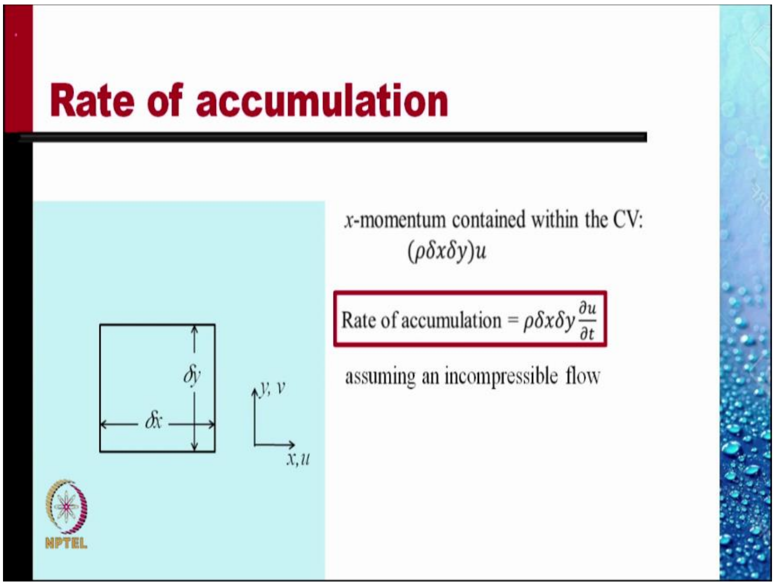

Let us consider the scalar component equation in the x direction, imagine a control volume with dimension δx and δy and unit depth in a two dimensional flow. Let the velocity component in the x direction be represented by u and the velocity component in the y direction be represented by v. Now the momentum theorem states that the material rate of change of x-momentum contained within the control volume is equal to the net x-force applied on the control volume.

The Reynolds transport theorem is used to obtain the material rate of change of momentum in terms of the rate of accumulation momentum plus the net efflux of momentum. Now the x momentum contained within the control volume is $\delta x \delta y$, the volume, times ρ, the mass contained within the control volume, times the horizontal velocity, the x component of velocity u.

So, $(\rho \delta x \delta y)u$ is the x momentum contained within the control volume. The rate of accumulation is obtained by taking the partial derivative with respect to t and this gives you $\rho \delta x \delta y \frac{\partial u}{\partial t}$ assuming an incompressible flow.

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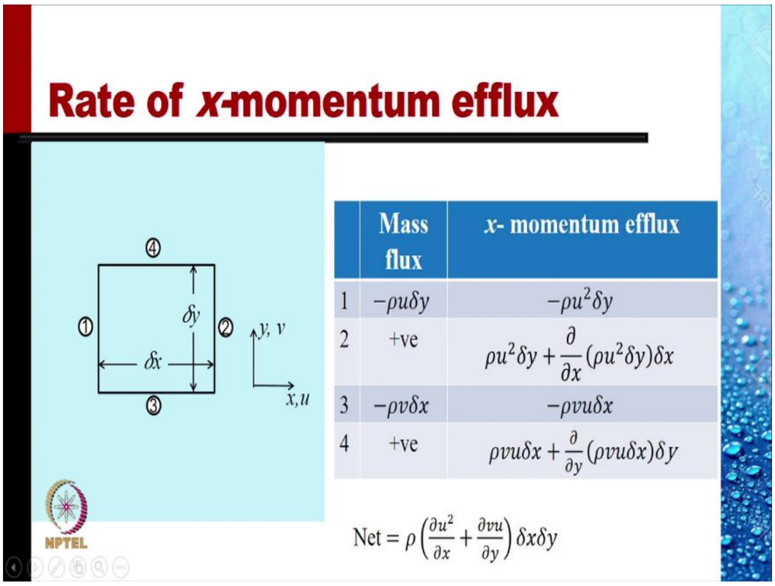

To obtain the x momentum efflux, consider the four segments of the control surface. Segment 1 with its outwards normal in the −x direction, the mass flux across this is $-\rho u\delta y$. We have used a minus sign because this is an influx. We use positive sign for efflux, and negative sign for influx.

So, the x momentum crossing the surface per unit time that is the x momentum efflux, is $-\rho u^2\delta y$. The efflux across surface 2 which is δx distance away from surface 1 horizontally, is positive, and is obtained by using the Taylor series expansion of $-\rho u^2 \delta y$, as $\rho u^2 \delta y + \frac{\partial}{\partial x}(\rho u^2 \delta y)\delta x$.

The mass flux across surface marked 3, which has an outward normal in the $-$y direction is $-\rho v \delta x$, negative because it is an influx. And the x momentum flux across this surface is this mass flux times the horizontal velocity u. So is $-\rho v u \delta x$, and then across surface 4. This is δy away in the y direction, is again obtained by the Taylor series as $\rho v u \delta x + \frac{\partial}{\partial y}(\rho v u \delta x)\delta y$.

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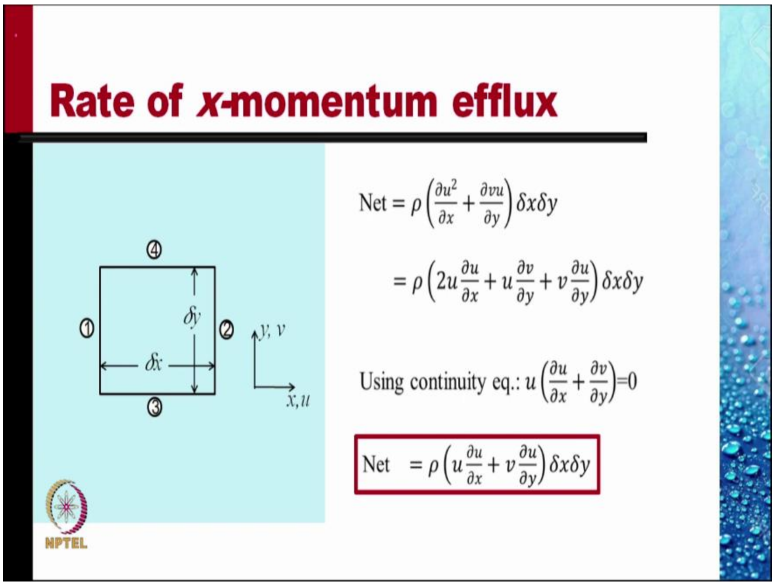

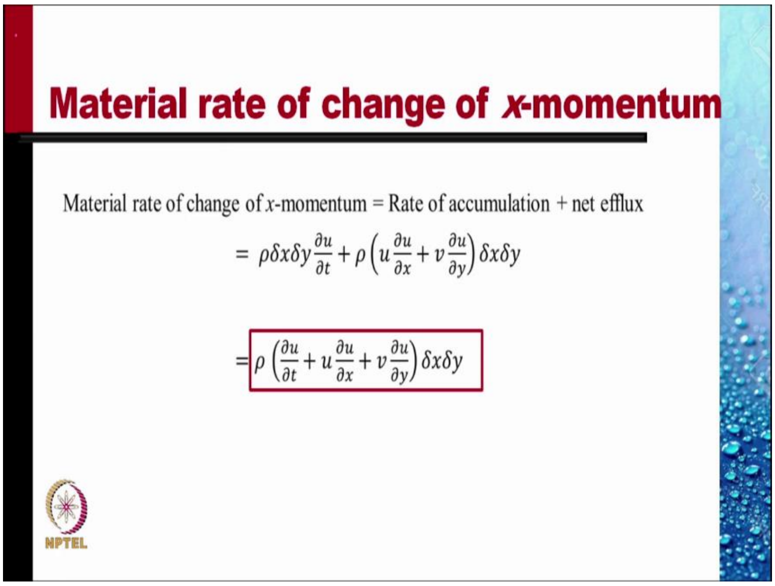

This is net efflux that we calculated, and we expand the term $\frac{\partial u^2}{\partial x}$ and $\frac{\partial v u}{\partial y}$ to obtain the net efflux as $\rho\left(2u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}\right)\delta x \delta y$. Now if we use the continuity equation that we obtained earlier, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. So multiplying this by u, the one of the first term and the second term cancel out, and so we get the net efflux as $\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)\delta x\delta y$.

So, the material rate of change of x momentum associated with the control volume, that is equal to the rate of accumulation plus the net efflux, becomes $\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)\delta x\delta y$ within the red box. This we carry forward. Now we have to find out the forces acting on this control volume.

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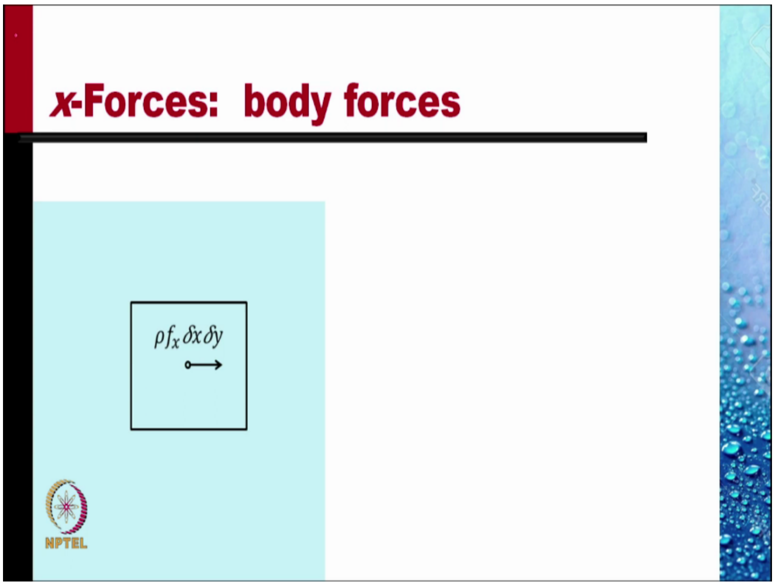

The forces of two kind, body forces and the surface forces. The body forces are given on per unit mass basis, and so the mass of the control volume being $\rho\delta x\delta y$ and if f_x is the component of body force in the horizontal direction per unit mass, then the total body force is simply $\rho f_x\delta x\delta y$.

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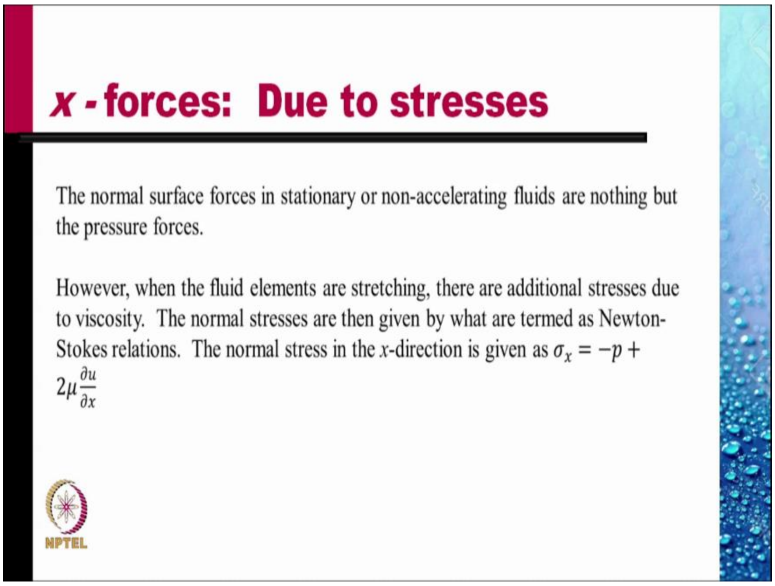

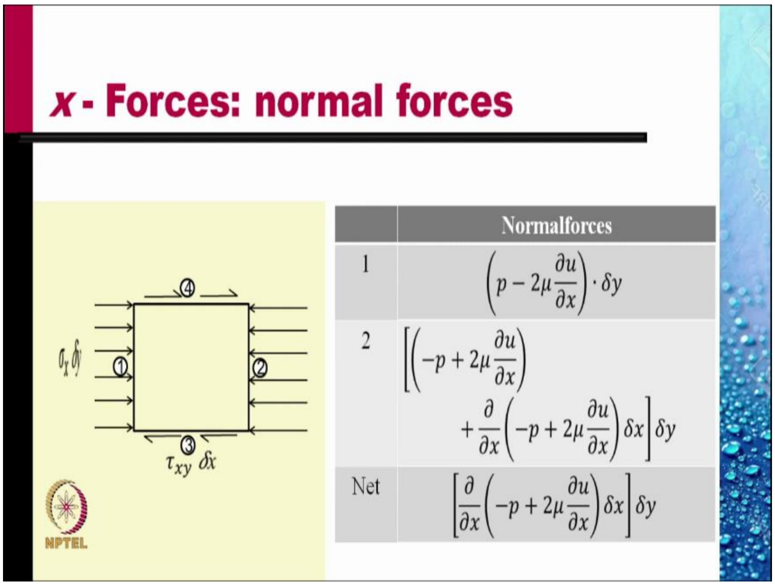

Now let us discuss the surface forces due to stresses. The normal surface forces in stationary or non-accelerating fluids are nothing but pressure forces. However, when the fluid elements are stretching there are additional stresses due to viscosity. The normal stresses are then given by what are termed as Newton-Stokes relation.

The normal stress in the x direction is given as $\sigma_x = -p + 2\mu\frac{\partial u}{\partial x}$. The $-p$, the pressure here is the hydrodynamic pressure, and t$2\mu\frac{\partial u}{\partial x}$ is the normal stress, the normal viscous stress because of stretching of the fluid element in the x direction.

If we analyze this, the normal forces in the x direction can be written like this: on surface 1 with normal in the positive x direction, the left face of the element, the normal forces would be the normal stress $\left(p - 2\mu\frac{\partial u}{\partial x}\right)$ multiplied by the area, which is δy into 1, the unit depth. Similarly, on the surface 2 on the right side, the force is now obtained by Taylor's expansion of the force in 1. This force is towards the left. The force on 1 is towards the right, so the net force is $\frac{\partial}{\partial x}\left(-p + 2\mu\frac{\partial u}{\partial x}\right)\delta x\delta y$.

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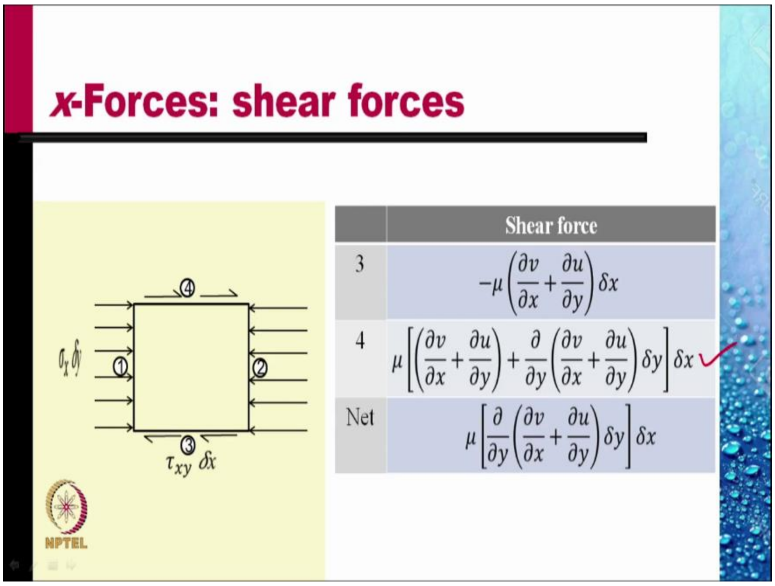

Similarly, the shear forces on surface 3, the bottom surface. The shear force is to the left, the shear stress τ_{xy}, by Newton's law, is $-\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\delta x$, minus sign because it is towards the left. Similarly, on surface 4, the top surface, by Taylor's expansion, we get this quantity. This is towards the right so it is positive; the net of the two is this.

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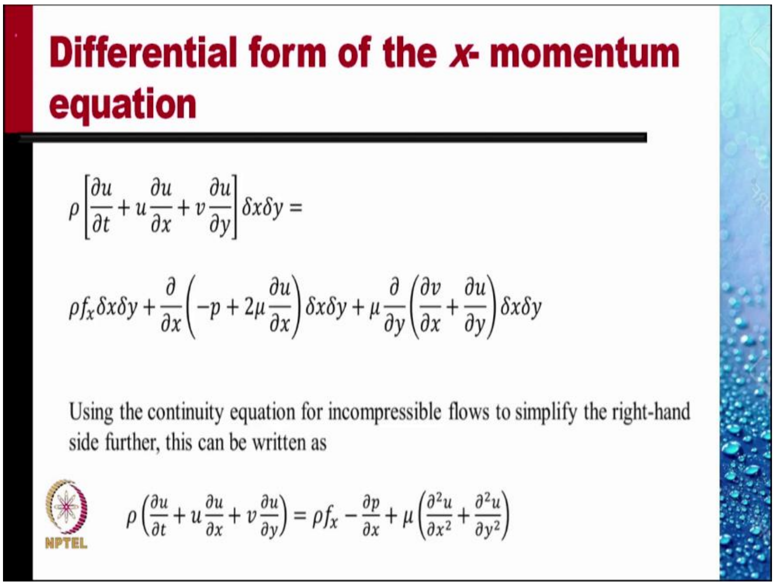

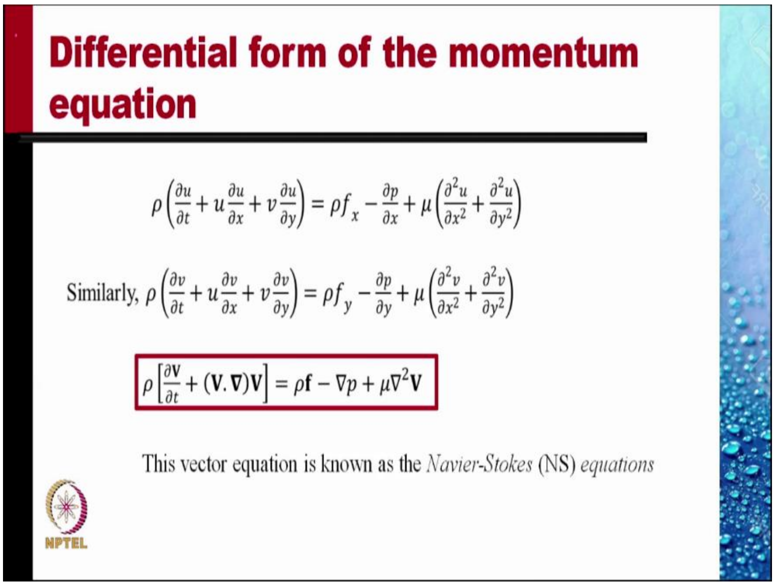

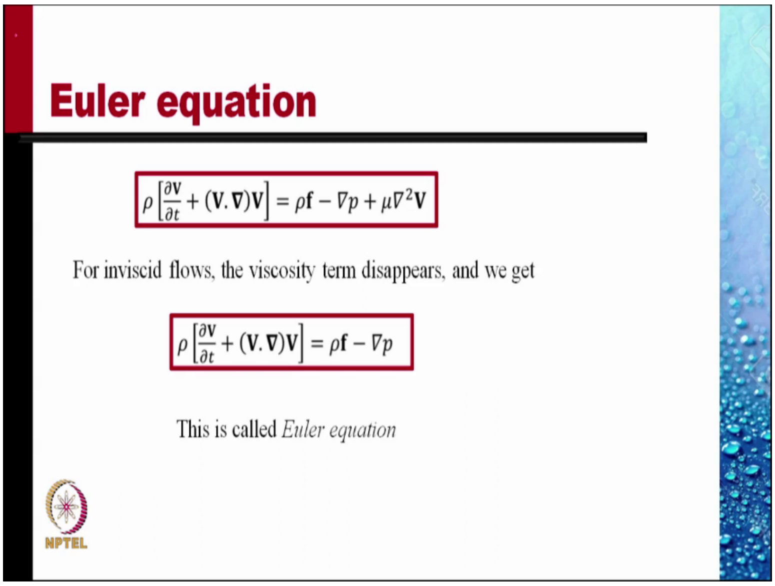

Putting in them all together, the net rate of change of momentum, which is the accumulation plus the efflux term, is equal to the net force. And if we use the continuity equation for the incompressible flow to simplify the right hand side further, this can be written simply as $\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)$ is equal to ρf_x, the body force terms, $-\frac{\partial p}{\partial x}$, which is the pressure force term, plus $\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$. This is the viscous force terms, the net viscous force terms for all the surfaces combined.

We can similarly obtain the equation in the y direction, and the only difference would be that u would be replaced by v aqnd f_x, the body force in the x direction would be replaced by f_y, $-\frac{\partial p}{\partial x}$ by $-\frac{\partial p}{\partial y}$. We could combine these equations into a vector equation, and the vector equation would be $\rho\left[\frac{\partial \boldsymbol{V}}{\partial t}+(\boldsymbol{V}.\boldsymbol{\nabla})\boldsymbol{V}\right]$, where ∇ operator is $\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}$ is equal to $\rho\boldsymbol{f}$ minus ∇p plus $\mu\nabla^2$, the Laplacian of **V**.

This famous vector equation is known as the Navier-Stokes equation, and this is a fundamental equation of fluid dynamics. For inviscid flows, the viscosity term disappears and we get after dropping the last term because of viscous forces, we get $\rho\left[\frac{\partial \boldsymbol{V}}{\partial t}+(\boldsymbol{V}.\boldsymbol{\nabla})\boldsymbol{V}\right]=\rho\boldsymbol{f}-\nabla p$. This is called Euler equation, and is applicable only when the flow can be considered as non-viscous.

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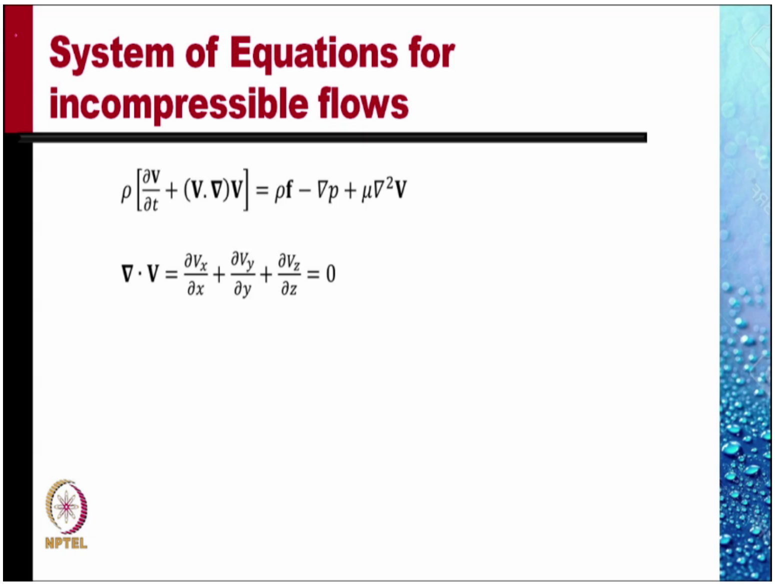

This equation, if we consider this in three dimensions has four unknowns, the velocity component V_x, V_y, and V_z, and the pressure p. But this system of vector equation consists only three scalar equations. So we need a fourth equation and and the continuity equation $\boldsymbol{\nabla}\cdot\boldsymbol{V}=0$

constitutes the fourth equation to complete the system of equations for incompressible flow assuming that ρ is constant throughout the flow.

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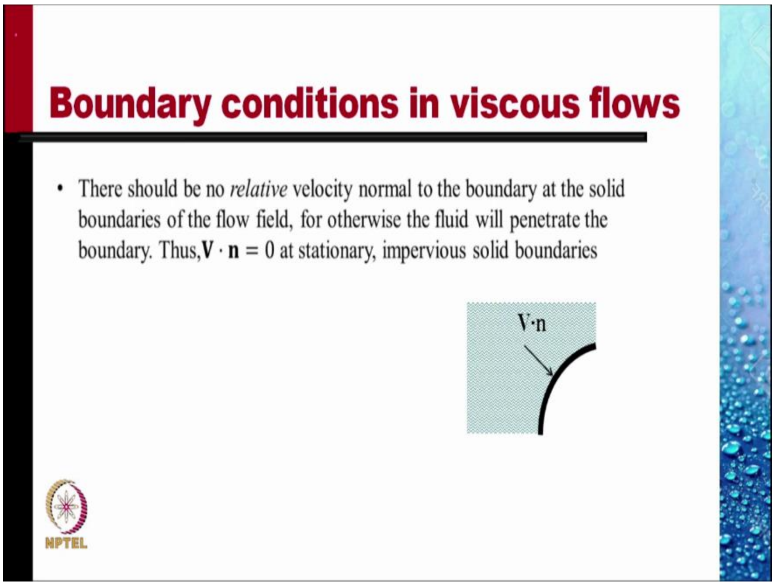

Let us now discuss the boundary conditions applicable to this equation in viscous flows. Clearly, there should be no relative velocity normal to the boundary at the solid boundaries of the flow field, for otherwise the fluid will penetrate the boundary. Thus, $\boldsymbol{V} \cdot \boldsymbol{n} = 0$. No fluid should flow out of a boundary or flow into the boundary at stationary impervious solid boundaries.

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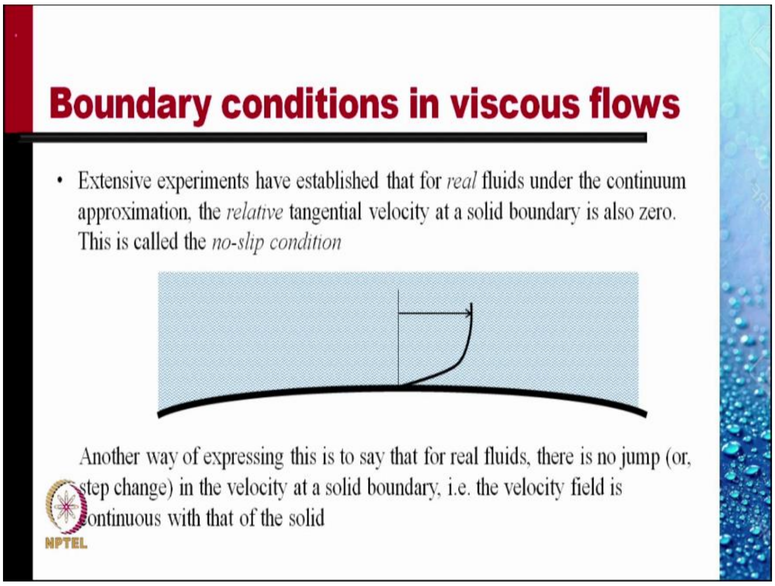

Extensive experiments have established that for real fluids under continuum approximation, the relative tangential velocity at a solid boundary is also 0. This is called the no-slip boundary condition. The fluid may be moving relative to a solid boundary, but at the boundary the velocity of the boundary would be same as the velocity of the fluid, or rather the velocity of the fluid would be same as the velocity of the boundary. If the boundary was stationary, the fluid at the boundary would be stationary too.

There are two exceptions to this, two known exceptions. One is that helium at 4° absolute behaves strangely and it may slip at the boundary. The other exception is for rarefied flows when the density of the fluid is so low that the body dimensions are of the same order as the mean free-path. Then the no-slip condition cannot be applied. There would be slip at the boundary.

Another way of expressing the no-slip condition is to say that for real fluids there is no jump or step change in the velocity at the solid boundary, that is, the velocity field is continuous with that of the solid.

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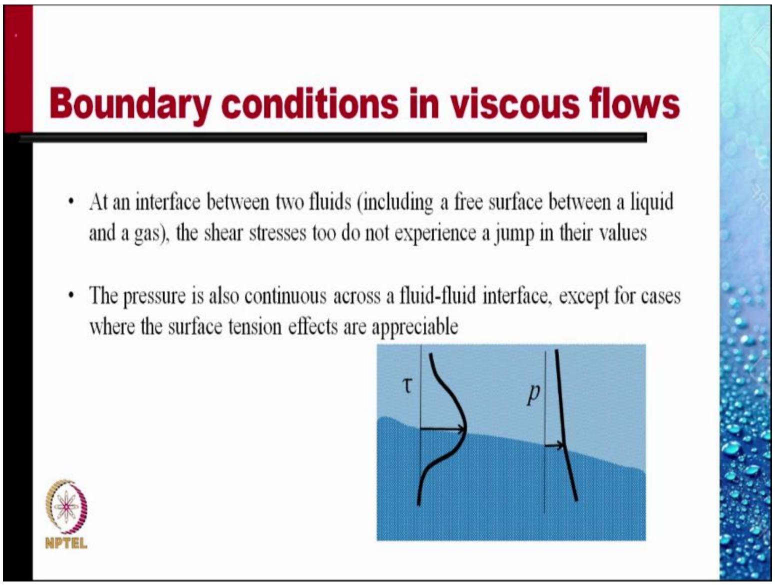

At the interface between two fluids including a free surface between a liquid and a gas, the shear stresses too do not experience a jump in their value. So whatever is the shear stress value at the interface and one fluid, the same shear stress is there in the other fluid at the interface, as shown in the figure here.

The pressure also is continuous across a fluid-fluid interface, except for cases where the surface tension effects are appreciable. So the continuity of pressure can be assumed only if the surface tension effects are absent. In most of the flows in the rest of this course, we will assume the surface tension to be negligible, so that the pressure can be taken as continuous.

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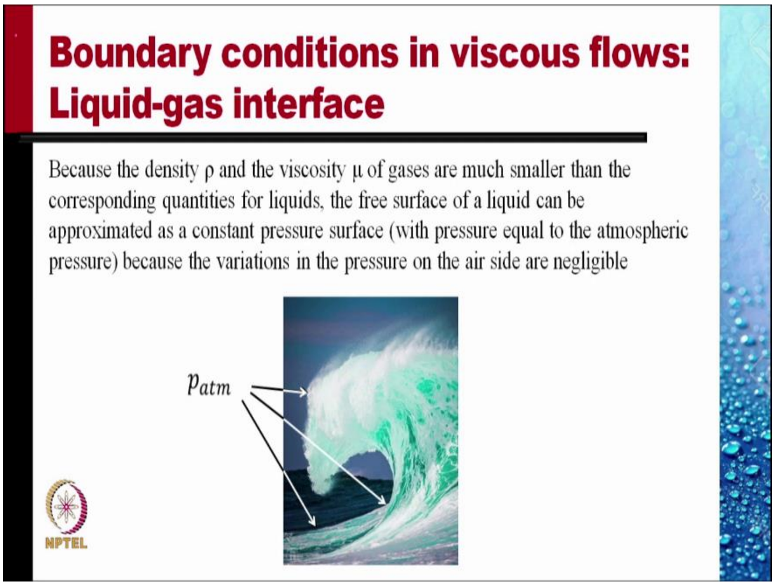

At a liquid-gas interface, because the density rho and the viscosity mu of gases are much smaller than the corresponding quantities for liquids, the free surface of the liquid can be approximated as a constant pressure surface. Because of continuity of pressure, the pressure within the liquid at the interface is equal to the pressure of the gas at that interface, and since the densities of the gas is very low, the pressure variations within the gas are negligible and therefore, everywhere the pressure can be taken as same.

In this picture of a wave in ocean, the pressure in the air can be taken as atmospheric, the same value over the distance of the wave height. And so the pressure within the liquid at all points at the interface at the surface of the wave would be equal to $p_{atmospheric}$.

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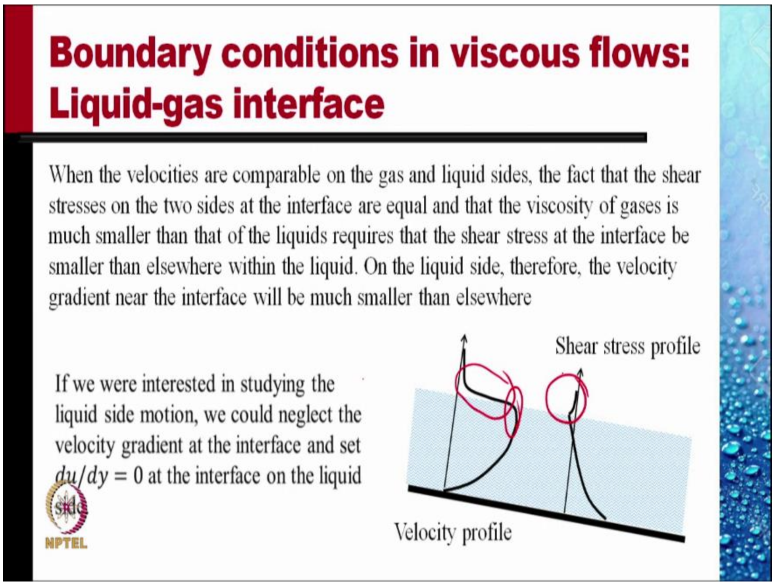

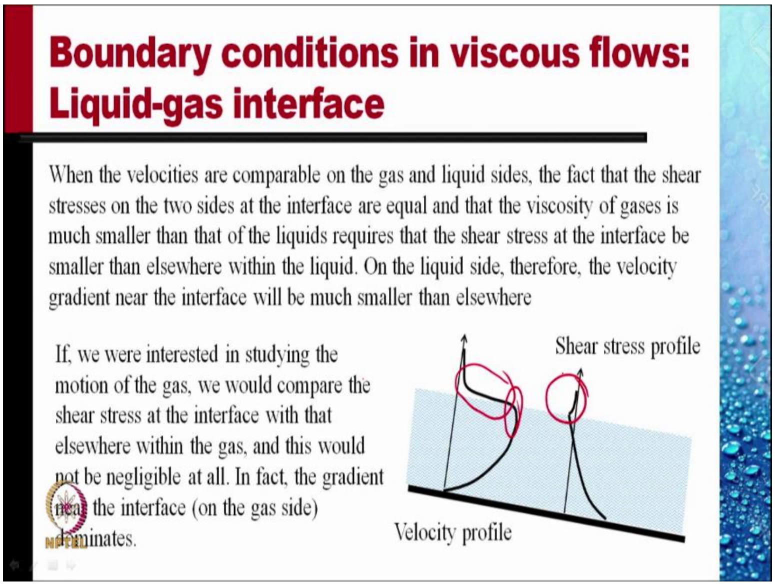

When the velocities are comparable on the gas and liquid sides, the fact that the shear stresses on the two sides of the interface are equal, and that the viscosity of gases is much smaller than of the liquid, requires that the shear stress at the interface be smaller than elsewhere within the liquid. On the liquid side therefore, the velocity gradient near the interface will be much smaller than elsewhere.

In the gas side even with the large velocity gradient, because the viscosity is very small of the gas, the shear stress will be low. So, this velocity gradient is going to be very small. In fact, as one approximation we can neglect this velocity gradient and we can assume that the velocity gradient here is 0.

If we were interested in studying the liquid-side motion, we could neglect the velocity gradient on the interface and set $\frac{\partial u}{\partial y} = 0$ at the interface on the liquid side, as stated earlier. If we were interested in studying the motion of the gas, we would compare the shear stress at the interface with that elsewhere within the gas, and this would not be negligible.

So, while considering the liquid phase, studying the liquid phase, we can assume this stresses to be 0 at the interface, but if we were interested in studying the motion of the gas near the interface, the shear stresses at the interface, within the gas, cannot be neglected. In fact, they would be the highest, and the velocity graded for the maximum there.

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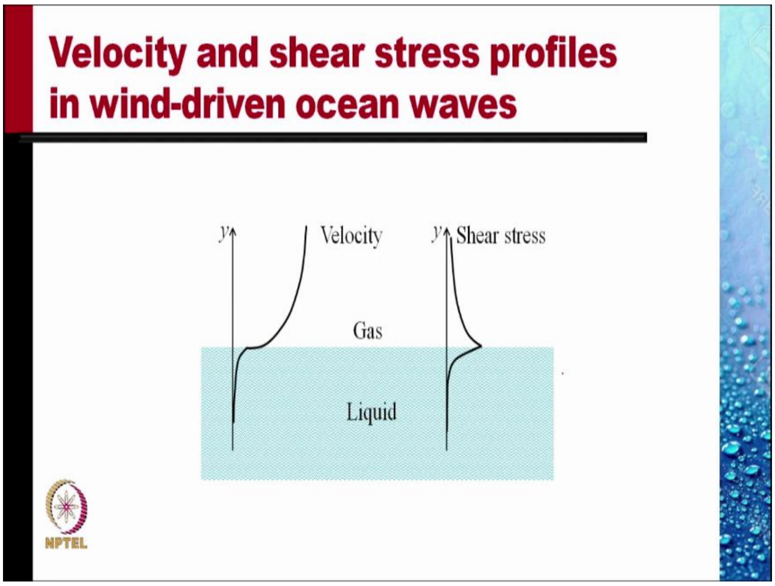

Boundary conditions in viscous flows

Type of boundary	Boundary conditions on: Relative tangential velocity	Relative normal velocity	Pressure	Shear stress
Impervious solid-fluid	Zero (no slip)	Zero	-	-
Fluid-fluid	Zero	Zero	Same on either side (for negligible surface tension)	Same on either side
Approximate for liquid-gas (for liquid side only)	Zero	Zero	$p_{interface} = p_{gas}$ = constant	Negligible, if gas-side velocities are not large

Now let us consider the air driven motion, like the ocean waves with the velocity profile would be as shown on the left. It is the air velocity which is driving the waves in the liquid. Of course, there will be continuity of velocity at the interface, the shear stresses would be maximum at the interface. In fact, it is these shear stresses which are driving the motion in the liquid, and these stresses would slow down the gas, the air, the atmosphere, above the ocean.

We summarize the boundary conditions in a viscous flow by this table. At an impervious solid fluid boundary there is no relative tangential velocity, zero slip. There is no relative normal velocity, impervious solid. There are no conditions on pressure and shear stresses. At the fluid-fluid boundary, the relative tangential velocity is again 0, no slip. Relative normal velocity is again 0. We are again assuming there is no penetration of one fluid into another fluid.

The pressure is same on either side for negligible surface tension. The shear stress is same on either side. Pressures and shear stresses and velocities are all continuous. The approximate condition for the liquid-gas interface for the liquid side only, the relative tangential velocity is 0, the relative normal velocity is 0 again, the pressure is the same, equal to the gas pressure, and gas pressure can be assumed constant, and the shear stresses are negligible if the gas side velocities are not very large. With these boundary conditions, we are now ready to solve problems.

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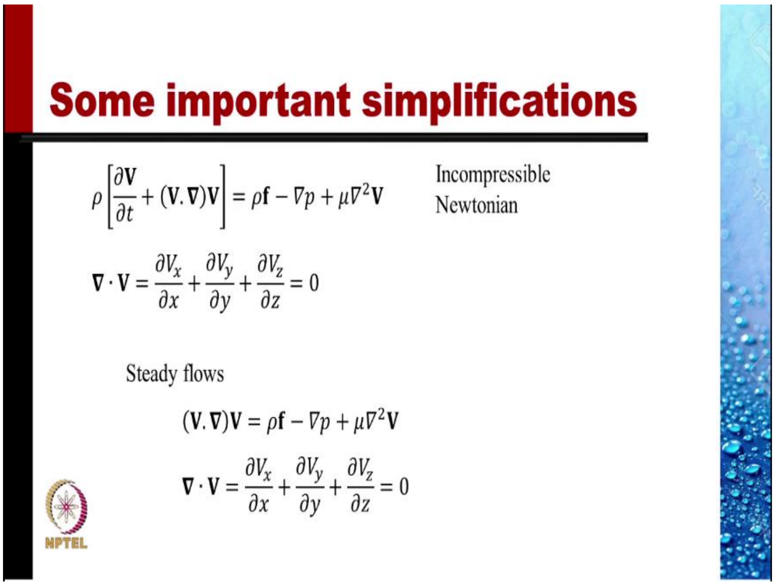

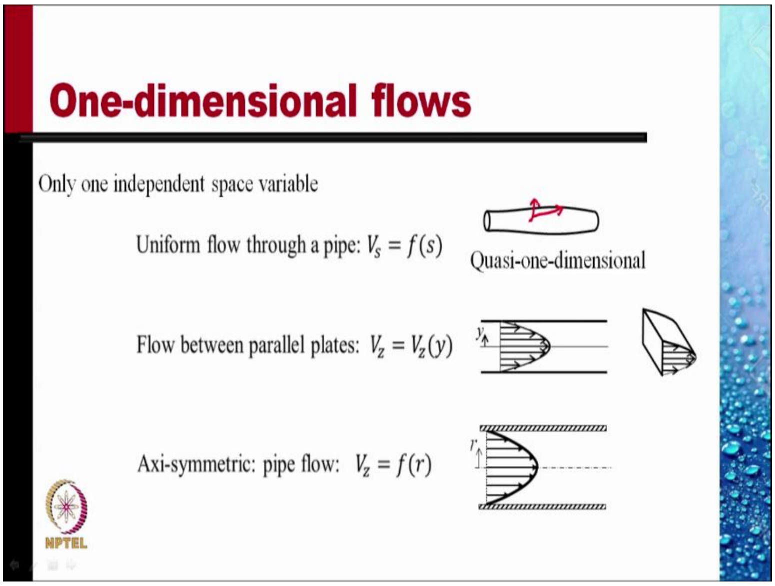

But before that we state some important simplifications that help us solve this equation. This system of four equations, the vector equation for momentum plus the continuity equation is a formidable system, second order non-linear equations, all coupled, it is not easy to solve. So, we look for simplifications wherever possible. First simplification is simply steady flow. The first term in the Navier-Stokes equation drops out, and we get the system with one less independent variable.

Then, simplification because of dimensionality of flows. If we have only one independent space variable, for example, we have flow through a channel, and we can assume the flow to be uniform, then at each section of the channel the velocity is uniform. That is, it does not vary across the cross section of the channel. So we can write V_s, the velocity in the s direction, the downstream direction, is a function only of s.

If it is a uniform channel, then V_s is constant, but if the cross section of this channel varies slightly so that is a very slowly varying channel cross section, then V_s is a function of s. It can vary and by simple application of continuity equation V times area at any section is the same as at any other section.

Such a flow is not truly one dimensionsal, because there has to be a component in vertical direction of the flow here. But if the channel is varying slowly, we can consider the crosswise component of velocity to be negligible, and the variations in the crosswise direction to be negligible, and the flow can be considered quasi-one-dimensional. This had been discussed earlier.

Flow between two parallel plates. The distance between the plates is constant and so there is only one component of velocity V_z, which is a function only of y, the distance in the vertical direction. It is not a function of x. In an axi-symmetric flow in a pipe there is only one component of velocity V_z, and this is a function only of r, the radius, that is the distance from the central axis. It does not depend upon the angle, it does not depend upon z. So, there is only one component of velocity V_z, and which is a function only of one independent space variable r.

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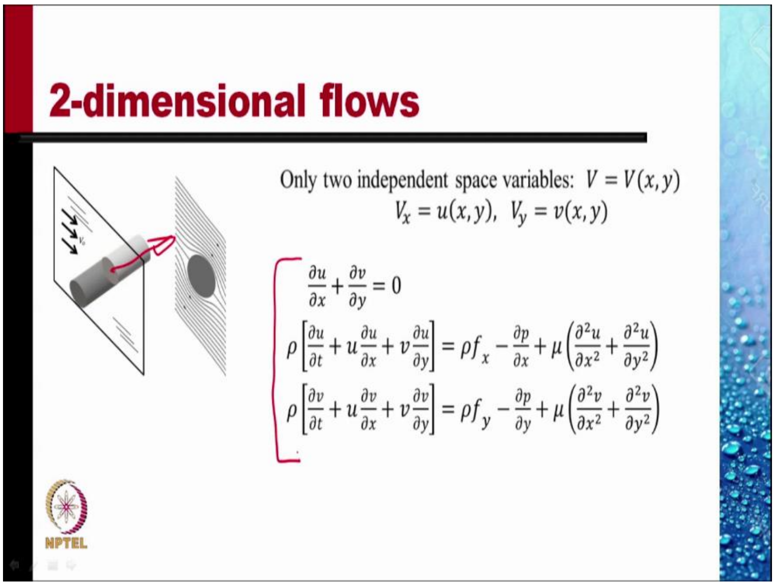

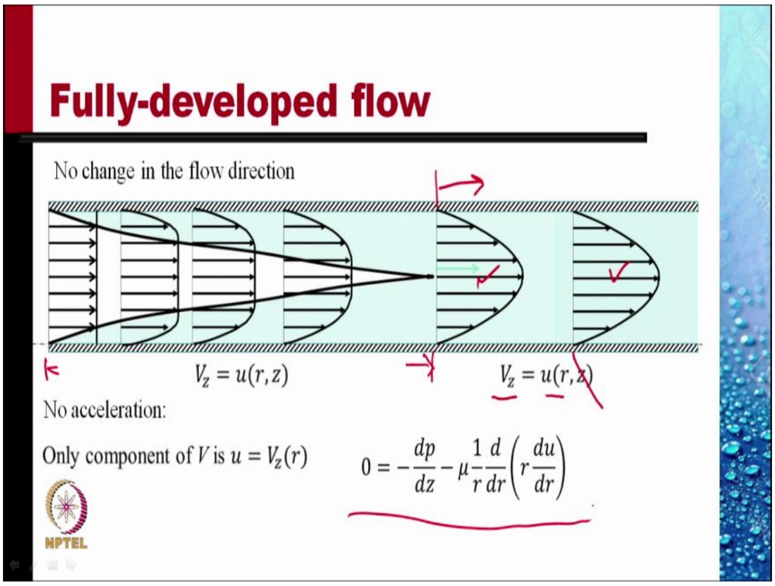

We introduced two-dimensional flows earlier where the flow varies within a plane, but is the same in the third direction. So, there are only two independent space variables x and y, and in the z direction there is no component of velocity, and there is no variation with the z direction. Then the equations governing the flow are as listed here. Only two components u and v, pressure p, only two space coordinates x and y, and of course, the time variable.

Then there is a concept of fully-developed flow, we have seen earlier that as a flow enters a pipe the velocity profile changes down the pipe, but only up to a distance which is termed as the entrance length. Beyond this entrance length the flow can be taken as fully developed. So this velocity profile is the same as this velocity profile. There is only one component of velocity V_z, and it is a function only of r, it is not a function of z.

Since the velocity does not change down the pipe, clearly there is no acceleration and the equation of motion simply reduces to this, the left side drops out. We have assumed that the weight, the body force, does not play a part. So the pressure forces balance the viscous forces, we will solve this equation shortly.

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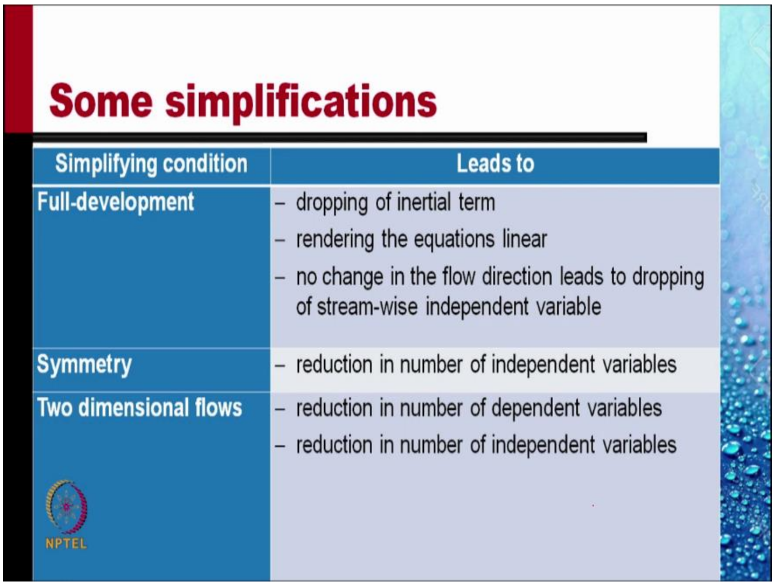

So, let us summarize these simplifications, the full development means dropping off the inertial term in the Navier-Stokes equation rendering the equation linear, because it is only the acceleration, the inertial term that contained the non-linear portion $\boldsymbol{V}.\boldsymbol{\nabla}\boldsymbol{V}$. So if there is no inertial term, there is no nonlinear term. That renders the equation linear. This is a big advantage, linear equations a lot easier to solve than non-linear equations. There is no change in velocity in the flow direction leading to dropping of the streamwise independent variables.

So, all terms with $\frac{\partial}{\partial x}$ are dropped. Of course we retain $\frac{\partial p}{\partial x}$, and we will elaborate on this a little later. Another simplification is because of symmetry which leads to reduction in the number of independent variables. In two-dimensional flows, there is a reduction in number of dependent variables. Now one less dependent variables and there is a reduction in number of independent variables, there is one less independent variable as well.