Advanced Numerical Analysis
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Lecture - 08
Gram-Schmidt Process and Generation of Orthogonal Sets

So, this is the process by which given a set of linearly independent vectors in a inner product space.

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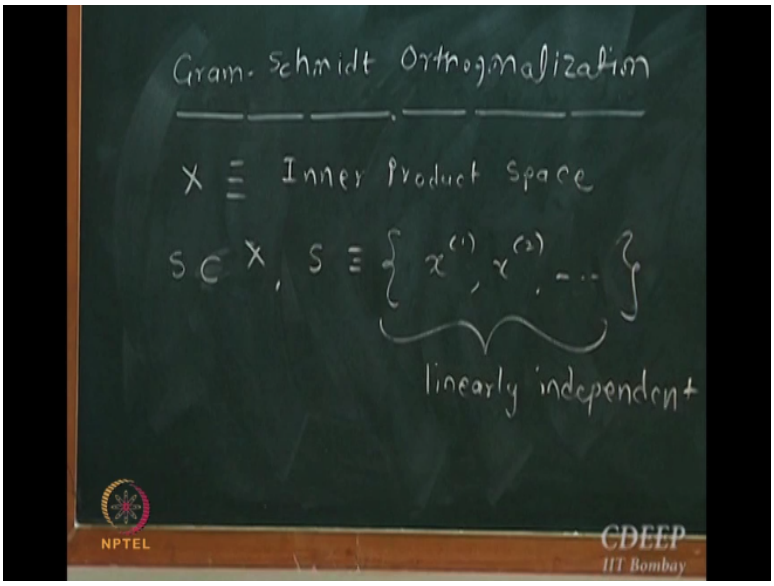

So, x is my inner product space and I am given some set of linearly independent vectors. So, I have this set s, which is a subset of x and s corresponds to say x1, x2 and so on. I have given a subset of vectors in a inner product space. This could have finite number of vectors; it could have infinite number of vectors. Right now, I am not worried about how many vectors are there in this set. They could be finite; they could be infinite.

All that I know is that, this vectors are linearly independent. But, this is not an orthogonal set, s is not an orthogonal set. What is an orthogonal set? The vectors are mutually orthogonal. You take any pair of vectors and find an inner product, inner product will be 0. So, that is where you have a set to be called as orthogonal set.

So, this is not an orthogonal set and I would like to generate an orthogonal set because, orthogonal sets are very, very useful when you do modeling, when you do applied mathematics, numerical computations. So, how do I do that? I start by defining a unit vector.

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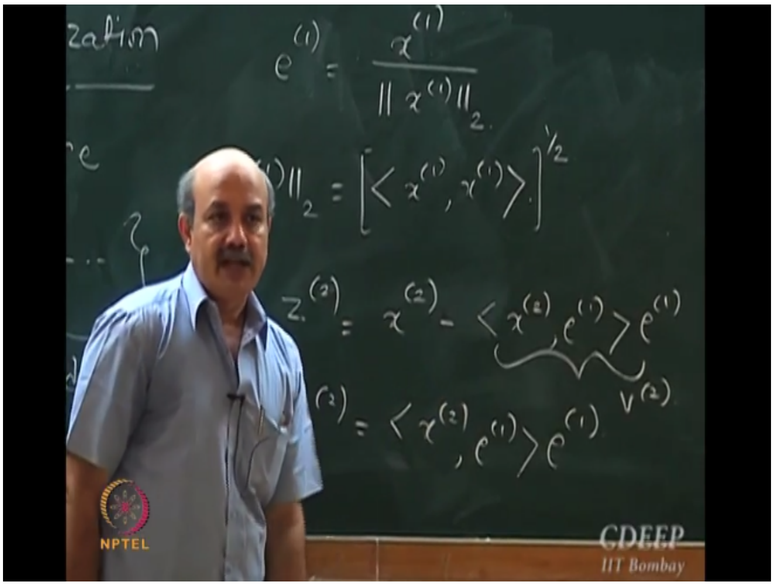

So, e1, my e1 is going to be x1 divided by norm x1 okay. Well, inner product defines a norm, so my norm x1 is nothing but 2 norm is nothing but inner product of x1 itself raise to half. So, this is my first vector, this is a unit vector. I want to create a set starting from this set, I want to create a set, which is not just orthogonal, but which is orthonormal okay. I want to create unit vectors, which are orthogonal to each other okay.

So, this is my first vector. What I do next is well, orthogonality allows us to split a vector into 2 components. One along a direction and one orthogonal to the direction okay. That is the concept which I am going to use in Gram-Schmidt Process. So, what is my first thing? First thing is I pick up now this vector x2 here and then I create a vector z2, which is x2-x2e1 inner product of x2e1. So, this gives me component of x2 along e1 okay times e1.

You can very easily check that this vector and so if I define another vector say v2, which is, I have 2 components of vector x2, z2 and v2 okay. This is one component x2-v2 is another component. This is nothing but, this is what I am calling as v2. x2-v2 and v2, I am decomposing the vector x2 into 2 orthogonal components.

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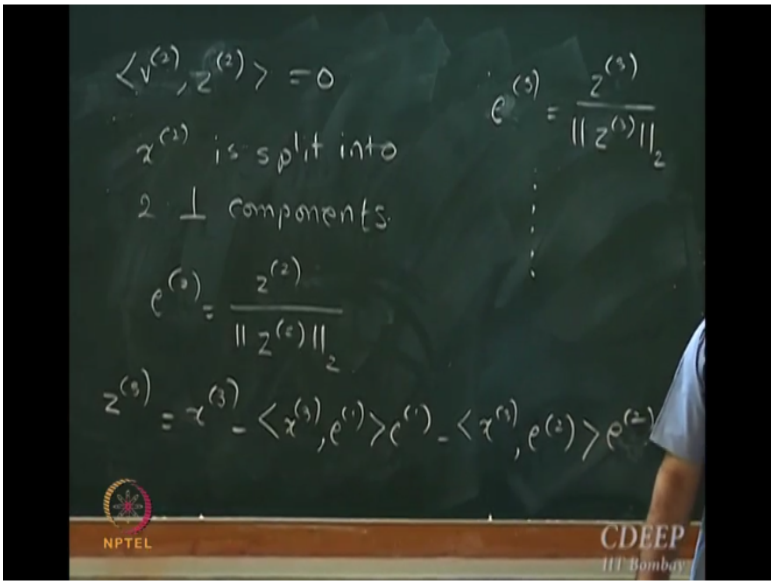

It is very easy to check that inner product of v2 and z2 = 0 okay. Just substitute and find out. We will get inner product to be 0, not at all difficult, these 2 are orthogonal components. I am splitting vector x2 okay. So, x2 is okay. So now what I do next? Well, I got 2 directions which are orthogonal. One is e1 and other is z2. See because v2 is just some scalar times e1 right. So, one direction is e1 and z2, these are orthogonal to each other and then I define e2, which is z2/norm z2 okay.

So, I got 2 directions e1 here starting from the first vector, then I removed the component along e1 from x2, I created z2 then I just normalized z2 to create e2 okay. So, now there are 2 directions e1 and e2, both of them are unit magnitude. This is a unit magnitude vector right and e1 and e2 are orthogonal okay. I just do this by induction. So, I take x3, I remove component along e1 and e2, whatever remains I make it unit magnitude. I go to x4, I remove from e1, e2, e3, just go on doing this.

So, this process is called as. So, my next step would be defined z3, which is x3-x3 inner product e1 e1-x3 inner product e2. This is my vector z3 and then using z3, I can define e3, which is z3/norm z3 and so on okay. So, given the set of vectors, which are not orthogonal, I can just follow this systematic procedure to split, to create a set, which is orthonormal okay and creation of this orthonormal set is called as Gram-Schmidt Process okay.

So, now let us start today doing something. Let us actually look at some examples and let us create some orthogonal sets starting from some non orthogonal sets okay.

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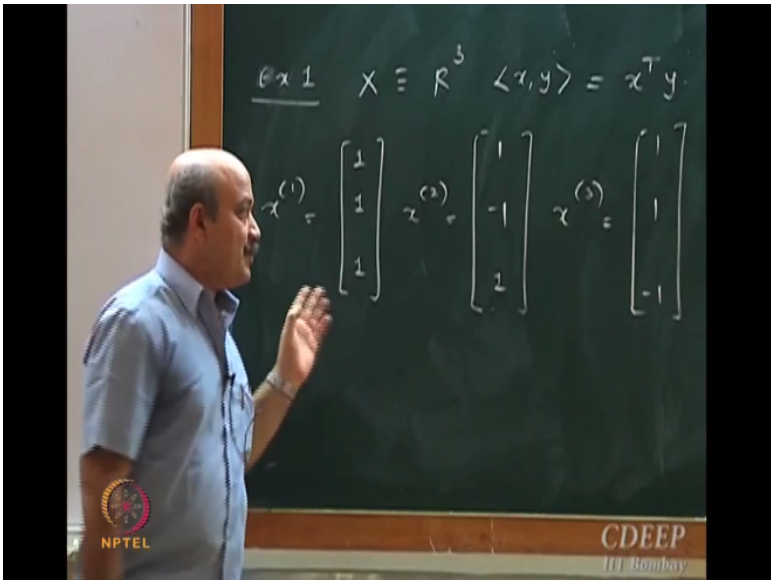

So, my first example is going to be in R3. My inner product space is simply R3 and a inner product between any 2 vectors is simply x transpose y okay. And I am given 3 vectors x1, which is 1 1 1, x2, now I want you to do this by hand, want to start doing it 1 -1 1 and x3 is 1 1 -1. Are these linearly independent? Are these 3 directions linearly independent in R3? These are linearly independent. Are they orthogonal? They are not orthogonal.

You take inner product of any 2, you will not get 0. So, these are not orthogonal directions. I want to construct a orthogonal set starting from this non- orthogonal set. I want to apply this process. So, just start doing this, what will be e1? E1 will be simply 1/root 3 1 1 1. What will be z2? Just calculate.

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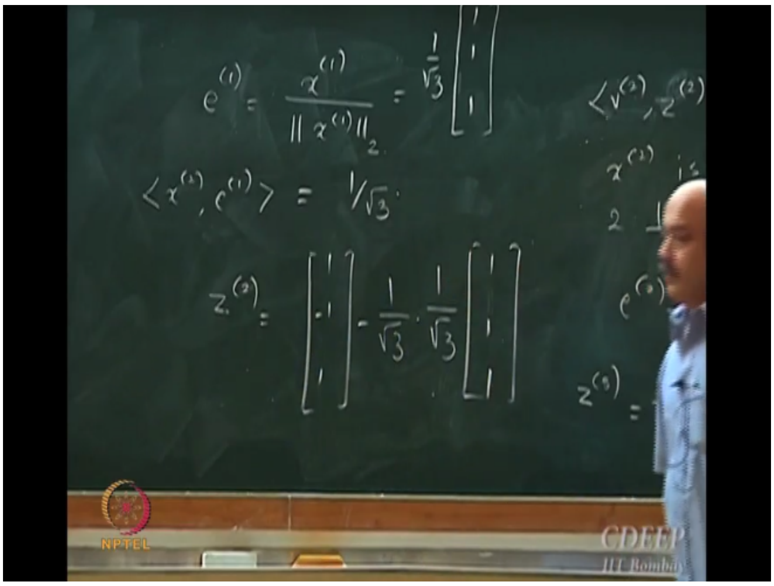

So, we have to start with what is inner product of x2e1 what is this quantity? X2 is this vector. 1/root 3. So, what is this second vector. What is z2? Will be 1 -1 1 - 1/root 3 times 1/root 3 1 1 1. So, what is this vector? 2 third - 4 third and 2 third.

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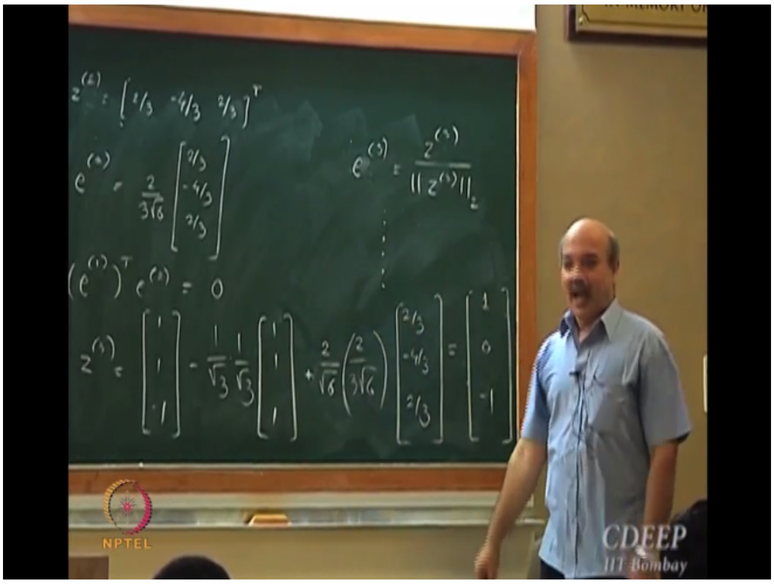

So, this gives you z2 is you said 2 third - 4 third 2 third transpose right okay. So, this is my z2. So, what is e2? You have to help me with this. 2/3 root 6 2 third - 4/3 2/3. This is my e2. Just check whether e1 and e2 are orthogonal. What do you get if you do e1 transpose e2 what do you get? Do you get 0? If you do not get 0, you have made a calculation error. You must get a 0 here if you take this inner product.

And those who have done this, just go ahead to e3. Compute e3. Does this turn out to be 0? It does. Just check. If you take inner product of this with e1, you should get 0 vector, not 0 vector 0 magnitude. Inner product should be 0. E1 transpose e2 should be 0, perfectly 0. If you are not getting it, there is some error somewhere. Is it 0? I do not hear. Yes, its 0 okay. What about next? What about x3, what about z3? 1 1 -1. What is inner product of x3 and e1?

1/root 3*1/root 3 1 1 1 then - what is the inner product here? So, what is the number that should appear here inner product. E2 with -2/root 6. So, this becomes 2/root 6. Is that fine? Because there was. Did we have a -? Is this correct? Is this - correct? This is correct. What about here, this becomes +. Does it become +? This is fine okay. So, what is this vector finally? Can somebody help me with this? What are these 3 numbers? Well, after you find this z3, you have to make it indeed magnitude, you have to divide it by its magnitude.

But, that is a simpler part. What will be. Has anyone completed? You tell me. This could be, I am just writing here, I am not doing the calculations. I am just writing here. Somebody is prompting me. So, you have to tell me whether this + is correct or it should be - minus here? + is correct here. So, what is the total? 1 0 -1. See, our friend says 1 0 -1, everyone agrees or there are some different calculations or.

By the way, 1 0 -1 is this orthogonal to this and this vector. It seems to be this is this 1 0 -1 is orthogonal to this vector, this direction. Forget about the multiplying factor, this direction is orthogonal to this. What about here 1 1 1. Yes, it is. So, these are mutually orthogonal because, this and this are orthogonal, this and this are orthogonal, this and this are orthogonal. So, 1 0 -1 seems to be. I am not sure about the multiplying factor.

Is that correct? 1/root 2 okay. z3 is 1 0 -1. So, e3 becomes 1/root 2 1 0 -1. So, I started here with 3 vectors in R3 okay, which were not orthogonal and then systematically I could construct set of 3 vectors, which are unit magnitude orthogonal vectors okay.

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So, if you try to, you have started with something like this in R3, 3 vectors which are linearly independent okay. Let us say x1, x2 and x3 starting from this, what you have done is you have created a set, which is orthonormal okay starting from a set, which was linearly independent but not orthogonal okay. So, we had a situation like this. We moved to orthogonal set okay. That is what we have done.

Now, specific vectors that you get here will depend upon how you define the inner product okay. I just wanted to repeat one calculation. Well, subsequently all the calculations will change but, if I change my definition here of the inner product okay. The subsequent calculations will change. The directions may not change in R3 but, magnitude calculations can change. And I want to emphasize this one small thing. Is this clear?

We started with a non orthogonal set and we came up with 3 directions, which are orthogonal to each other okay. So, now, let me just do one small change here. My second example is again R3.

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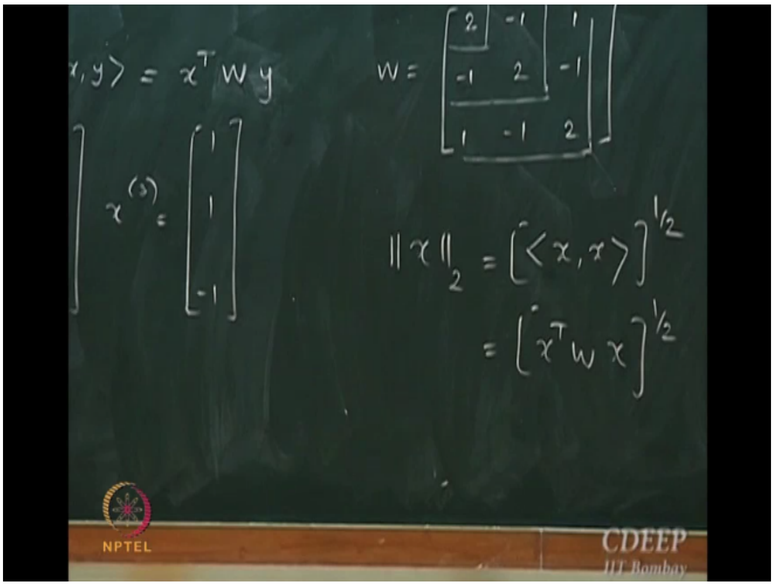

But, I am going to change the definition of my inner product to x transpose w y, where w is a symmetric positive definite matrix. I am going to pick up one particular matrix here. I am going to pick up this matrix w. well, there are many ways you can pick up a symmetric positive matrix. Take a matrix, which is simply diagonal elements, which are positive okay. So, that is one way. I am going to do with this matrix 2 -1 1. Is this a symmetric puzzle design matrix?

This is a symmetric matrix, that is for sure. Is this puzzle definitely? Do you know test for finding out puzzle design matrix? Any other test? Principle minors. This is greater than 0. 2 is > 0. Is this determinant greater than 0? This determinant is greater than 0. What about this determinant? All 3 put together. Calculate, is the determine greater than 0? There is simple algebraic test to find whether a matrix is positive definite or not.

Look at these matrices constructed by first element and first 2 cross 2 matrix, then 3 cross matrix, is it positive? Okay, it is a simple test to find whether a matrix is puzzle definite or not. So, this is a positive definite matrix and x transpose w y will define a inner product on R3, this inner product is different from what we defined previously okay. So, now just remember what is our 2 norm? 2 norm is x inner product x raise to half. So, in this case, it will be x transpose wx raise to half.

All their calculations will have to be done using x transpose w. so, what will be the unit direction now? What is my first vector?

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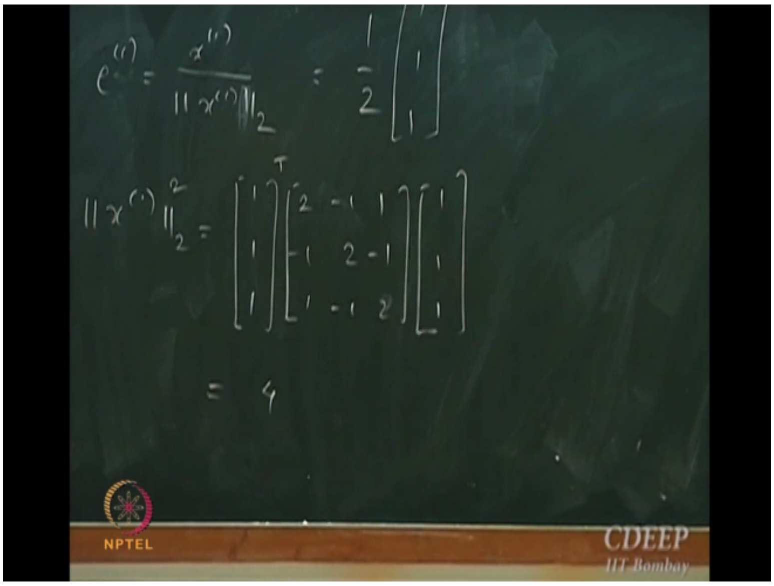

What is the very first vector that you, what will be e1? E1 is x1 divided by norm x1. What is norm x1? So, you have to work with 1 1 1 transpose this matrix. So, norm x1 square is 2 -1 1, -1 2 -1, 1 -1 2 * 1 1 1. So, what is this quantity? 1 is 2, this multiplication comes to be 2, square is 4. So, this comes out to be 4 and then square root of this 2 okay. So, what is the first direction? the first direction e1 becomes half 1 1 1, this is different from what we got earlier right.

Earlier we defined inner product in a different way. So, the norm which was defined through the inner product was different and the unit vector was different. With this definition of inner product, say this definition of inner product here x transpose w y, where w is a symmetric puzzle definite matrix, that induces a 2 norm. the 2 norm of 1 1 1 using this definition of inner product turns out to be 2. 2 norm square is 4 right.

And unit vector, the direction is same but, the vector is different right. Earlier we got 1/root 3 1 1 1 okay. Now, I am getting 1/2. So, what I want to do is, further what I want to stress here is all further calculations will have to be done using this inner product. Do not forget this okay. So, in exam if I give you a problem which has a matrix w, see, earlier we had a special case, w was identity matrix 1 1 1 okay.

If I give you a different matrix w, you have to keep using that matrix every time you calculate inner product in that example. Because R3 with this inner product is a different inner product space than what with w=i. R3 with w=I and R3 with w = this matrix are 2 different inner product spaces okay. Calculations will be different, very, very important. Is this clear? I am not doing the further calculations. We will move on to some other example okay.

Well, now I want to graduate from finite dimension spaces to infinite dimension spaces and then we will see how we start meeting some of the old friends that you have known in your undergraduate curriculum. So, now inner product space is any set which satisfies certain axioms right. And we have generalized the concept of inner product space. Now, I am going to look at set of continuous functions.

My inner product space is going to change my inner product definition is going to change. So, my third example and this is where now you have to do some work out okay. And you have to help me on the board and how do we come up with vectors, which are unit vectors okay. Then we start developing vectors which are orthonormal. We start with a non-orthogonal set and develop an orthogonal set, same idea okay.

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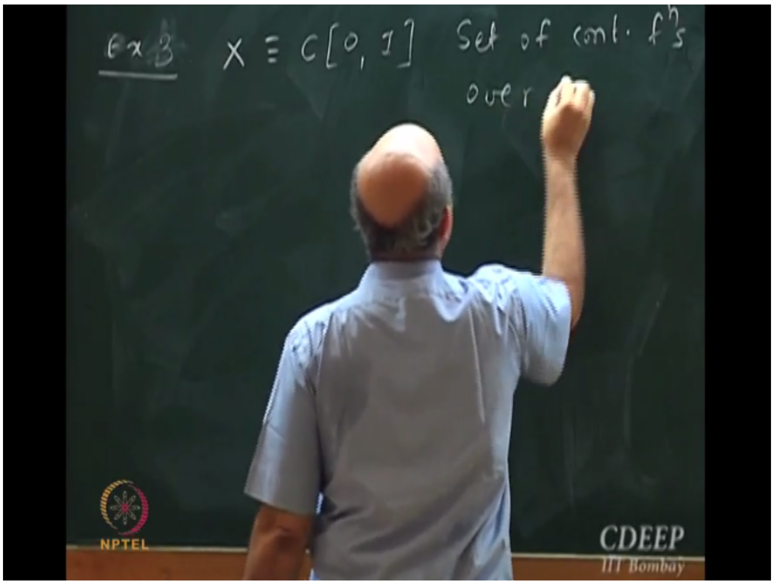

Now, my x is set of continuous functions over 0 to 1. My inner product space is set of continuous functions over intervals.