Fluid Mechanics and Its Applications
Professor. Vijay Gupta
Sharda University
Honorary Professor
Indian Institute of Technology, Delhi
Lecture 12A
Further Applications of Momentum Equations

Now let us do another example this time using a deforming control one. This is a little complicated problem in which an L-shaped tube of constant area *A* is filled completely with water, and is held as shown. There is a plug at the lower end that is holding this water like this. The pressure increases hydrostatically down the vertical leg and is constant throughout the horizontal leg.

Then, at time $t = 0$, we remove the stopper at the lower right end and the water starts to drain out. The gauge pressure at the exit becomes 0 when the water starts draining out. We cannot use one control volume throughout this volume, because if we use one control volume there would be horizontal forces and vertical forces in each one of them. Instead, we break this volume into two: control volume 1 and control volume 2.

The pressure throughout the vertical leg is in the *x* direction horizontal on the curved wall, except at the two ends, the top and the bottom ends. The pressure on the cylindrical part of the control surface would not affect the vertical momentum equation, the *z* momentum equation, because those forces are horizontal.

Similarly, the pressure distribution throughout the horizontal leg, that is, the cylindrical part of the control surface of control volume number 2, is not in the *x* direction, and will not enter the *x* momentum equation. So we will write the *z* momentum equation for control volume 1, and the *x* momentum equation for the control volume 2.

For control volume 1, we have shown the forces. There are two forces: one is the weight of the fluid within the control volume, and the other is the pressure force acting upward on the surface at the lower end of this control volume. When the level of fluid is changing at rate $\dot{h}$, the fluid volume within both the vertical and horizontal leg is moving with the speed *V* is equal to $\dot{h}$. If dh/dt is the rate at which the level is falling, $\dot{h}$, considered positive upward, then the velocity is $\dot{h}$ here. $\dot{h}$ would be negative. So velocity would be negative.

The *z* momentum contained within this control volume, control volume number 1, at time *t* is ρAh, the mass times $\dot{h}$, the velocity, $\rho Ah\dot{h}$. This is the momentum at time *t*. So what is the rate of accumulation? Rate of change of this momentum $\partial(\rho Ah\dot{h})/\partial t$, which is shown as $\rho A(h\ddot{h} + \dot{h}^2)$.

In this deforming control volume there is no flux or momentum at the upper end, but the efflux at the lower end is $(-\rho A\dot{h})\dot{h} = -\rho A\dot{h}^2$, $-\rho A\dot{h}$ is the volume coming out, is negative because it is downwards, times $\dot{h}$. And so the net efflux of momentum is $-\rho A\dot{h}^2$.

Putting the two together, we get $p_{1,g}$ is $\rho gh + \rho h\ddot{h}$. ρAh is purely the hydrostatic pressure. To this is added a pressure $\rho h\ddot{h}$, the acceleration of the level *h*, since $\ddot{h}$ is negative, so this is negative.

Similarly, we work on the second control volume, the horizontal leg. Here it is easy to see that the momentum influx is the same as momentum efflux, same volumes, same rate the mass is coming in, same rate the mass is leaving; the velocity at inlet is the same velocity outlet, so net efflux is 0.

Also we evaluate the accumulation term. The total mass contained within this tube is constant at ρAL_2, But the velocity is changing and the velocity is $-\dot{h}$, and so the rate of accumulation is $-\rho AL_2\ddot{h}$, because $\frac{\partial}{\partial t}$ of the momentum contained within the control volume, and the momentum contained within the control volume is ρAL_2, the mass times the velocity, and the velocity is $-\dot{h}$.

So, this is the rate of accumulation, and that should be equal to the net external force applied on the system, and the only external force in the *x* direction is $p_{2,g}A$, and from this we get $p_{2,g}A = -\rho AL_2\ddot{h}$, or $p_{2,g} = -\rho L_2\ddot{h}$. Now what next? The next is, if the volume of the bend is very small, the pressure p_1 should be equal to the pressure p_2, and we equate the two to obtain this equation $\rho gh + \rho h\ddot{h} = -\rho L_2\ddot{h}$. And this equation is simplified to obtain $\ddot{h} = -\frac{gh}{h+L_2}$. The problem is solved. This tells you the rate at which the level would fall and the acceleration of that level.

Let us do one more example. In this example, we try to find out the pressure loss in a sudden expansion. We discussed in an earlier lecture that as a liquid flows in a pipeline with a sudden expansion of the diameter of the pipe, then the flow separates at these locations and an eddy is trapped in this dead water region, an annular dead water region. The flow comes out, the velocity remains the same as here, and this flow then reattaches quite a ways downstream.

We have shortened the distance in this picture for illustrating it. We have to find out what is the pressure difference between section AA and section BB. Let us take a control volume like this. We draw the picture of this control volume and the forces. Because the velocity does not change at location A, the pressure all across at section AA would be p_1, and all across section BB would be p_2. So, we can calculate the net horizontal force as $(p_1A - p_2A)$ in the horizontal direction. And that should be equal to the net efflux of momentum.

The influx of momentum at AA is $V_1A_1 \cdot V_1$, influx into A_1, the velocity is V_1. The efflux of momentum at BB is $\rho V_2A_2 \cdot V_2$. The net external force $p_1A_2 - p_2A_2$ is equal to net efflux. So efflux at BB which is $\rho V_2A_2 \cdot V_2$ plus the efflux at AA, but the efflux at AA is negative of the influx at AA, so it is $-\rho V_1A_1 \cdot V_1$, the mass flux times the velocity V_1. The mass balance equation permits relating V_2 with V_1 as $V_2 = \frac{A_1}{A_2}V_1$. And we use this in the above equation, we get $\frac{p_2 - p_1}{\rho V_1^2} = \frac{A_1}{A_2}\left(1 - \frac{A_1}{A_2}\right)$.

Uptill now in all calculations of momentum efflux, we have assumed the flow to be uniform across the inlet and the outlet. At 1 the velocity is 1, at 2 the velocity is 2. But usually in control volumes the liquid or the fluid enters through a pipe, and we know very well that there is a velocity profile across this pipe. If the flow is laminar, fully developed, the profile is parabolic. If it is turbulent is most likely to be a one-seventh power law. We can still work with a one-dimensional assumption, but with a momentum correction factor.

We define a moment correction factor β just like we defined an energy correction factor γ when we were dealing with the energy equation. So we define β as equal to actual moment of flux divided by moment of flux when the same mass crosses at the average velocity, which we have been doing so far.

So, the actual momenttum flux would be $\int_A \rho V^2 dA$. And the momentum flux when the whole mass crosses at V_{av} is $\rho V_{av}^2 A$, where $V_{av} = \frac{1}{A}\int_A VdA$. We had done that before. So, this correction factor can be evaluated by evaluating the integrals.

For a circular port of radius *R*, capital *R*, the velocity is a function of the radius *r* alone. So, consider a ring of thickness *dr*, at radius *r*, and the total momentum flux would be

$2\pi\rho \int_0^R rV^2 dr$. This is the numerator and the denominator would be simply $\rho\pi R^2 V_{av}^2$. That is, β is equal to the numerator divided by the denominator.

Now if we evaluate this factor β for laminar flow first. For laminar flow, the velocity profile is parabolic, that is, $V = V_{max}(1 - r^{*2})$, and the V_{av} we have shown before is $\frac{1}{2} V_{max}$, so that $\frac{V}{V_{av}} = 2(1 - r^{*2})$. The above integration yields β is equal to 1.33. If we had used a turbulent flow, a very common profile is the one-seventh power law profile of the form $V = V_{max}(1 - r^*)^{1/7}$. And with this profile we can obtain V_{av} as 0.817 V_{max}.

This we did when we were evaluating the energy correction factor, so that the $\frac{V}{V_{av}} = 0.817(1 - r^*)^{1/7}$, and the integration above gives$\beta = 1.020$, a very small correction over 1. So, if the flow at the inlet or the outlet or at both locations is turbulent, as it is more likely to be, since most economical flows are turbulent, we may not apply momentum correction factor, because this will result in an improvement only by 2 percent, not significant.