Fluid Mechanics and Its Applications
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Lecture 12
Momentum Equations

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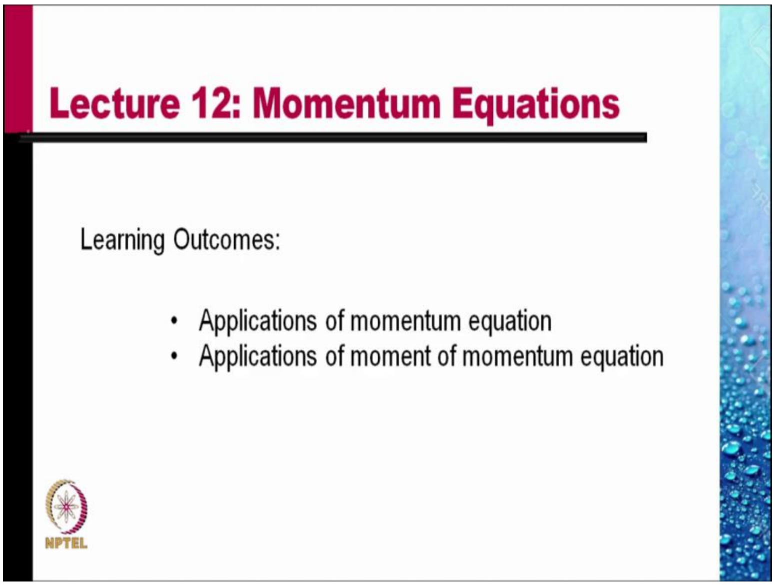

Welcome back. Today we will cover the momentum theorem. Momentum theorem follows from the second law of Newton that the rate of change of momentum of a body in motion is equal to the net external force applied on it.

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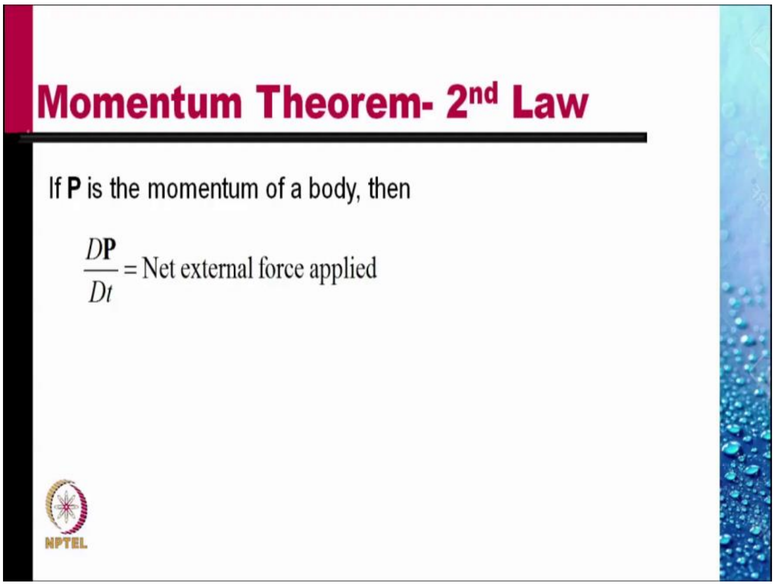

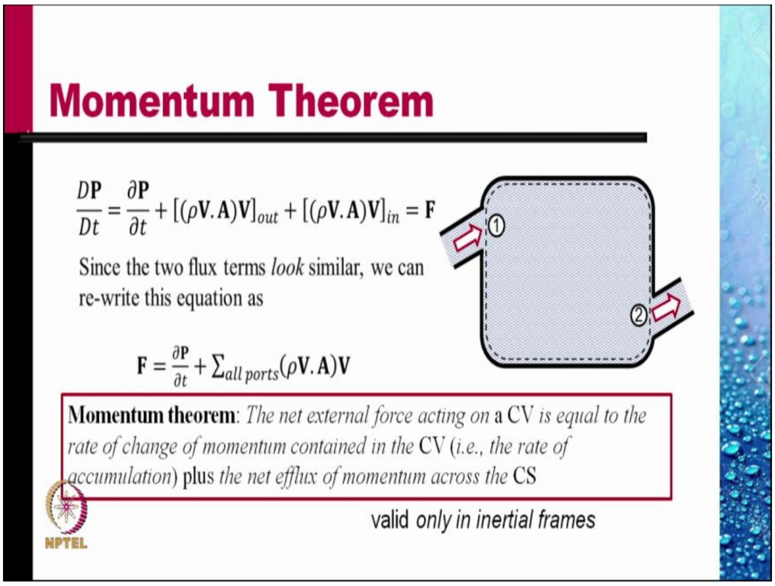

So if capital P is the momentum of a body then DP/Dt, the rate of change of momentum following the particle is equal to the net external force applied. In fluids, we use control volumes, and for the control volumes, the material rate of change of momentum P would be given by the local rate of change of momentum $\partial P/\partial t$ plus the net efflux of momentum. In the efflux terms, we have written $\rho\mathbf{V}\cdot\mathbf{A}$, $\rho\mathbf{V}\cdot\mathbf{A}$ is the mass flux at the outlet, into $\mathbf{V}$ is mass flux into velocity, is the momentum.

Since, the area vector $\mathbf{A}$ of a surface is pointing outwards, $\mathbf{V}\cdot\mathbf{A}$ positive would imply that this is an outflux. $\mathbf{V}\cdot\mathbf{A}$ negative would imply that velocity is opposed to the area. An area is taken positive outwards, so $\mathbf{V}\cdot\mathbf{A}$ negative means it is an influx. So if we consider the signs of $\mathbf{V}\cdot\mathbf{A}$, then both the terms, outflux and influx written in this manner, would be with the positive signs, and so we can write $\mathbf{F}$, the net external force applied, would be equal to the rate of accumulation of the momentum within the control volume plus the net efflux of momentum from all the ports of the control volume.

This is valid only in inertial frames of reference, because the Newton's second law of motion is applicable only in the inertial frames. What are initial frames? Frames that are either stationary, or are moving with constant velocities, that is, frames that are not accelerating. A rotating frame would definitely be accelerating.

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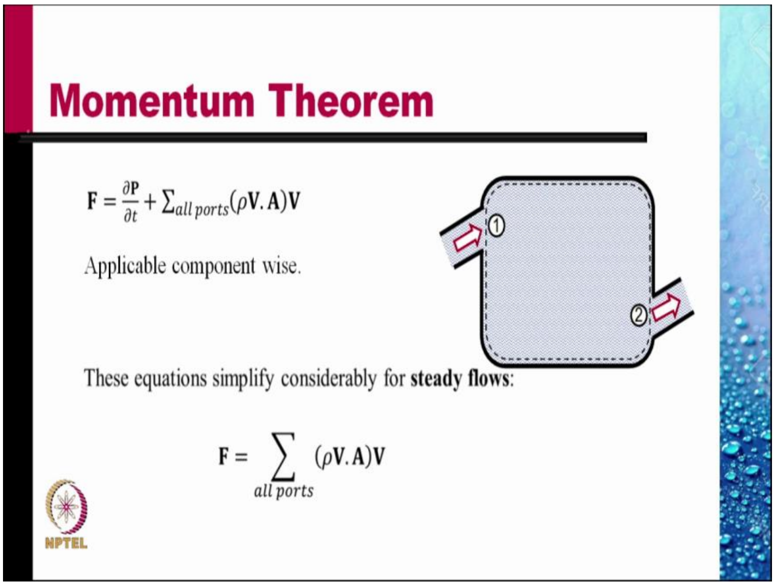
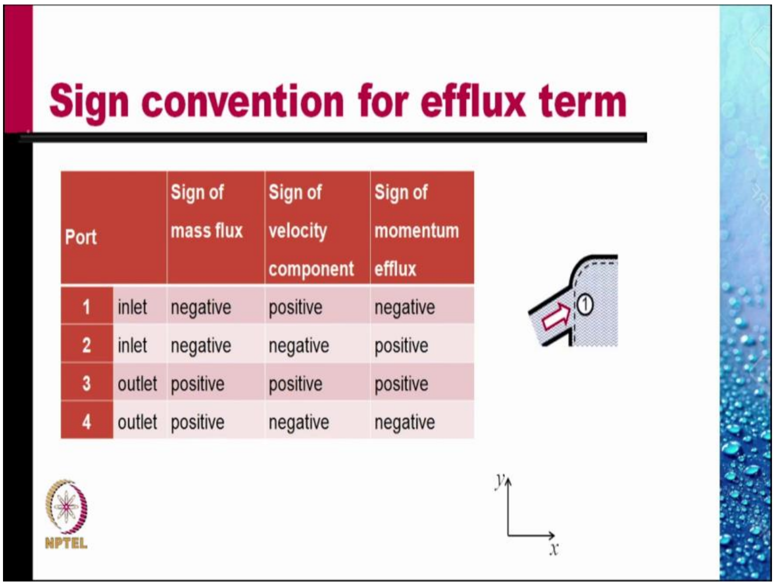
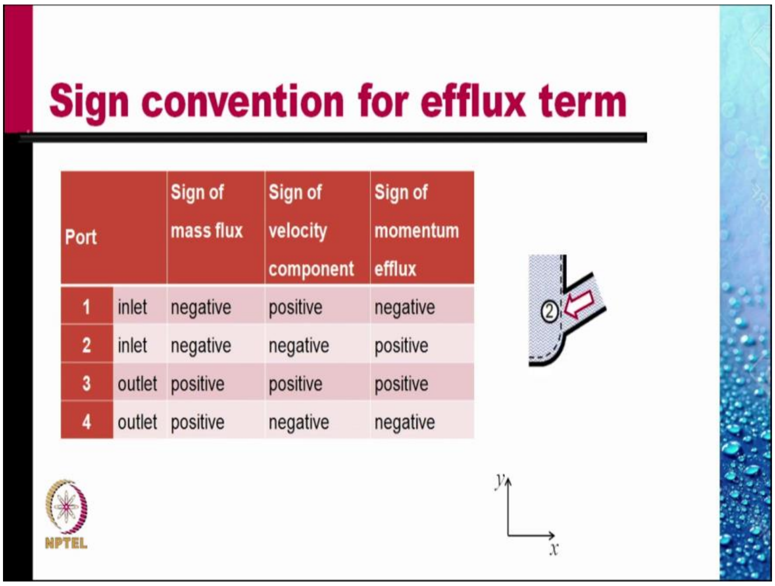
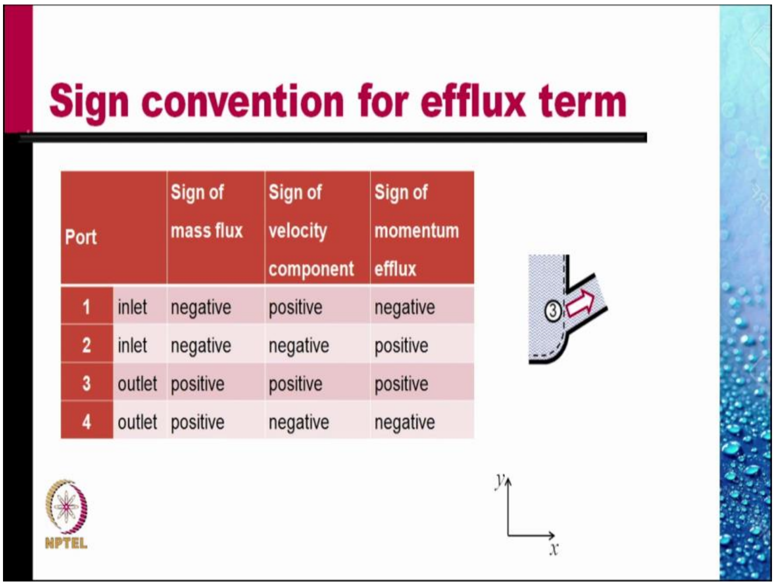




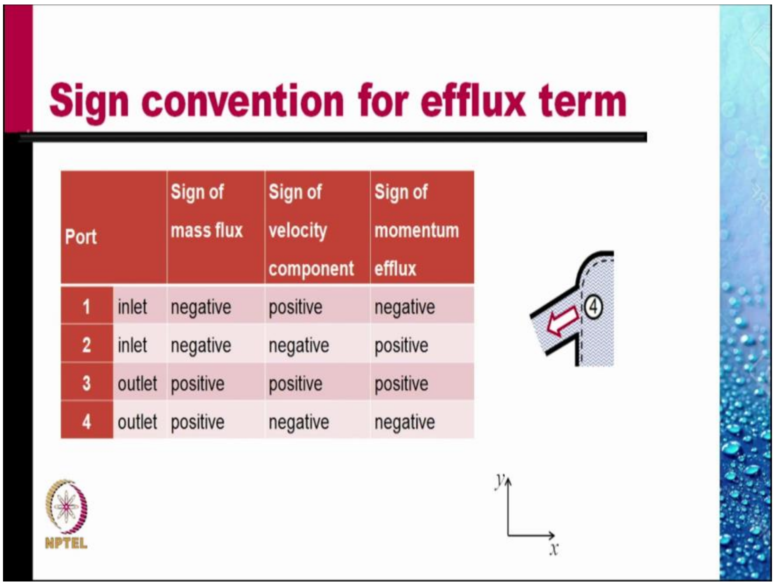
Port		Sign of mass flux	Sign of velocity component	Sign of momentum efflux
1	inlet	negative	positive	negative
2	inlet	negative	negative	positive
3	outlet	positive	positive	positive
4	outlet	positive	negative	negative



Port		Sign of mass flux	Sign of velocity component	Sign of momentum efflux
1	inlet	negative	positive	negative
2	inlet	negative	negative	positive
3	outlet	positive	positive	positive
4	outlet	positive	negative	negative



Port		Sign of mass flux	Sign of velocity component	Sign of momentum efflux
1	inlet	negative	positive	negative
2	inlet	negative	negative	positive
3	outlet	positive	positive	positive
4	outlet	positive	negative	negative



Port		Sign of mass flux	Sign of velocity component	Sign of momentum efflux
1	inlet	negative	positive	negative
2	inlet	negative	negative	positive
3	outlet	positive	positive	positive
4	outlet	positive	negative	negative

Now this is a vector equation and this should be applicable component-wise also. So we can write this for the x component, the y component, and the z component. Now for steady flows, the rate of accumulation of momentum P would be 0, and so the net external force applied would be the sum of $\rho \mathbf{V} \cdot \mathbf{A}$ times $\mathbf{V}$ at all ports, the net efflux of momentum. A very simple equation.

We need to worry about the sign convention for the efflux term. And as was said before, at the inlet the sign of the mass flux is negative. In this picture that we have shown, because the area vector is outwards and, that is, towards the left and down and the velocity vector is towards the right and up. So velocity vectors is positive in the x and y components, whereas the area vector is downwards, so the sign of the mass flux is negative. And the sign of the velocity component is positive. The sign of the momentum of efflux would be negative.

If however, the port was like this: this is again is an inlet port, here the sign of mass flux $\mathbf{V} \cdot \mathbf{A}$ is negative but the velocity component is also negative, so the sign of the momentum efflux is positive. At an outlet like this, the mass flux is positive, the velocity component is positive, so that the momentum efflux is also positive. But if the exit is like this, the sign of the mass flux is positive, area vector is towards the left, velocity vector is toward the left, and so the mass flux is positive, and the velocity component is negative, so that the momentum efflux has a negative sign. We will follow this convention in writing all the momentum effluxes.

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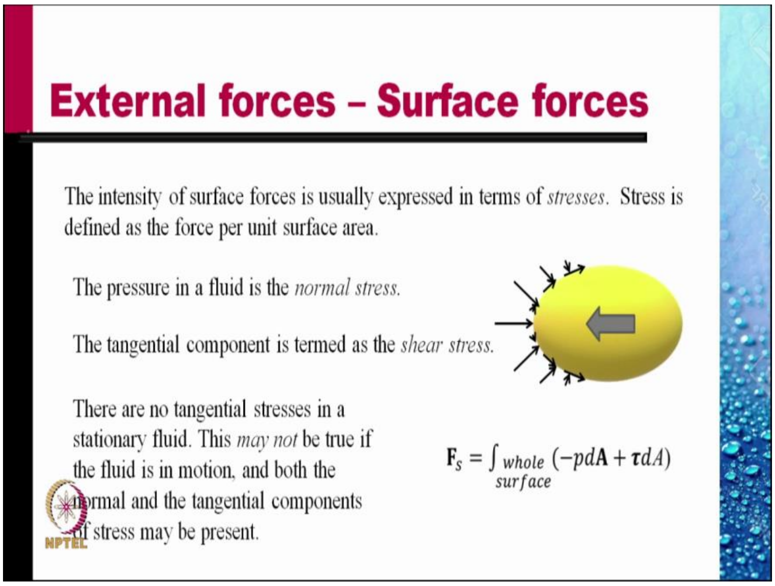

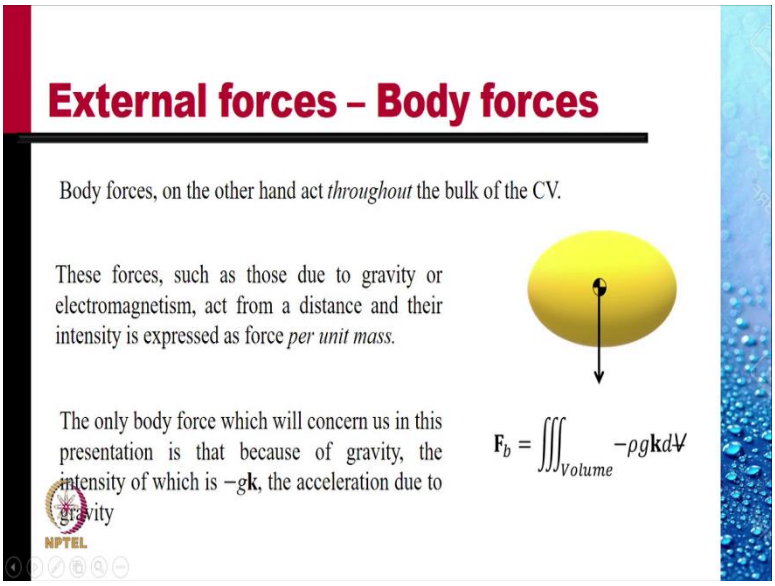

Now let us consider the external forces. First we talk of surface forces. The intensity of surface forces is usually expressed in terms of stresses. Stress is defined as the force per unit area. If your body is a fluid, the pressure in a fluid is a normal stress. At the nose of this body this pressure is acting inwards. this is normal stress. The tangential component is termed as a shear stress. There are no tangential stresses in a stationary fluid, but this may not be true if the fluid is in motion, and both the normal and tangential components of stresses may be present in such a fluid.

So, the total external body forces would be expressed as integral over the whole surface of $-pdA$. dA is pointed outwards, p is towards the surface, towards the body, so it is $-pdA + \tau dA$, the shear stress times dA. Next, we consider the body forces: body forces on the other hand act throughout the bulk of the control volume. These forces, such as those due to gravity or electromagnetism act, from a distance and their intensity is expressed as per unit mass.

The only body force which will concern us in this presentation is that because of gravity, the intensity of which is $-g\mathbf{k}$, $\mathbf{k}$ is the unit vector in the vertical direction, gravity is vertically downwards, so the intensity of the gravity forces, that is, weight per unit mass is $-g\mathbf{k}$, where g is the acceleration due to gravity. So, the total body force is obtained by integrating over the volume $\int_V -\rho g\mathbf{k}dV$. So, this expression gives you the total body force.

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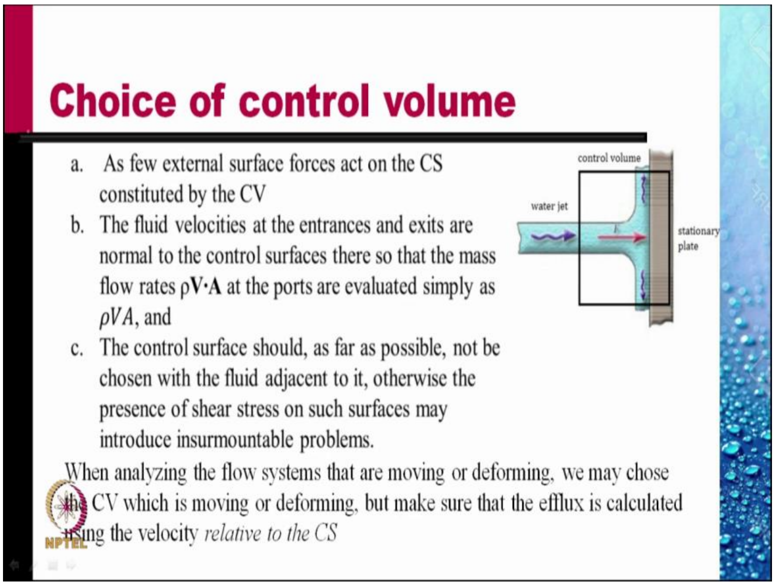

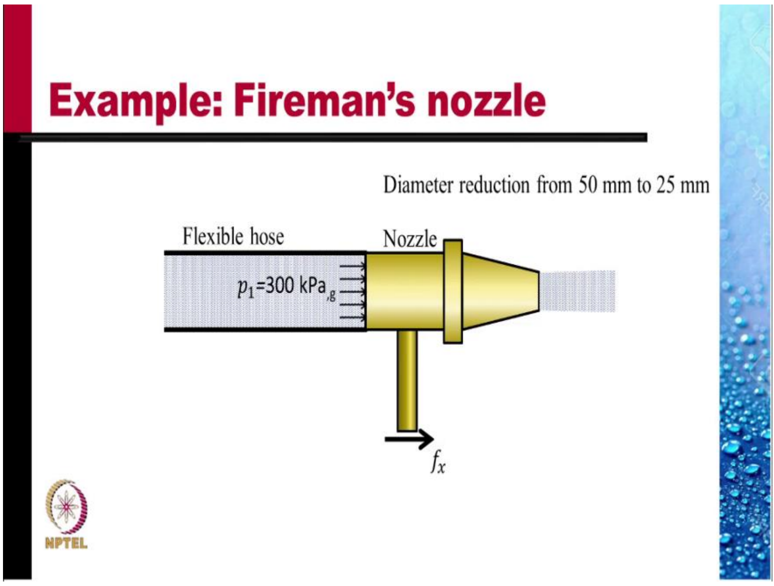

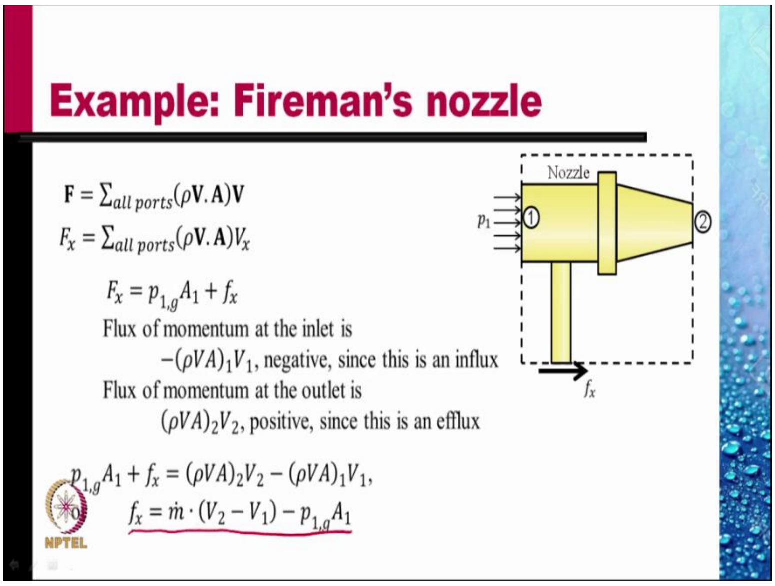

Now we have to choose the control volume. The choice of the control volume is dictated by the following suggestions, a, we should choose a control volume such that as few external surface forces act on the control surface constituted by the control volume. Obviously, the lesser the forces, the fewer terms we will need to incorporate within the equations.

The fluid velocities at the entrance and exits are normal to the control surface there, so that the mass flow rate rho $\rho\mathbf{V}\cdot\mathbf{A}$ at the ports are evaluated simply as ρVA, not the vector, but the scalar, ρVA. Otherwise, $\rho\mathbf{V}\cdot\mathbf{A}$, we will have to worry about the angle between the area and the velocity.

The third recommendation is that the control surface should, as far as possible, not be chosen with fluid adjacent to it. Otherwise, the presence of the shear force or such surfaces may introduce insurmountable problems. When analyzing the flow systems that are moving or deforming, we may choose the control volume which is moving or deforming, but make sure that their flux is calculated using the velocity relative to the control surface.

Let us do a few examples. This example here is of a fireman's nozzle. In a flexible hose, the water is coming. At the nozzle inlet the pressure is 300 kPa gauge. The nozzle reduces the flow diameter from 50 mm to 25 mm, as shown. There will be a force in the nozzle because of this pressure and because of the efflux of water. So, we will have to hold this nozzle rather firmly. We need to apply a force to hold it. Let the external force applied be f_x in the *x* direction.

Now we will write the momentum equation for this, and we will use the *x* component of momentum equation because we are interested in evaluating the force f_x in the *x* direction. This is the control volume that we take. We draw all the external forces on this, there are only two. The pressure p_1, the gauge pressure p_1, and the force f_x. There is atmospheric pressure all around, but atmospheric pressure all around leads to 0 force, a uniform pressure all around the volume leads to 0 net force.

So, the equation in the *x* direction would be f_x, the total external force, would be the total net efflux of *x* momentum, because this is steady flow, so there is no rate of accumulation. The total external force is $p_{1,\text{gauge}}\ A_1$, the area at 1, and f_x, both positive. The flux of momentum at the inlet is $-(\rho VA)_1$ times velocity at 1. This is negative, because mass flux is negative and velocity is positive, so the product is negative. The other efflux is at the outlet.

The efflux of momentum at the outlet is rho $(\rho VA)_2$ times velocity at 2, and this is positive since the mass is coming out, so it is positive, and the velocity is in the positive direction. So, the equation of momentum reads $p_{1,g}A_1 + f_x = (\rho VA)_2V_2 - -(\rho VA)_1V_1$. This is simplified to obtain this result.

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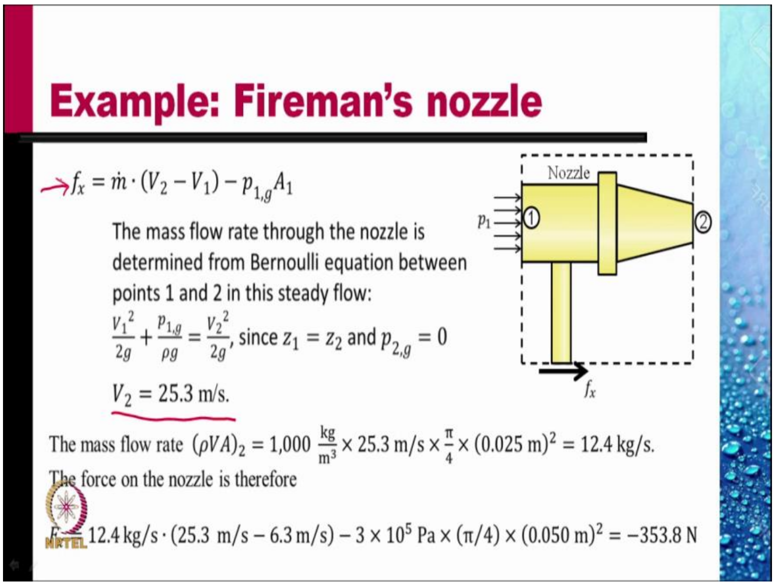

We can apply Bernoulli equation between point 1 and 2. And if we apply Bernoulli equation with points 1 and 2, $\frac{V_1^2}{2g} + \frac{p_{1,g}}{\rho g} = \frac{V_2^2}{2g}$, since z_1 is z_2 and $p_{1,g} = 0$. Using the value of $p_{1,g}$ which was given as 300 kPa, given the value of ρ for water, and given the area ratio 1 : 4, diameter at 1 is twice the diameter is 2. So V_1 is one-fourth of V_2. Using this information we get the velocity at the exit to be 25.3 m/s.

Once we know V_2 we can calculate V_1 and plug it in the equation on top, and we get a result that the force is $-$ 353.8 N. Why minus sign? Minus sign means that this force would be in the direction opposite to the direction that we assumed this force to be in. We assumed this to be positive, pointing towards the right. The force would be to the left. So, a fireman would have to apply a force to the left, that is, the nozzle by itself would tend to go towards the right. So a fireman would have to apply this rather large force. 353.8 N is a fairly large force.

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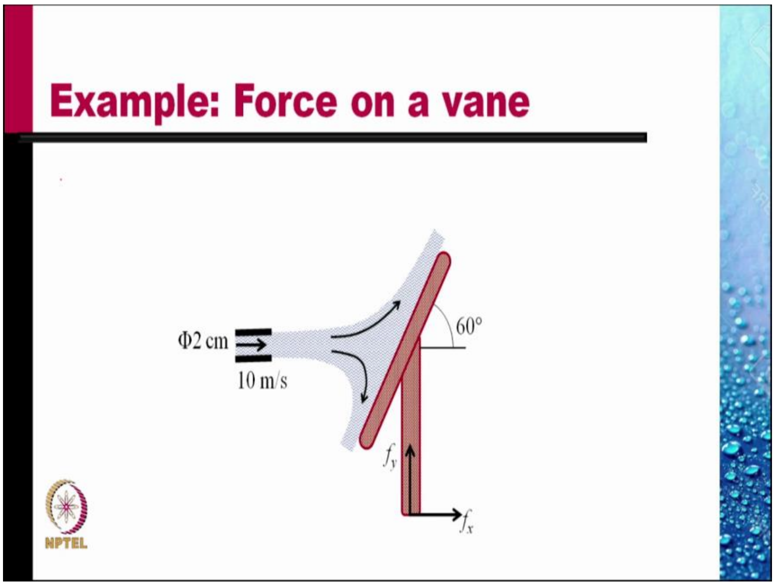

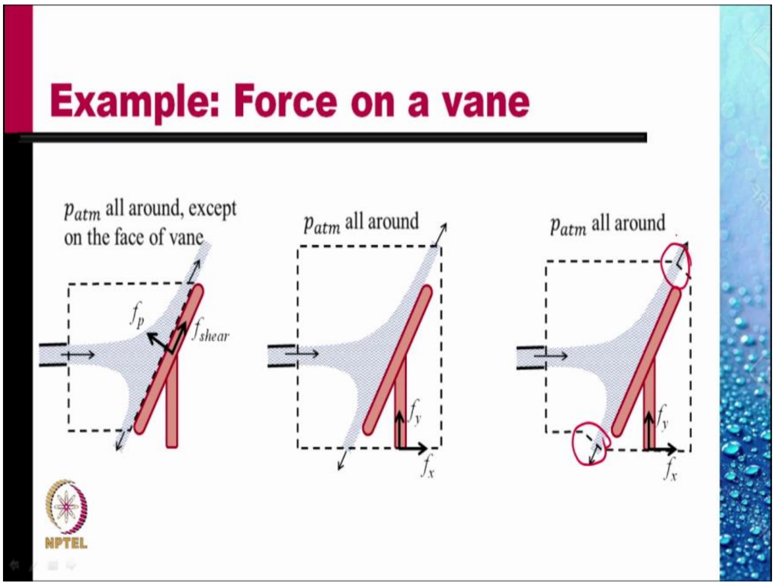

Let us do another example, a simple example: force on a vane. Let there be a vane at 60°, a jet of water coming from an orifice of diameter 2 cm at 10 m/s impinges on this vane, and is divided in two equal streams. One going up the vane, the other going down the vane. We have to determine what would be the force that would be needed to keep this vane in place.

Let that force have two components f_x and f_y as shown. Now to determine f_x we need to write the x component of the momentum equation, and to determine f_y we need to write the y component of the momentum equation. What control volume do we choose? We could choose a control volume like this, where the control surface runs next to the vane. This is not a very good choice, since shear forces act on this control surface all along the vane, and so too would the pressure force.

It is not easy to determine those forces. So you may choose a control volume like this. In this case there is atmosphere pressure all around like there was in the first case as well. The jet of water is out of the nozzle, so it will have a pressure equal to atmospheric at all three locations, and then the solid forces within the support of the vane would become external forces f_x and f_y as shown.

This is a fairly good control volume, but we can improve upon this a little bit by considering the area of this control volume as normal to places where there is water influx and water efflux. So, just minor changes here and here, to make the control surface normal to the velocity at this end, so we will work with this control volume.

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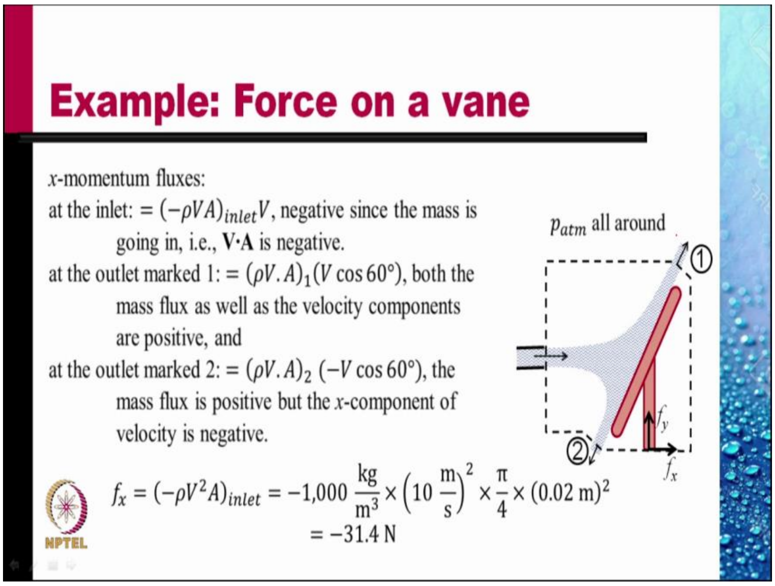

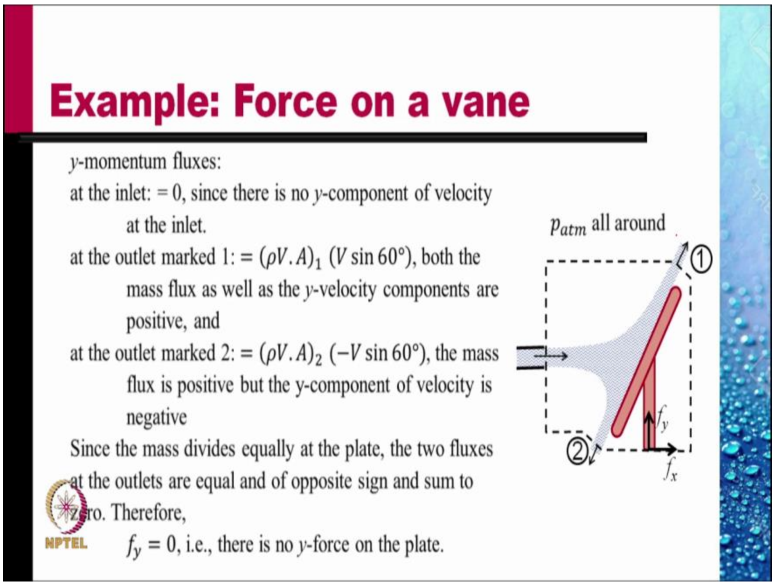

We will first work with the x momentum. x momentum flux at the inlet is $-(\rho VA)_{inlet}$, minus sign because it is the influx of mass, into V, the velocity there. At the outlet marked 1, $(\rho VA)_1 \cos 60°$. There is a x component of velocity which is positive, so if V is the velocity then the x component $V \cos 60°$, the angle of the vane is 60°. So, both the mass fluxes as well as the velocity components are positive, and so this flux is positive.

At the outlet marked 2, $(\rho VA)_2$the mass flux is positive, it is coming out. But the x component of velocity is negative, $-V \cos 60°$. The mass flux is positive but x component velocity is negative, so that the momentum efflux here is negative. So the x momentum equation then the external force, the only external force on this control volume is f_x horizontal. Force is f_x. So f_x is the net efflux of momentum. You see the flux at 1 and the flux at 2 cancel one another out if the jet divides equally.

So, the mass flux at 1 and 2 are same, the velocities are equal but opposite, so the efflux at 1 cancels out the efflux at 2. So there is only a net influx, or a negative efflux of ρ, which is 1000 kg/m^3, into velocity squared, velocity is 10 m/s, into the area. Area is 2 cm dia, so is $\frac{\pi}{4}(0.02\ m)^2$, and that is -31.4 N.

So, the force that we need to apply to hold this in place would be in the negative direction of 31.4 N. We have to hold the vane back. It will tend to move in the direction of the jet so we will have to hold it back.

Let us do the y momentum equation again. At the inlet, the y momentum is zero because there is no component of y velocity there. At the outlet marked 1, the mass flux is ρVA positive, and the y component of velocity is $V \sin 60°$ in the positive y direction. Both the mass fluxes as well as the y velocity components are positive, hence this efflux is positive.

At the outlet marked 2, the mass flux is ρVAat 2, positive, and the velocity in the y direction is negative, $-V \sin 60°$, and so the efflux at 2 is negative. And as you can see clearly. the two effluxes have equal magnitude but opposite signs, so they cancel out. So the net y efflux across the whole of the control surface is 0, and so the net external force f_y is 0. There would be no force in the vertical direction due to this jet of water playing on this vane.

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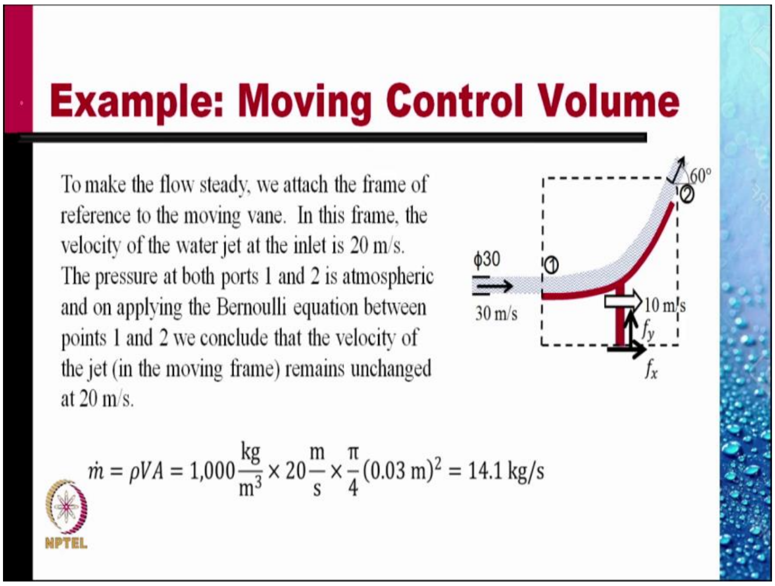

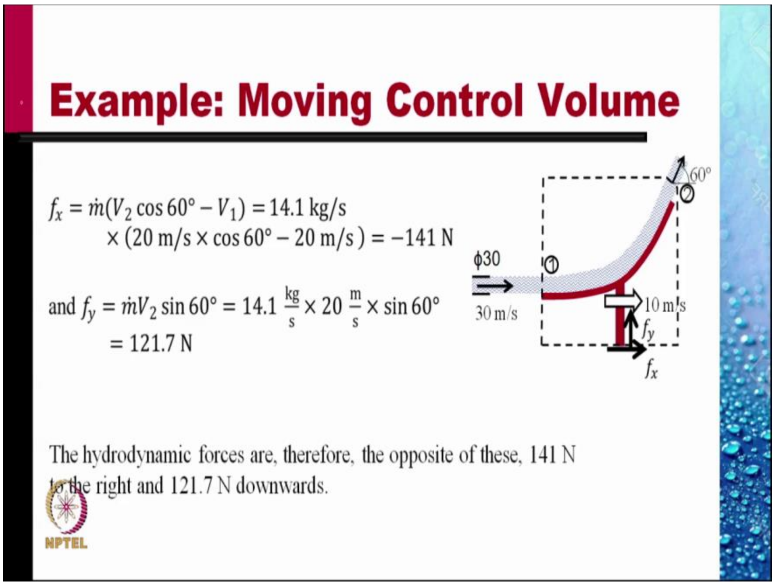

Let us do another example, this time with a moving control volume. Let there be a curved vane on which the water enters horizontally, tangent to it, and leaves at an angle of 60° after turning upwards. And let this wane be moving with a velocity of 10 m/s to the right. To analyze this, let us take a control volume as shown. This control volume where the external forces are f_x and f_y as shown.

There is a flux or momentum 1 and there is a flux of momentum at point 2. As mentioned before, since the control volume is moving, but it is moving at a constant speed, so the frame is inertial if we attach the frame to this control volume. So, we attach the frame to this control volume, and then the velocities are to be measured with respect to this control surface.

So, now the inlet velocity at location 1 will be 30 m/s minus 10 10 m/s, or the relative velocity of 20 m/s. Within this frame, we apply the Benoulli equation. The pressure at point 1 and point 2 are the same, and if we can neglect the difference in the elevation, the velocity at 1 and 2, in this moving frame, would be same, that is, 20 m/s.

The mass flow rate, the mass flux, is $\dot{m} = \rho VA$ is equal to 14.1 kg/s. This is negative at 1 and positive at 2, influx at 1 and efflux at point 2. So, the force f_x is equal to $\dot{m}(V_2 \cos 60° - V_1)$, and this is equal to 14.1 kg/s into 20 m/s cos 60° minus 20 m/s, which is the value of V_1. And this gives you a net value -141 N to the left. This is the force we need to apply to hold the vane moving at a speed of 10 m/s, that is, to prevent any acceleration of this vane.

So, that means, effectively, the water jet is applying a force towards the right, and to overcome this we have to apply a negative force, -141 N. Similarly, the force in the *y* direction, f_y is equal to the net efflux of *y* momentum, and *y* momentum comes out only at 2. So this is $\dot{m}V_2 \sin 60°$, and then comes out to be 121.7 N, that is, we have to apply a force upwards. So, the force because of water jet on the vane is downwards.