Fluid Mechanics and Its Applications
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Lecture 8B
Stream Function

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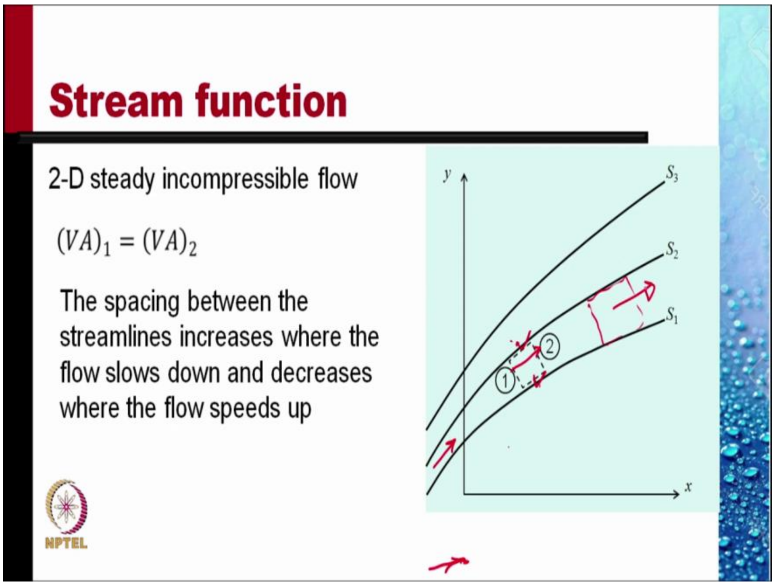

Let us talk of another concept, the concept of stream function, a very important concept, a very useful concept. Let us study stream functions for only two-dimensional steady incompressible flows, so that the equation of continuity $\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0$ applies.

Let S_1, S_2 and S_3 be three streamlines. Streamlines do not intersect let us consider a small control volume within the two streamlines S_1 and S_2. This surface coincides with the streamline S_1, this surface coincides with the streamline S_2. Surface 1 is normal to the velocity and so is surface 2. As a streamline is a line which is tangential to the velocity, so there is no component of velocity across these two surfaces. If the flow is steady and incompressible, there is no accumulation of mass within the control volume, and so the mass entering at 1 should be equal to the mass leaving at 2. Since the locations of surface 1 and 2 are arbitrary, I could take the control volume anywhere between the two streamlines.

What conclusion can we draw? That the mass flowing within two streamlines is constant at all locations. So, the mass flow rate here is a same and mass flow rate here, is the same as mass flow rate here. Everywhere we have the same mass flow rate $(VA)_1 = (VA)_2$, the density being constant. So, the spacing between the streamline increases where the flow slows down, and decreases where the flow speeds up. If area A_1 is more than area A_2, then V_1 should be less than V_2.

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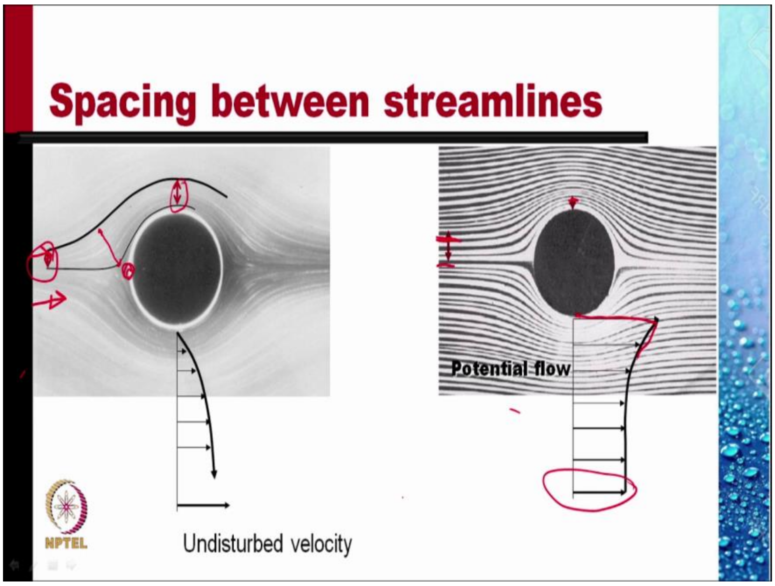

This is illustrated very nicely by this example. Consider the creeping flow about a cylinder. A flow at very low velocities such that Reynolds number is much less than one. This is a picture that we had shown you earlier. The flow looks quite symmetric. There is no separation. If you mark a few streamlines, we see a central streamline going like this, and another streamline going like this.

The spacing between two streamlines here is less than this spacing there. So, what does that mean? The velocity on the shoulder of the cylinder is less than the velocity upstream. This spacing here is the largest. So, velocity here should be smallest. In fact, this point is a stagnation point where there is no velocity. If we draw the velocity profile at the shoulder, this is the velocity profile. Notice that even at substantial distances away from the surface, the velocity of flow is still less than the undisturbed velocity out here. We will talk about this later.

On the other hand, consider the flow of a fluid, which is non viscous. There is no fluid which is completely non-viscous, but it is possible to simulate this flow in an apparatus call the Hele Shaw apparatus, where the fluid behaves as if the Reynolds number is infinite. There is no boundary layer. There is no separation in this flow.

Notice, that in this case, the distance between two streamlines is much larger upstream than on the shoulder. What does this mean? This means that the velocity on the shoulder is much larger than the velocity upstream. The velocity profile is like this. This is very close to the undisturbed velocity. Velocity at the shoulder is much larger than the undisturbed velocity.

Of course, in an actual flow the velocity at the surface must be 0. So, this velocity must have a very sharp reduction to zero velocity. This is what the boundary layer is, that we discussed earlier, and which we will discuss later on in the course.

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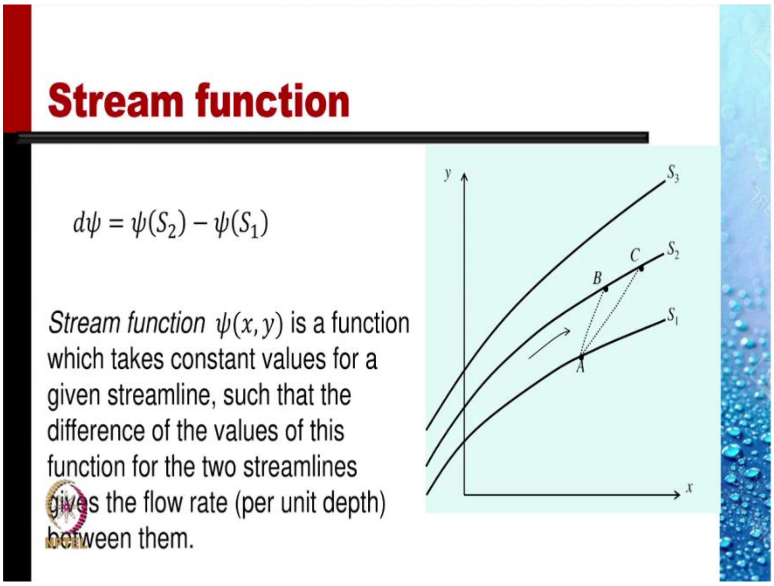

Let us again consider the same picture, the same S_1, S_2, S_3. The flow between S_1 and S_2 is moving towards the right as shown. Consider the point A on S_1, and points B and C on S_2. Clearly, the flow across *AB* must be same as flow across *AC*. So, as before, the flow between two streamlines is constant. The two streamlines actually act like a conduit for flow, no flow crossing the streamlines.

Now, we can associate a function ψ with streamlines such that $d\psi$ is $\psi_{S_2} - \psi_{S_1}$. Since, if we associate a function ψ with S_1, then the value of ψ with S_2 would be $d\psi$ greater. Stream function ψ is a function which takes constant value for a given streamline such that the difference of values of this function for two streamlines gives the flow rate per unit depth between them in two dimensional flows.

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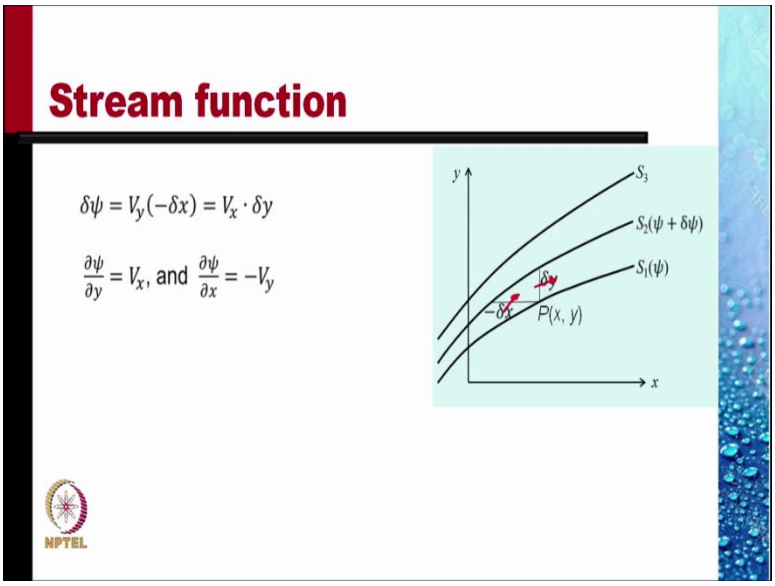

Let us look at this in further details. Consider a point P at location (x, y) on steamline S1, the value is ψ. Let us draw a vertical and horizontal line from point P to the next streamline S_2. The volume flow rate across this must be same as volume flow rate across this, and this volume flow rate must be $d\psi$, because the value of the steam function associated with S_2 is ψ+$d\psi$. So, $d\psi$ is the volume that flows between the two.

Clearly, $d\psi$ is $V_y(-dx)$ and which is also equal to $V_x(dy)$, because it is only V_y velocity component which will create a flow across the horizontal line segment, V_x being along the line segment itself. And V_x would be the only component that will create a flow across the vertical line segment. This can be manipulated to give $\frac{\partial\psi}{\partial y} = V_x$, and $\frac{\partial\psi}{\partial x} = -V_y$.

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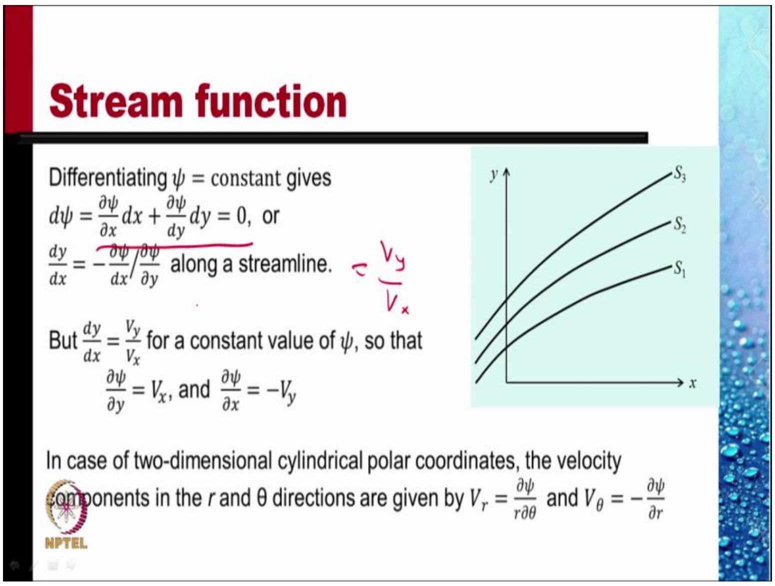

We can obtain these equations by formal mathematics also. Differentiating $\psi = constant$ gives you $d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0$, or $dy/dx = -(\partial\psi/\partial x)/(\partial\psi/\partial y)$ from this equation along the stream line, because ψ is constant.

Now, dy/dx along a streamline is nothing but V_y/V_x. So, $-(\partial\psi/\partial x)$ can be treated as V_y and $(\partial\psi/\partial y)$ can be treated as V_x, the same as before. Notice an important use of stream functions. We can replace the two variables V_x and V_y by a single variable ψ, for incompressible steady flows in two dimensional, where reduces the number of unknown variables by 1.

Two variables V_x and V_y can be replaced by ψ. This gives an important advantage in calculating complex flows. In case of two dimensional cylindrical polar coordinates, the velocity components in the *r* and θ directions are given by $V_r = \frac{\partial \psi}{r\partial \theta}$ and $V_\theta = -\frac{\partial \psi}{\partial r}$. Here ψ is the stream function.

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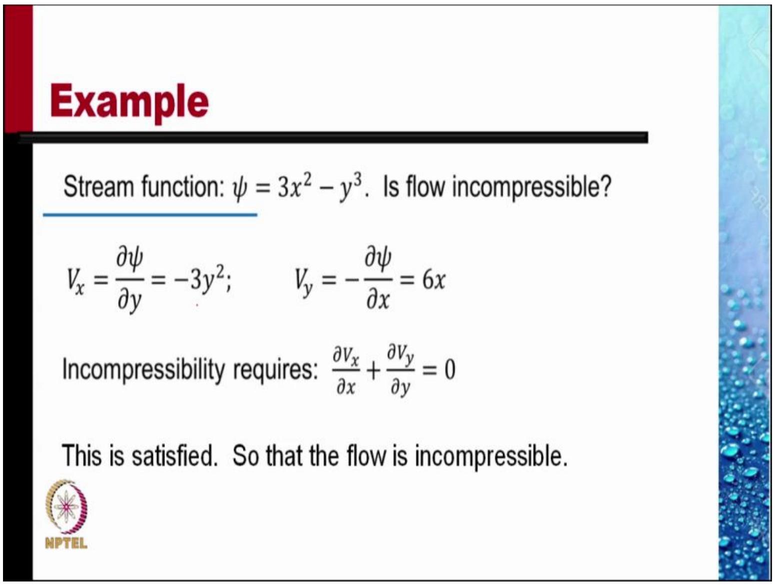

Let us do an example, a stream functions given as $\psi = 3x^2 - y^3$. Is the flow incompressible? How can you find out if the flow is incompressible or not from a stream function? Yes, we can find out the velocity component from this stream function V_x and V_y. Once we know the velocity component V_x and V_y, we can test with the continuity equation whether they satisfy the continuity equation or not.

$V_x = (\partial\psi/\partial y)$ and here equal to $-3y^2$ and $V_y = -(\partial\psi/\partial x)$ and this would be $-6x$. In-compressibility requires $\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0$, and this is satisfied, and so the flow is incompressible. $\frac{\partial V_x}{\partial x}$ is 0, $\frac{\partial V_y}{\partial y}$ is 0. So, that the continuity equation is satisfied and the flow is incompressible.

This is very interesting property of streamlines in stream function.

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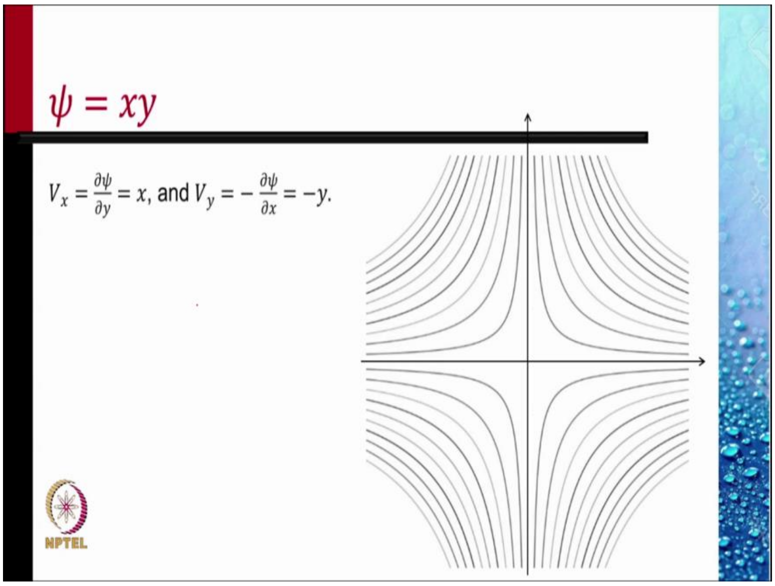

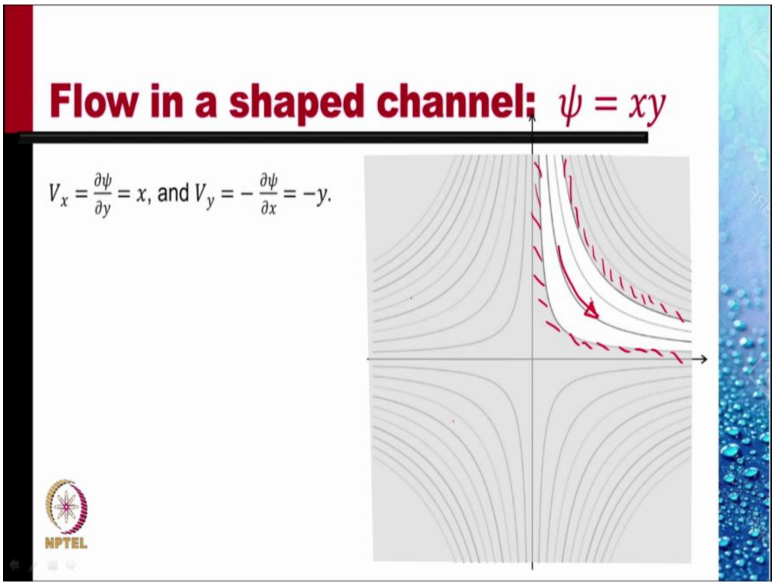

Let us consider the function $\psi = xy$. $\psi = constant$, that is $xy = constant$ lines are plotted in this x-y plane, as the rectangle hyperbolas of this shape. $V_x = \frac{\partial\psi}{\partial y}$, which is x, and $V_y = -\frac{\partial\psi}{\partial x}$ which is $-y$. Now, we have stated that no flow crosses a streamline.

So, if we replace the x axis by a wall, then $\psi = xy$ represents the flow in a half plane above the wall, assuming the viscosity is not there, assuming that the no slip condition does not apply. This is a particular kind of flow which will be considered later, called the irrotational on the potential flows.

So, $\psi = xy$ can be used to represent the flow towards a wall. The flow coming vertically downwards towards the wall, and then the flow would split into two portions. This would be stagnation point, the velocity would decrease as we go down, it would increase as we go away from the origin in the two directions.

Not only this, we may replace the y axis also by a wall so, that this first quadrant is the only region where the flow takes place. $\psi = xy$, the stream function also represents flow in a corner. This is a flow that comes here and goes towards this in one corner with these replaced by solid walls.

So, what is the stream function of this corner flow, $\psi = xy$, simple, and the velocities can be obtained at any x and y by these formula. Not just this there is one more possibility there, then we replace this stream line by a wall and replace this stream line, another streamline by a wall. Then $\psi = xy$ represents flow through this channel as well. And the velocities at any point x and y within this channel of this specific shape would also be obtained by differentiating the stream function xy.

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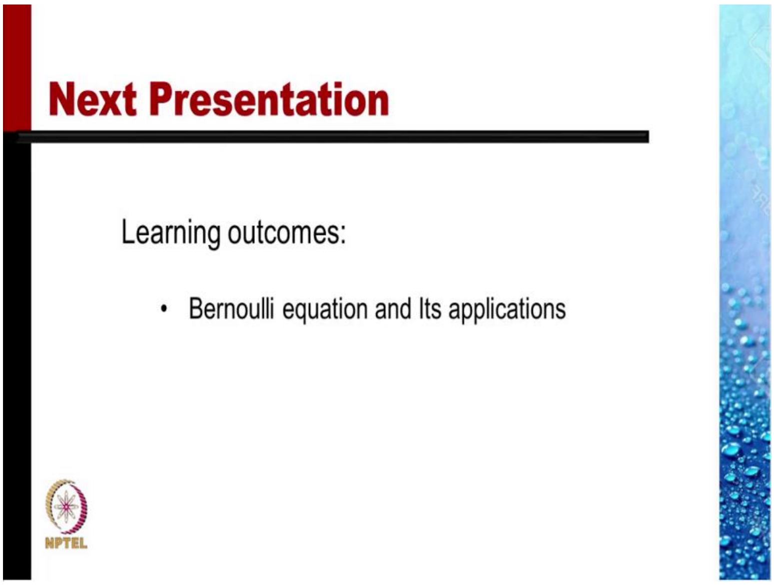

Thank you very much.