Introduction to Interfacial Waves
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Lecture - 54
Applications of Faraday waves - atomisation and spray formation

We were looking at the stability of a interface on a pool of liquid which was subject to vertical vibrations. We had done the analysis in the last video and we had found that the interface the time dependence of the interface is governed by a Mathieu equation. We had looked for standing wave kind of solutions and we had obtained expressions for the perturbation velocity potential.

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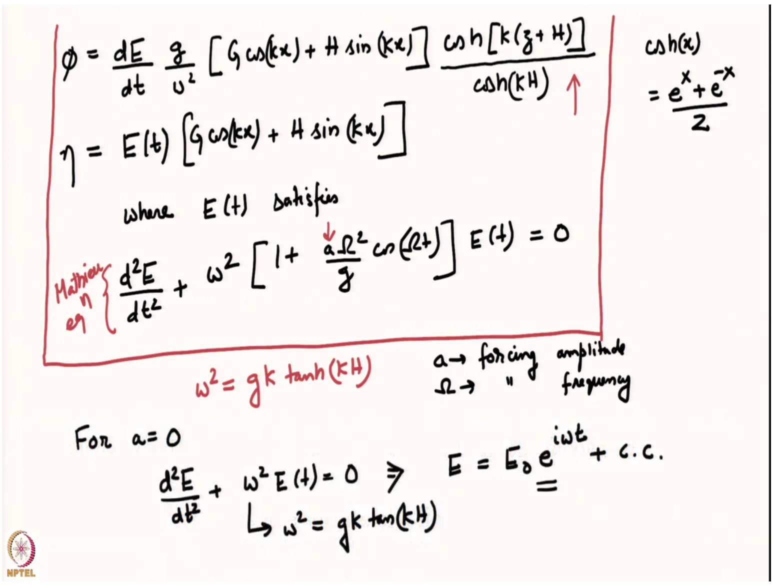

And the perturbation on the interface eta. In particular, we had found that the time dependence E of t is governed by this equation and we had understood that this was the

Mathieu equation that we had encountered in the course earlier. Now, let us look at the application of this Mathieu equation to this particular problem. So, as you can see in the for 0 forcing. So, for a is equal to 0. So, a was the forcing amplitude and omega was the forcing frequency.

So, for a equal to 0, we should recover our previous results. So, you can see that this equation for E just becomes a simple harmonic oscillator equation; for small a is equal to 0. And we have seen that this equation can be solved in the form E is equal to E 0 e to the power i i omega t plus some complex conjugate.

Note that in this case, we already know omega. So, for the Faraday wave case, we had only put gravity in the problem. So, this had turned out to be g k tan hyperbolic k H. There was a pool of depth capital H and so, this was the dispersion relation. So, we already know omega and if you substitute it into this problem, you will see that this is a solution of this equation where E 0 is a complex constant. So, we just recover what we had understood earlier from normal mode analysis.

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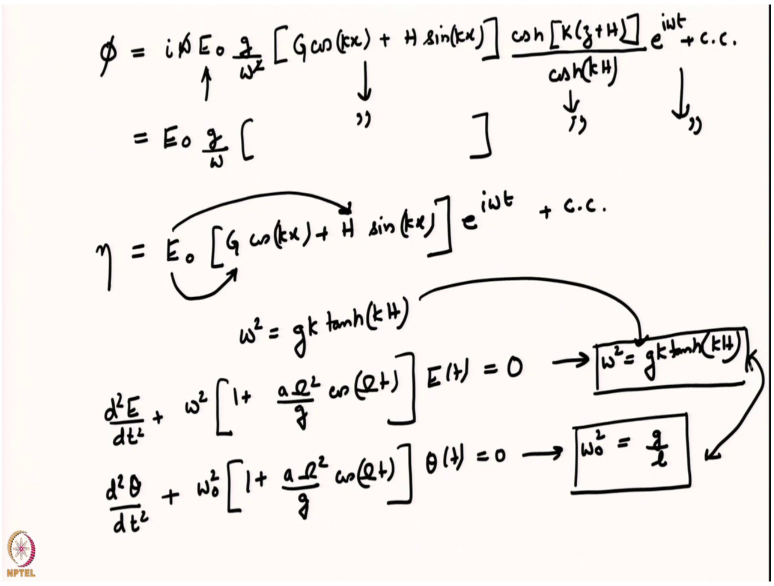

Also, note that the perturbation velocity potential takes the form i omega E 0 g by omega square. I am doing a dE by dt. So, there is a dE by dt. So, if E is of this form, then taking a time derivative of this will bring i omega. So, that is why I have an i omega in my expressions. So, I have i omega E 0 into g by omega square into the rest G cos k x plus H sin k x and, because this E 0 is a complex constant. So, I will have to also add its complex conjugate now.

And you can simplify this and write this as E 0 into g by omega. So, I am missing of cos hyperbolic factor here. So, plus c c. And then, I can simplify this the omega and the omega cancel out and the rest remains the same. So, you can see that this E 0. So, similarly there is an expression for eta also, which is just E 0 into G cos k x plus H sin k x.

So, the same thing here; whatever is here; whatever is here the same thing here into e to the power i omega t plus c c and this can be absorbed into these constants this E 0 and they will become new constants, but they will become new complex constants. Similarly, this E 0 in the expression for phi can be absorbed into G and H and we can write it as a new complex constant. And you can see that for the unforced case, we are just recovering what we had known earlier. This is for the dispersion relation omega square is equal to g k tan hyperbolic k H.

Now, I encourage you to try the derivation of the Faraday wave problem with surface tension in into it. Interestingly, it also has applications particularly with surface tension into it, because as we will see shortly; there are applications in drug delivery where the device dimensions are very small and so, the waves that are created are mostly Faraday.

The Faraday waves that are created are mostly capillary waves and gravity plays a very small role there. So, I encourage you to try this derivation of the Mathieu equation having both gravity and surface tension and you will find that you exactly get the same equation. Except that now, you will have to add in the dispersion relation that I have written in the bottom of the slide here. So, you will get g k tan hyperbolic k H or rather g k plus t k q; the whole thing multiplied by tan hyperbolic k H.

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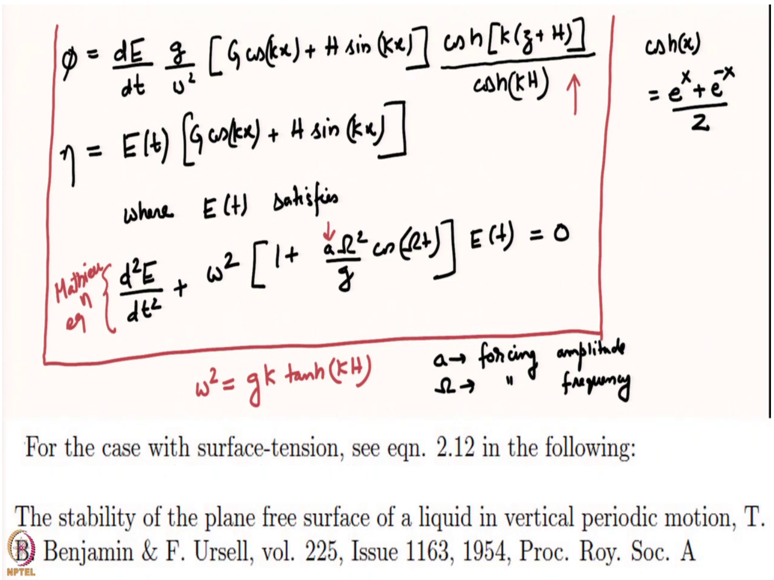

This is the dispersion relation for free oscillation that we had already seen earlier. So, it will just be a modified omega square. The equation will remain the same. So, now, having verified that we are recovering the correct results in the unforced case, let us now understand this Mathieu equation and try to relate it.

Particularly, we had derived the Mathieu equation earlier for the Kapitza pendulum. Now, we have obtained the same Mathieu equation for linearized surface waves on a pool of liquid whose undisturbed depth is H and which is being vibrated up and down with a frequency capital omega and amplitude small a.

So, now let us try to understand what is the implications of this Mathieu equation. We know that the Mathieu equation can be written in a non-dimensional form that we have already written earlier. And there are we can draw a stability chart of the Mathieu equation. There are tongue like structures on the stability chart and inside those tongues one obtains unstable response.

So, let us understand what is the meaning of instability here. So, our Mathieu equation. So, we have got a Mathieu equation d square E by dt square plus omega square 1 plus a omega square by g cos omega t into E of t is equal to 0. We had got a similar Mathieu equation earlier and there it was for theta, because that was the angular displacement of the Kapitza pendulum.

And there we had got in dimensional variables, we had got here omega square was this omega. So, omega square is equal to g k tan hyperbolic k H. And here, the omega naught square was the natural frequency of the pendulum. So, g by l. So, you can see that this and this play analogous roles in the two problems.

Now, let us understand what is the implication for the stability chart of the Mathieu equation. Let us draw those tongue shape structures again and let us understand what is the meaning of instability here. So, we had seen that generically I can nondimensionalize my equations. So, I am going to write it as an generic variable sum theta tilde.

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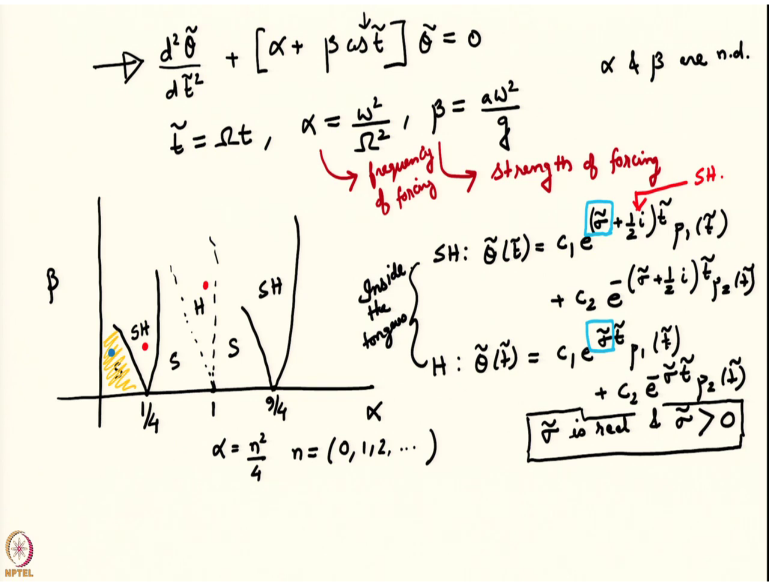

So, you can nondimensionalize E here you can nondimensionalize this variable this equation. And let say that I have nondimensionalized this equation and I am writing it in terms of some non-dimensional E which I call it as theta tilde. So, we have seen earlier that you can write it in this nondimensional manner, where now t tilde is a non-dimensional t and it has been nondimensionalized by the time period of forcing or rather the forcing frequency then, alpha is again a nondimensional variable the ratio of the natural frequency to the forcing frequency.

In the natural frequency in this case, is related to the dispersion relation and beta which is a nondimensional measure of the strength of forcing. So, there will be a small a here and then, small a is nondimensionalized by omega square by g. So, you can see that alpha and beta are non-dimensional. Similarly, time is non-dimensional and theta tilde has is a suitably scaled non-dimensional version of E.

So, now, with this the equation we have seen the stability chart of the equation on the alpha beta plane. Recall, that this is a measure beta is a measure of strength of forcing strength of forcing or amplitude of forcing. Now, what is the maximum displacement on either sides of the equilibrium position. And this is a measure of the frequency of forcing. It is a non-dimensional measure.

Now, we have seen earlier that the beta alpha chart for this equation, we have done a floquet analysis of this equation and we had got a number of qualitative conclusions about the solution to this equation on the beta alpha plane. In particular we had seen that there are the special points where alpha is equal to n square by 4; n being 0 1 2 and so on.

So, let us start with 1. So, this is 1 by 4 and so, there is a tongue which goes out like this. Then, at 2 square by 4 which is just 1, there is another tongue. It also goes out like this and then at 3 square by 4; which is 9 by 4, there is one mode. I had told you that the response to this the solution to theta tilde from this equation is stable, if it is in these regions between the tongues.

But if it is inside the tongue, we have a unbounded response. We have an exponentially growing response, but it is an oscillatory exponential growth. So, in particular we had seen that this was the sub harmonic tongue, this was the harmonic tongue this was again a sub harmonic tongue and so on.

Sub harmonic here, implies that the oscillatory response is one half of the forcing frequency. Here, the note that the forcing frequency is 1, because of nondimensionalization. This corresponds to a time period of 2 pi and so, we will see here, what is the response once we are inside the tongue shortly.

So, we had seen that in the sub harmonic tongues, the solution inside the tongue is of the form the solution to the Mathieu equation plus a similar quantity with a negative sign. Note in particular; that this is the sub harmonic tongue and hence, there is this factor of half i, which basically just means that if the forcing is forcing frequency is 1, then the response frequency from this term is half of that.

So, this is why this is the sub harmonic response p 1 and p 2 are two periodic functions with the same time period as the forcing frequency. So, in this case in scale variables it is just 1. So, which it has the same time period. So, the frequency is 1, the time period is 2 pi. Similarly, in the harmonic inside. These are all inside the tongues. And in the harmonic inside the harmonic tongues also, there is instability.

So, the that is C 1 and once again, p 1 and p 2 are periodic functions of time with the same time period as the forcing time period. And you can see that in both the solutions there is a e to the power this part of the solution will cause divergence in time, because sigma recall in these solutions was greater than sigma tilde is real and sigma tilde is greater than 0.

We had written this earlier, while analyzing the Kapitza pendulum. So, you can see that in both if we are inside the sub harmonic or the harmonic tongue, we are going to get an exponential growth in time. But it will be an oscillatory kind of an exponential growth and the dominant response in a sub harmonic tongue will be one half of the points of the oscillation will be one half of the forcing frequency; whereas, in a harmonic tongue, it will be the same as the forcing frequency.

The frequency of oscillation will be the same as the forcing frequency. So, now let us look at some numerical solutions to this equation in various points on the tongue. So, in particular I have chosen three points. So, one point here as I as the region indicates. So, this point is stable, because it lies in this region. Then, I have chosen something which is inside the first sub harmonic tongue. So, I will indicate it with red and it is inside the first sub harmonic tongue and I have chosen another point inside the first harmonic tongue.

So, the points in red are supposed to give me an oscillatory response, but whose amplitude grows exponentially in time. The point in blue here, this point lies outside the tongue and it is in the stable region. So, here I am supposed to get an oscillatory response is not periodic, but it is just oscillatory, but does not its amplitude does not grow in time it does not amplitude does not diverge in time.

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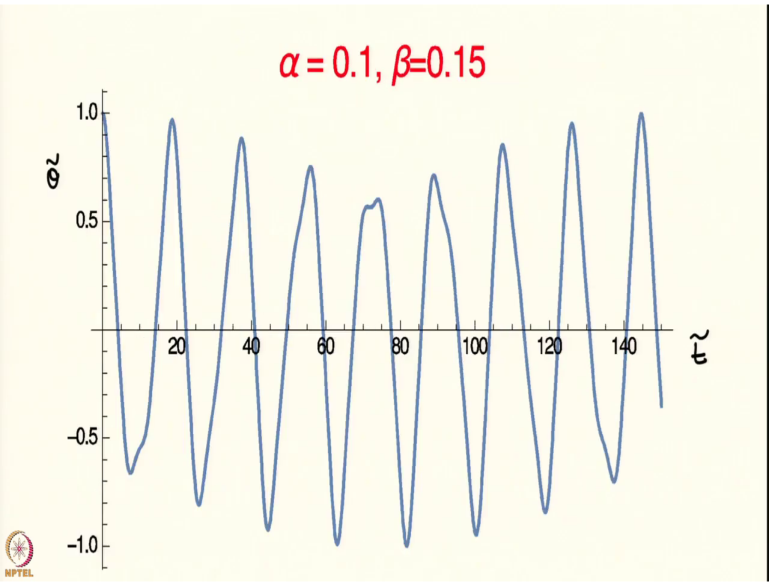

So, let us look at those numerical solutions. So, this is the solution. So, what I am plotting here, is just theta tilde as a function of t tilde. So, it is just a numerical solution to the Mathieu equation that I have written here, for some choice of alpha and beta. So, you can see that this corresponds to. 1 alpha equal to 0.1 and beta equal to 0.15.

So, it is. So, you can see that alpha is less than one fourth it is. 1. So, it is very close to the origin. So, it is somewhere here and I have chosen beta to be about 0.15. So, this is outside the first sub harmonic tongue and it should give me an oscillatory response. You can see the oscillatory response. The amplitude changes with time, but it there is no exponential growth.

This corresponds to initializing your interface with some Fourier mode and then, the Fourier mode will just the interface as a response to the Fourier mode, will just go up and down in time with a certain time dependence which is indicated in this figure. Pay attention that this is predictions from linearized theory. So, this will not be very good, if the amplitude of the perturbation grows very large. Now, this is the stable response.

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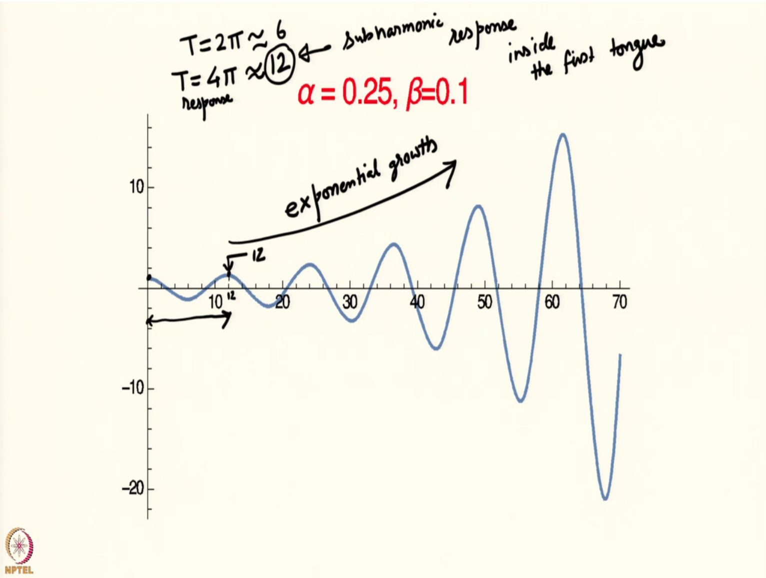

Let us choose an unstable point. Recall, that the first unstable tongue on the beta alpha plane starts from alpha is equal to 1 by 4. So, I have chosen alpha equal to 1 by 4 alpha is equal to. 25 and I have chosen beta to be a small number. So, I am just inside the first sub harmonic tongue. Recall, that our equation the forcing term was this. The forcing frequency in the scale coordinates is just 1.

It is omega t tilde, but omega is just 1 here. This implies that the time period of forcing is 2 pi. In the non-dimensional sense, the time period of forcing is 2 pi. Now, this is we are inside the sub harmonic tongue. So, you can see that. So, here the time period of forcing is 2 pi. Because we are inside the sub harmonic tongue, the response will have a e to the power i t by 2. So, this is the sub harmonic response and it has a half factor. So, this implies that the oscillation will happen at one half of the forcing frequency.

The forcing frequency is 1. So, the oscillation frequency will be half the time period will be double. So, the time period of the response here, will be 4 pi or approximately 12. You can see that this is a this is an exponential growth. This is happening, because we are in an unstable regime.

So, what is happening is you we have put a Fourier mode on the surface. The amplitude of the Fourier mode as the container is being vibrated up and down the amplitude of the Fourier mode is growing exponentially in time that is, because there is a e to the power sigma tilde factor here, where sigma tilde is real and greater than 0. So, that is causing the amplitude to grow.

Let us look at the frequency or rather the time period. So, as we said earlier the time period here is, because we are inside the sub harmonic tongue, the time period here is expected to be double. This is approximately 6 and so, this is the time period of the response of the surface is going to be double. You can see that this is approximately.

So, this is 1. So, you start from here and this is approximately one oscillation of the interface and you can see that this is about this point is about 12. So, this is 12 that is 14 16 18 and 20. So, this is about approximately 12. So, this is a sub harmonic response inside the first tongue.

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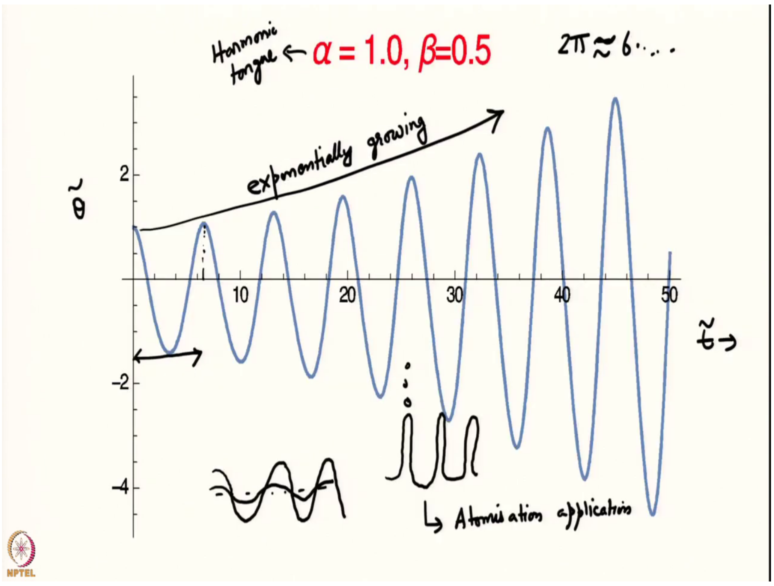

Let us go to the second tongue. Recall, that in the second tongue; the second tongue was a harmonic tongue. So, this we just looked at the solution here. This was the sub harmonic response. Now, we are here in this red point and this is inside the harmonic tongue. By definition this implies that here too the response will grow exponentially in time; however, the response will be at the forcing frequency, because there is no factor of half here. So, this p 1 is just a function which has the same frequency as the forcing.

So, here I expect the oscillation. So, here too you can see. So, this alpha is 1. So, this is the harmonic tongue. Once again, this is theta tilde as a function of t tilde. You can see that the amplitude is growing exponentially. What about one oscillation time period. You can see that it is about 6 point something which is basically 2 pi ok. So, this is approximately 6 point something 6.28 ok.

So, this is a after one oscillation of the interface. So, we are shaking it with a certain frequency. The interface responds with the same frequency in the unstable regime. In the previous case, we are shaking it with a certain frequency. The interface responds with half the frequency in both cases the amplitude grows with time and grows exponentially fast.

Now, what does this growth lead to. This is where the engineering application comes in. So, this growth. So, let us plot it here. So, if we have a. So, let us have the interface here. I have a Fourier mode let say and I am causing I am inside one of the; one of the unstable tongues. So, I am going to grow exponentially. The interface, the amplitude of the Fourier mode is going to grow exponentially. So, it is going to grow bigger and bigger with time.

Eventually, it will cause the production of these jet-like structures from the interface and there will be droplets ejected from the tip of these jets. This is where the application of atomization comes in. A quantity of practical interest is to predict the sizes of these droplets. We will see how, but let us first understand; how to plot the stability charts on the dimensional on the plane of dimensional quantities.

We have right now, plotted the stability chart on the alpha beta plane. Recall that alpha and beta are non-dimensional. Let us correlate this with the dimensional stability chart.

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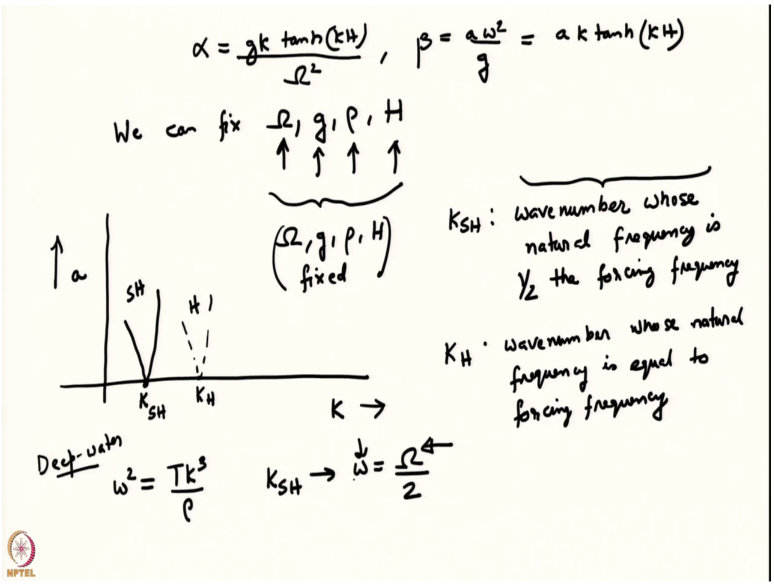

So, the dimensional. So, alpha by definition is g k tan hyperbolic k H divided by omega square. It is just small omega square by capital omega square. Beta is a omega square by g. And if you plug in the formula for omega square then it just becomes a k tan hyperbolic k H. Now, we can fix omega the forcing frequency; acceleration due to gravity is anyway fixed. We can choose a liquid of certain density rho and we can fix the depth of the pool.

So, all these let us say are known numerically. So, we know numbers or numerical values for each of these things. So, you can see that we can use the alpha beta chart the chart on the alpha beta plane to draw a corresponding chart on the a k plane. So, this is a dimensional chart and I will have a and a k here on this chart also. So, this is the. So, this is a measure of strength of forcing and this is the wave number which is being which is being imposed on the interface.

So, here also you will get these tongue-like structures sub harmonic and so on. The important question to ask is what are these wave numbers. So, if I call this as K sub harmonic and this as K harmonic, then you can readily see that what is K sub harmonic. So, K sub harmonic is that wave number. So, all this is for a given value of rho g omega g rho and H. So, these things are fixed and once you fix these things, one we can draw this chart.

So, K sub harmonic is the wave number whose natural frequency is one half the forcing frequency. Similarly, K H is that wave number whose natural frequency is equal to forcing frequency. So, this can be used for practical applications. Recall, I told you that in atomization applications, we are typically inside the tongue and that causes the amplitude to grow in time and it may lead to a jet from whose tips droplets come out.

So, we can see that the sizes of those droplets has to be of some correlation to the wave number of the mode which is unstable. One can typically use these unstable wave numbers to estimate the size of the droplet. Let us see how. So, we are going to simplify the dispersion relation and we are just going to use, because this atomization applications is typically used at very small scales. So, gravity is ignorable as I told earlier and it is mostly surface tension driven oscillations.

Recall that I told you that you will get the same Mathieu equation, if you put in surface tension into your analysis. And even if you drop gravity, you will just get the Mathieu equation where omega square is just the dispersion relation for gravity waves. We are going to take the deep water approximation just for simplicity and so the dispersion relation will just become omega square is equal to T k cube by rho.

There will be a factor of tan hyperbolic K H which is just one in deep water and I am ignoring gravity. So, this is my dispersion relation and this is the omega square which will appear as a coefficient of the corresponding Mathieu equation that we have already did. Now, let us see what is K SH and K H. So, K SH is by definition that wave number whose omega is equal to omega by 2.

The natural frequency is small omega the forcing frequency is capital omega. So, as I have written here, K SH is that wave number whose natural frequency is capital omega by 2. So, we can quickly make an estimate of K SH.

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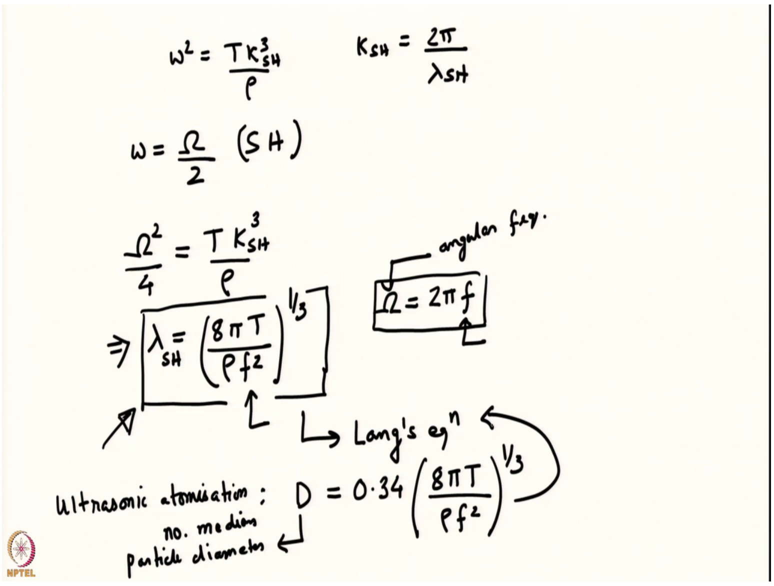

So, omega square is equal to T k cube by rho. We will put omega is equal to capital omega by 2 for the sub harmonic response. So, this makes it K SH. So, this makes it omega square by 4 is equal to T K SH cube by rho. If I go from K SH is equal to 2 pi by lambda SH, then you can show very easily from this equation that this equation just becomes lambda is equal to 8 pi T by rho small f square to the power one third.

Here, I have related capital omega, capital omega is the angular frequency. So, I have just related it to small f. So, this is the angular frequency radians per second small f is has the dimensions of per second whereas, capital omega has the dimensions of radians per second. So, I have used the formula. I have used this formula to go from capital omega to small f here.

Now, this equation is a very well-known equation. It is called the Lang's equation and it essentially arises. So, this is basically lambda SH. And it essentially tells us gives us a way of estimating what are the what is the wavelength of the most unstable mode, if I shake a container at frequency small f.

Now, this can be used for estimating the median diameter of the droplets which are ejected during atomization process. This application is a very well-known application. It is called ultrasonic atomization. And Lang's equation actually is a has a correction on this. So, Lang's equation says that there are number median diameter number median particle diameter of the liquid particles which gets ejected as a result of atomization.

So, this is number median particle diameter. When I say particle, I mean fluid particles which I have in indicated here. So, these are small these small circles are the fluid particles and they are correlated to the size of the mode which first became unstable as we entered the tongue the sub harmonic tongue.

Typically, in if the pool is sufficiently deep, the it is the first sub harmonic tongue we which actually shows the response. So, the wave number gets selected from the first sub harmonic tongue. So, D just has a prefactor which is empirically obtained and it is. 0.34 and it is the rest is just the same. And this is actually the Lang's equation. This formula finds a lot of applications in obtaining rough estimates of the sizes of the droplets which are ejected as a result of ultrasonic atomization.

This particular application has lot of applications in the field of drug delivery. I have accumulated. I have provided in the next slide a few references and if you are interested, you can go through those references to understand. Just as we have estimated K SH, you can also estimate a K H.

All you have to do is, just take the dispersion relation and replace small omega by capital omega. For the K H, the response is harmonic and so, small omega is exactly equal to capital omega and that will give you the wavelength of the mode, whose natural frequency is equal to the forcing frequency.

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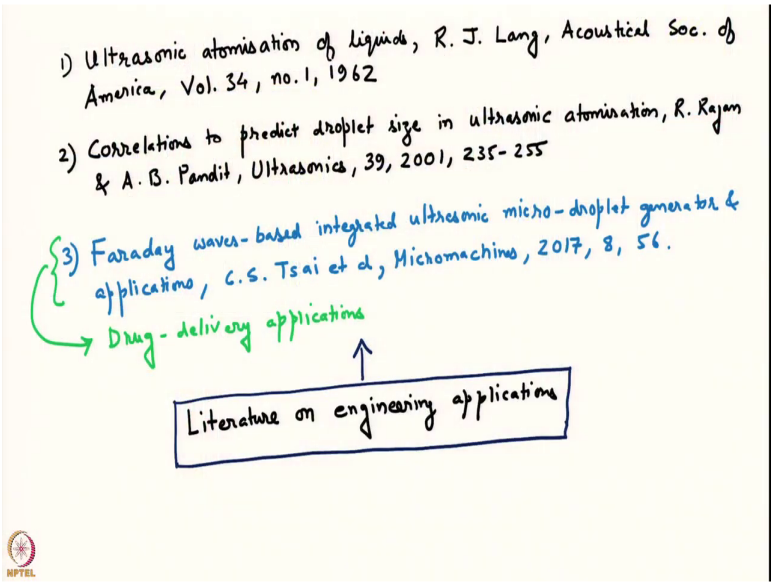

So, these are the papers some of. There is an extensive literature on this subject of ultrasonic atomization. The first two papers if you read these you will get some idea about where does the Lang equation come from. We have discussed it briefly and the factor of capital omega by 2 that comes, because one is typically inside the first unstable mode comes from the sub harmonic term.

And so, the first two papers discuss contain an extensive discussion on ultrasonic atomization. And the third reference in blue, contains a lot of very interesting applications of ultrasonic atomization in drug delivery. Here, the container is being oscillated at a frequency which is close to ultrasound frequency and hence, the word ultrasound ultrasonic atomization is used.