Fluid Mechanics and Its Applications
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Lecture 12B
Moment of Momentum Equation

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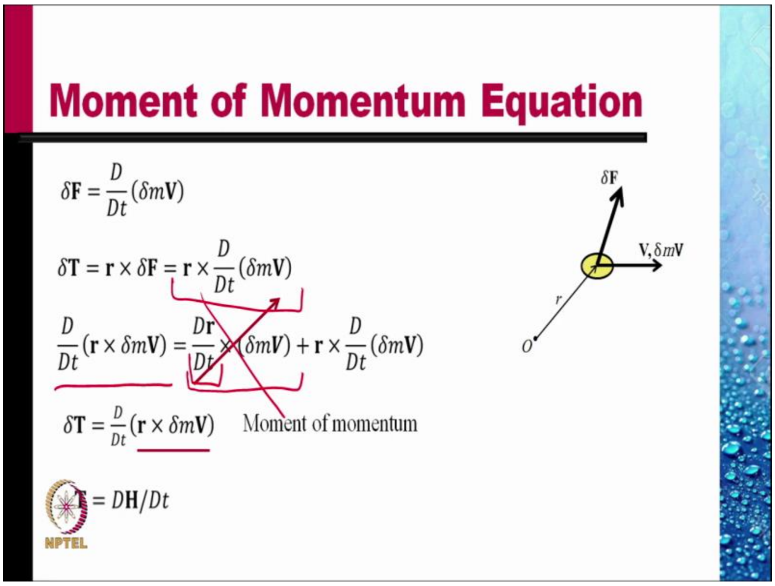

Let us do one more section, the moment of momentum equation. Consider a particle moving with the velocity **V**, let its mass be δm, this momentum would be $\delta m\mathbf{V}$. Then by Newton's second law, the net force $\delta\mathbf{F}$ on this particle is nothing but the rate of change of momentum $\frac{D}{Dt}(\delta m\boldsymbol{V})$. Let us take the torque of the force by cross multiplying vectorially at the left. So this gives you delta torque $\delta\boldsymbol{T} = \boldsymbol{r} \times \delta\boldsymbol{F} = \boldsymbol{r} \times \frac{D}{Dt}(\delta m\boldsymbol{V})$.

Now if we take derivative of $\boldsymbol{r} \times \delta m\boldsymbol{V}$, by chain rule, we get $\frac{D\boldsymbol{r}}{Dt} \times (\delta m\boldsymbol{V}) + \boldsymbol{r} \times \frac{D}{Dt}(\delta m\boldsymbol{V})$. Now this is nothing but velocity, time derivative of the position vector **r** is nothing but velocity. So this term is like **V×V**, cross product of two collinear vectors **V** and **V**, so this is 0.

So, $\boldsymbol{r} \times \frac{D}{Dt}(\delta m\boldsymbol{V})$ can be replaced by $\frac{D}{Dt}(\boldsymbol{r} \times \delta m\boldsymbol{V})$, and if we do this we get $\delta\boldsymbol{T} = \frac{D}{Dt}(\boldsymbol{r} \times \delta m\boldsymbol{V})$. This is the moment of momentum; $\boldsymbol{r} \times \delta m\boldsymbol{V}$ is moment of momentum. We integrate this and we say the total torque acting on a body is the time rate of change of the moment of momentum; this is the moment of momentum equation for a particle.

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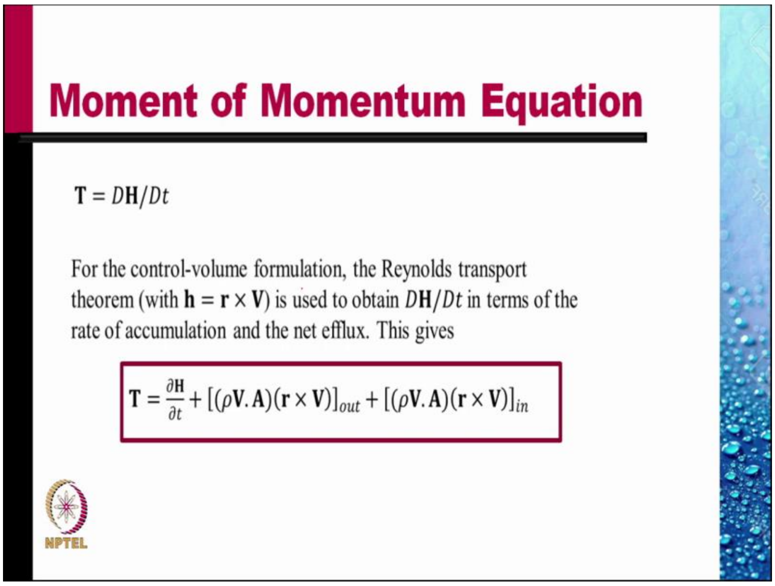

For a control volume formulation, the Reynolds transport theorem with the specific value of the moment of momentum $\boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{V}$, that is, the moment of momentum per unit mass. Then $D\boldsymbol{H}/Dt$ is obtained in terms of the rate of accumulation and the net efflux of moment of momentum.

So, we get the equation for a control volume as $\mathbf{T}$ is equal to $\frac{\partial \boldsymbol{H}}{\partial t}$, which is the rate of accumulation of moment of momentum within the control volume, plus the net efflux of moment of momentum across the control surface at inlet and outlet. We again would use the same sign convention as we use for momentum.

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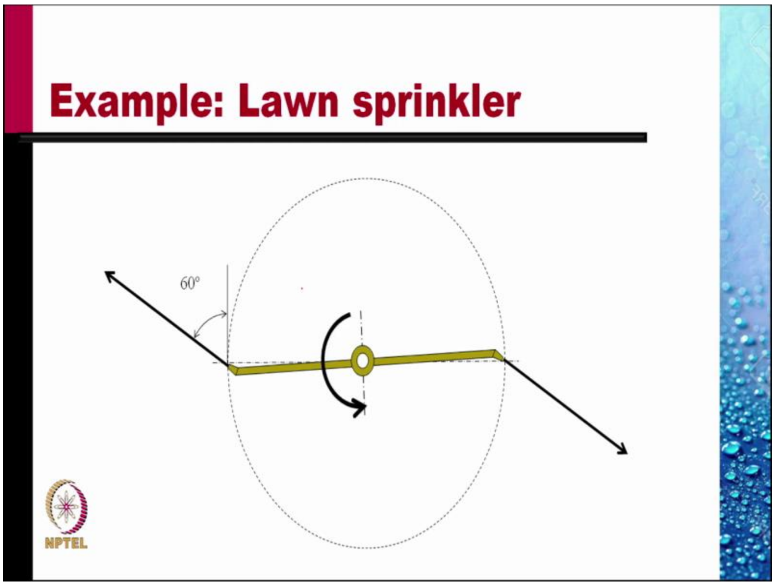

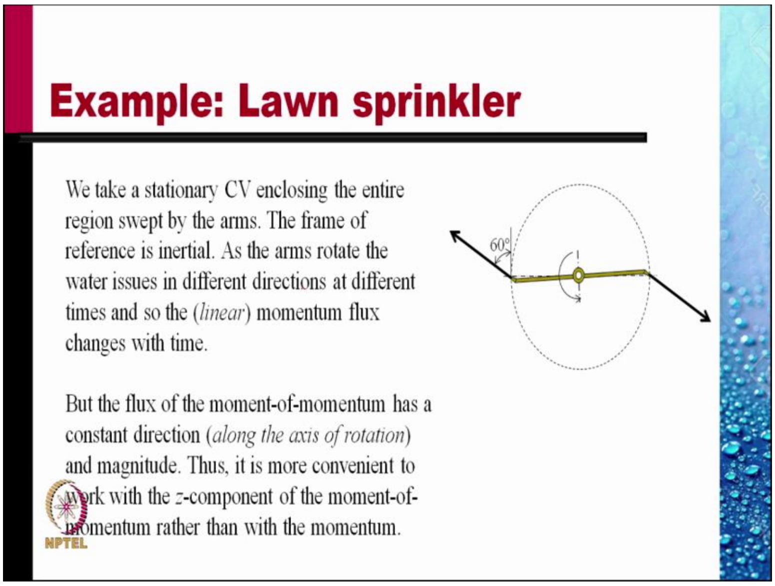

We apply this law to a very simple example. Consider a lawn sprinkler. It consists of two arms rotating about a central pivot, a frictionless pivot. There is water coming out at 60° to the edge, and because of this water coming out, the arms rotate. We have to determine that if there is no friction, what is the rate at which the arms rotate?

The equilibrium rate of rotation would be obtained when the net torque acting on these sprinkler arms would be 0. So, we have to use the moment of momentum equation to determine what should be the angular rate or rotation of these arms so that there is no net torque acting on this sprinkler.

For this, we consider a control volume. We take a stationary control volume enclosing the entire region swept by the arms as shown. The frame of reference is inertial, because the frame is not rotating. As the arms rotate, the water issues in different directions at different times, and so the linear momentum flux changes with time. The flux of momentum has a constant direction along the axis of rotation and magnitude. Thus, it is more convenient to work with the *z* component the moment of momentum equation rather than with the momentum equation.

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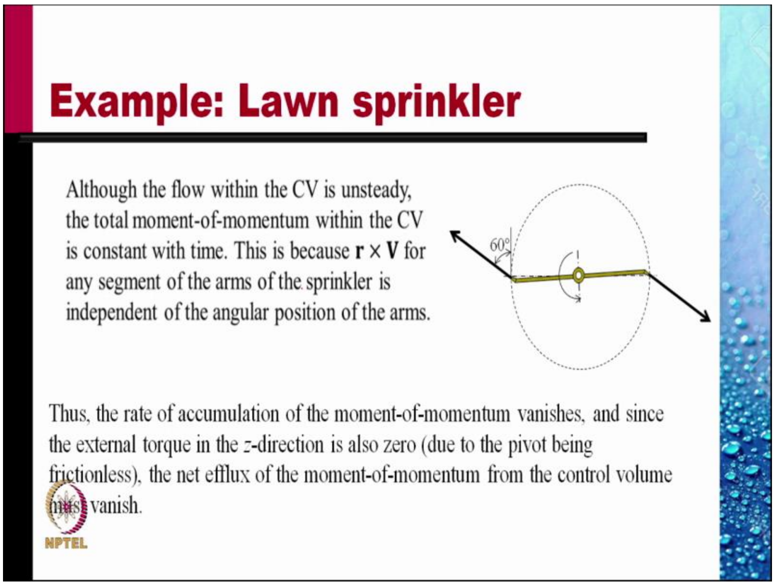

Although the flow within the control volume is unsteady, the total moment of momentum within the control is constant with time. This is because $\mathbf{r} \times \mathbf{V}$ for any segment of the arm of the sprinkler is independent of the angular position of the arm. Whatever angular position, the value of $\mathbf{r} \times \mathbf{V}$ is the same, and so angular moment of momentum does not change with time.

Rate of accumulation is 0. The rate of accumulation is 0, the net torque we want is 0. So the sprinkler arms will rotate at such a speed that the net efflux is 0, net efflux of moment of momentum is 0. Now the two orifices at 1 and 2, at both, the signs of the moment of momentum are the same. The sum should be 0. Each has the same value. So it means the moment of momentum efflux at any orifice is equal to 0.

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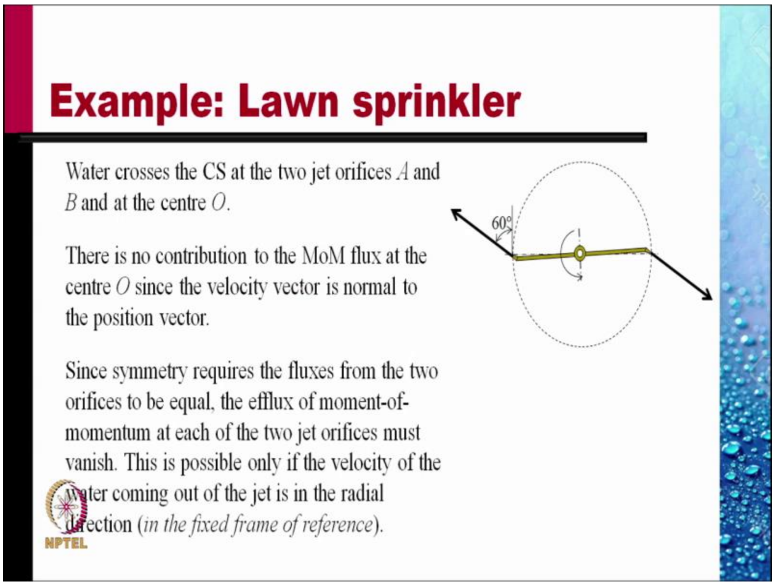

The symmetry requires the fluxes from the two orifices to be equal. The efflux of moment of momentum at each of the two jet orifices must vanish. This is possible only if the velocity of the water coming out of the jet is in the radial direction. Only if the momentum is in the radial direction would **r** cross momentum be 0, or moment of momentum would be 0. So, in the frame of reference that we have, the water should be coming out radially.

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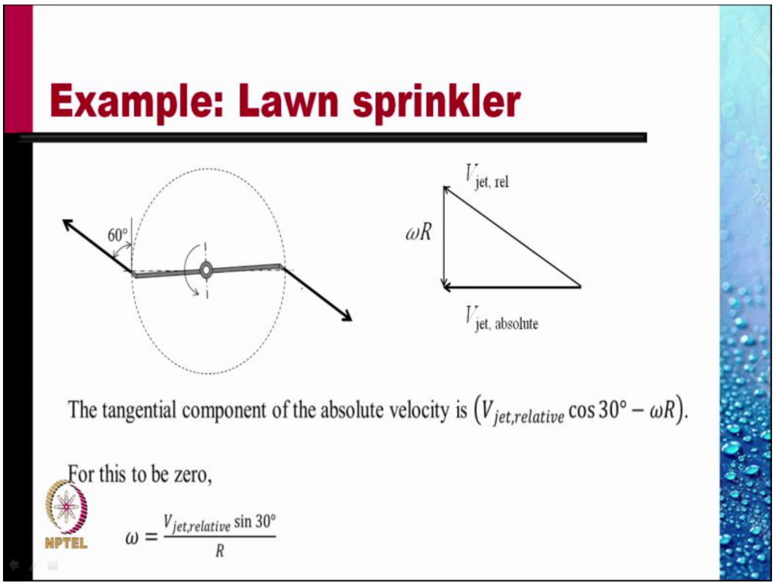

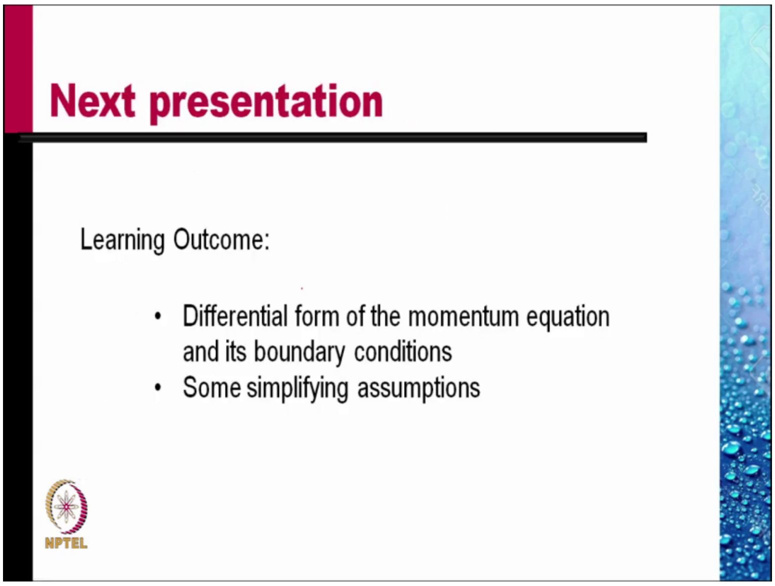

How do we do this? This is $V_{\text{jet, rel}}$ with respect to the sprinkler arms at 60° to the vertical. But the jet arm is rotating, and it has a component ωR downwards, if this is moving downwards. So the net velocity, the effective velocity, is the difference of these. If this is to be radial, then the ωR should be equal to the vertical component of the relative velocity of the jet.

The tangential component of the absolute velocity is $(V_{jet,\, relative} \cos 30° - \omega R)$, or ω should be $\frac{V_{jet,\, relative} \sin 30°}{R}$. So, we know the $V_{\text{jet, rel}}$ at what rate it is coming out. Then we can determine by applying the Bernoulli equation from the pressure at which the water is being supplied to the sprinkler. We can calculate the relative velocity of the jet, and then we can find out what would be the rate of rotation of the sprinkler if it is frictionless.

Thank you very much.