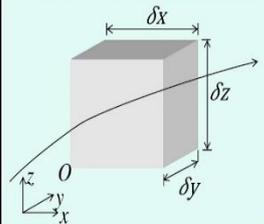


**Fluid Mechanics and Its Applications**  
**Professor. Vijay Gupta**  
**Indian Institute of Technology, Delhi**  
**Lecture 8A**  
**Differential Form of Mass-Balance Equation**

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### Differential Form of the Mass Balance Equation



Rate of accumulation =  $\frac{\partial \rho}{\partial t} (\delta x \delta y \delta z)$

Face with normal in direction	Mass crossing into CV in time $\delta t$ across the face
$-x$	$\rho V_x (\delta z \delta y) \delta t$
$+x$ , a distance $\delta x$ away	$-\left(\rho V_x + \frac{\partial(\rho V_x)}{\partial x} \delta x\right) (\delta z \delta y) \delta t$
$-y$	$\rho V_y (\delta z \delta x) \delta t$
$+y$ , a distance $\delta y$ away	$-\left(\rho V_y + \frac{\partial(\rho V_y)}{\partial y} \delta y\right) (\delta z \delta x) \delta t$
$-z$	$\rho V_z (\delta x \delta y) \delta t$
$+z$ , a distance $\delta z$ away	$-\left(\rho V_z + \frac{\partial(\rho V_z)}{\partial z} \delta z\right) (\delta x \delta y) \delta t$



Let us now go to the differential form of the mass balance equation, the mass conservation equation: an equation that we can apply to every point of the flow field. For this purpose, we consider within a fluid, a cubical fluid element or a cuboid fluid element or dimension  $\delta x$ ,  $\delta y$ ,  $\delta z$  as shown. There is a velocity field with components  $V_x$ ,  $V_y$ , and  $V_z$ . The rate of accumulation of mass within this volume would be, the volume is fixed, volume is  $(\delta x \delta y \delta z)$ .

So, the mass contain this within this is  $\rho(\delta x \delta y \delta z)$ . The only way this mass within the control volume can change is if the density changes. If  $\partial \rho / \partial t$  is the rate of change of density within this control volume, then the rate of accumulation would clearly be  $\text{del } \rho \text{ by } \partial \rho / \partial t (\delta x \delta y \delta z)$  at this time. And this rate of accumulation should be equal to the net influx.

Let us calculate the net influx. Let us consider the face with the normal in the  $-x$  direction, which is the face there. And what is the rate at which the mass is crossing this control surface in time  $\delta t$  across this face is the density  $\rho$ ,  $V_x$  times the area which is  $dz dy$ , and times  $\delta t$ .

Next, let us consider the opposite surface with a normal in  $+x$  direction, a distance  $\delta x$  away. This is the velocity. Such a velocity is positive, and the mass is moving out of this surface. So, this is efflux. That is why we put a minus sign, and we use the Taylor's expansion to get this expression for the net efflux across this surface in times  $\delta t$ . Similarly, we write for the

surfaces with normal in  $-y$  direction, in the  $+y$  direction, in the  $-z$  direction, and in the  $+z$  directions to get these six expressions for influx and efflux. The net influx is obtained by adding these six terms.

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**Differential Form of the Mass Balance Equation**

$$\frac{\partial \rho}{\partial t} (\delta x \delta y \delta z) \delta t$$

$$= - \left( \frac{\partial(\rho V_x)}{\partial x} + \frac{\partial(\rho V_y)}{\partial y} + \frac{\partial(\rho V_z)}{\partial z} \right) \cdot (\delta x \delta y \delta z) \delta t$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho V_x)}{\partial x} + \frac{\partial(\rho V_y)}{\partial y} + \frac{\partial(\rho V_z)}{\partial z} = 0$$

$$\frac{D\rho}{Dt} + \nabla \cdot (\rho \mathbf{V}) = 0 \quad (\nabla \cdot \rho \mathbf{V}) + \rho(\nabla \cdot \mathbf{V})$$

$$D\rho/Dt + \rho \nabla \cdot \mathbf{V} = 0$$

For an incompressible fluid:  $\nabla \cdot \mathbf{V} = 0$ , or  $\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0$

So that the accumulation  $\partial\rho/\partial t$  into the volume times  $\delta t$  is given by the remaining terms of the efflux. The negative sign for converting this into influx, and the volume  $\delta x \delta y \delta z$  cancel.  $\delta t$  cancels, and the resulting equation is  $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho V_x) + \frac{\partial}{\partial y}(\rho V_y) + \frac{\partial}{\partial z}(\rho V_z) = 0$ .

In vector form we can write this as  $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$ . This can be expanded into  $\nabla \rho \cdot \mathbf{V} + \rho(\nabla \cdot \mathbf{V})$ . But this term and this term: what are they? This reminds us of the total derivative formula, Euler acceleration formula that we derived in this last lecture.

This is nothing but  $D\rho/Dt$ , the material rate of change of density, and this results in  $D\rho/Dt$ , the material rate of change of density, plus  $\rho(\nabla \cdot \mathbf{V})$  is equal to 0. This is the differential form of the mass balance equation. Clearly, if the fluid is incompressible so  $\frac{D\rho}{Dt} = 0$ , the material later change the density should be 0, and then we get a simple expression  $\nabla \cdot \mathbf{V} = 0$ , or  $\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0$ .

Since, quite a bit of fluid mechanics deals with incompressible flows, so this equation is the equation quite often used as the equation for mass balance in the differential analysis. This equation is also known as the continuity equation. This is applicable strictly for

incompressible flows or where we can assume the flow to be nearly incompressible: quasi incompressible flows.

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### Example

The tangential velocity component in a two-dimensional incompressible flow is given by  $V_\theta = \frac{K}{r}$  with  $K$  as a constant. What could be the possible radial velocity component?

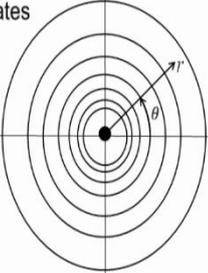
$\nabla \cdot \mathbf{V} = 0$

$\nabla \cdot \mathbf{V} = \frac{1}{r} \frac{\partial (rV_r)}{\partial r} + \frac{1}{r} \frac{\partial V_\theta}{\partial \theta} = 0$  in cylindrical polar coordinates

$rV_r = F(\theta, t)$ , or  $V_r = \frac{F(\theta, t)}{r}$

Simplest solution:  $F(\theta, t) = 0$

A free-vortex



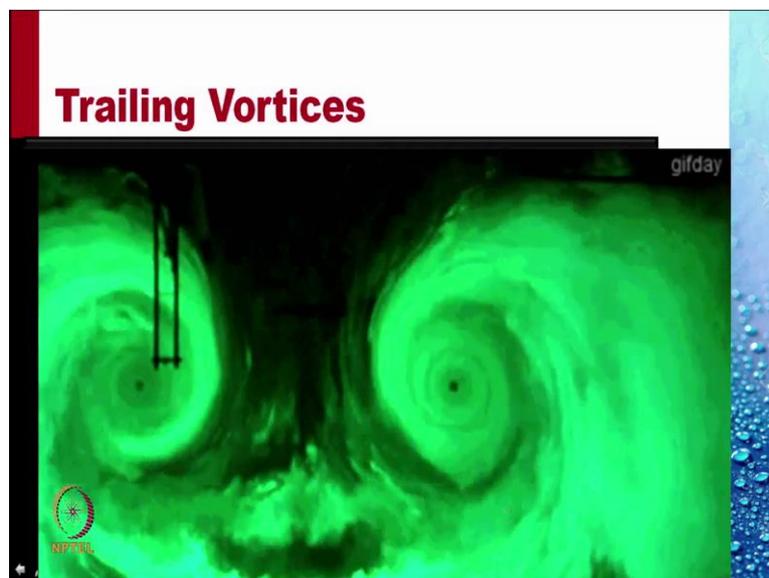
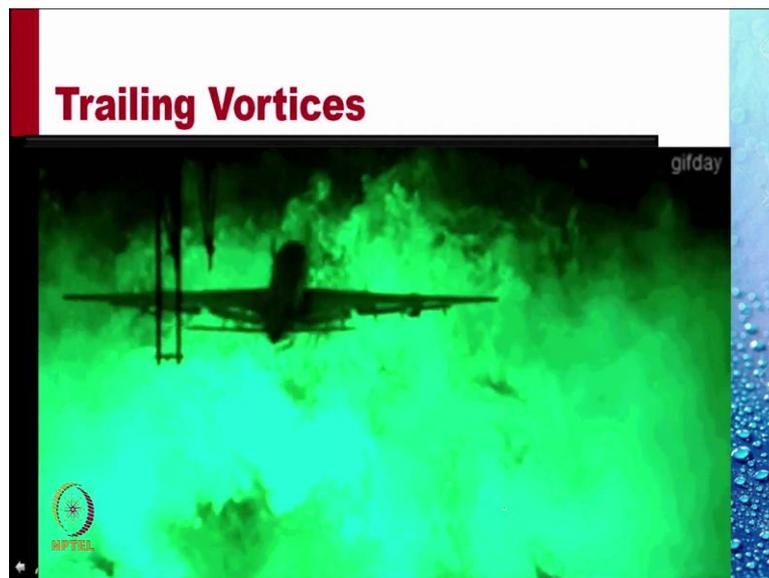

Let us give an example of how we can use this continuity equation. The tangential velocity component in a two dimensional incompressible flow is given by  $V_\theta = \frac{K}{r}$  with  $K$  as a constant. What could be the possible radial velocity component? Clearly, we are now dealing with cylindrical polar coordinates  $\theta$  and  $r$  in two dimensional incompressible flow.

So, we will use the form of the equation  $\nabla \cdot \mathbf{V}$  in cylindrical polar coordinates, and in cylindrical polar coordinates  $\nabla \cdot \mathbf{V}$  is  $\frac{1}{r} \frac{\partial}{\partial r} (rV_r) + \frac{1}{r} \frac{\partial}{\partial \theta} V_\theta$ . And, since  $V_\theta$  is a function of  $r$  alone, not of  $\theta$ , the second term is 0. That means,  $rV_r$  is independent of  $r$ . From this, this is 0. This means that  $rV_r$  must be independent of  $r$ .

It should not change with  $r$ , that is,  $rV_r$  can we function of theta and  $t$  only, not of  $r$ . This gives  $V_r = \frac{F(\theta, t)}{r}$ .  $V_r$  varies inversely as  $1/r$ . There could be dependence on theta and dependence on time. If we talk about steady flow, there will be no dependence on time. So, there could be a dependence on theta. All kinds of solutions are possible for this.

The simplest of course, is when  $F(\theta, t)$  is 0. If  $F(\theta, t)$  is 0, then  $V_r = 0$ , there is no radial velocity. The fluid particles are going around in circular path, with only  $V_\theta$  velocity, which also decreases as  $r$  increases. So, the velocity at near the origin is very large in the velocity dies out as  $1/r$  as  $r$  increases. This is known as a free-vortex in fluid mechanics. We will have occasion to use this free-vortex extensively in later calculations.

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The flow behind an aircraft consists of two trailing free vortices. You can see in this picture the vortices rotating. The flow is rotating. The velocity is very large at the center and is slowing out as  $r$  increases.

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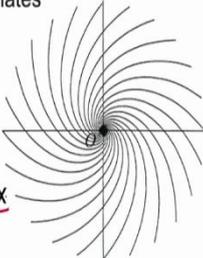
### Example

The tangential velocity component in a two-dimensional incompressible flow is given by  $V_\theta = \frac{K}{r}$  with  $K$  as a constant. What could be the possible radial velocity component?

$$\nabla \cdot \mathbf{V} = \frac{1}{r} \frac{\partial(rV_r)}{\partial r} + \frac{1}{r} \frac{\partial V_\theta}{\partial \theta} = 0 \text{ in cylindrical polar coordinates}$$
$$rV_r = F(\theta, t), \text{ or } V_r = \frac{F(\theta, t)}{r}$$

Another solution:  $F(\theta, t) = -k$

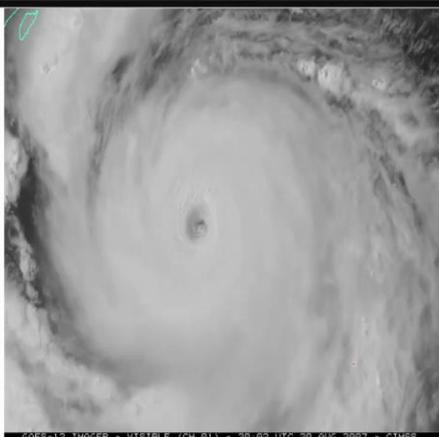
A bath-tub vortex



Another solution is when  $F(\theta, t)$  instead of being 0 is a constant. Let us say  $-k$ . Then  $V_r = -\frac{k}{r}$ . So, there is an  $r$  velocity which also decreases as  $r$  increases, but is negative: is directed inwards toward the origin. A bathtub vortex, a vortex that you see in a bathtub, or when, or when you pull out the plug in a sink, and the water goes around in circles and is sucked in, is called a bathtub vortex.

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### A hurricane can be modelled as a bath-tub vortex



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An example of the vortex is a hurricane. Hurricane can be modelled as a bathtub vortex. The fluid is moving circularly, and there is a component of velocity towards the center of the hurricane.