Medical Image Analysis
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Level Sets, Geodesic Active Contours, Mumford-Shah Functional, Chan-Vese

Hello, and welcome back. So, in the last set of videos we are going to look at the level sets based image segmentation algorithms. Specifically, we are going to lok at some geodesic active contours, which is an improvement to the snakes model we saw earlier. And then we are also going to look at the channel based segmentation which is derived from the Monferrato function.

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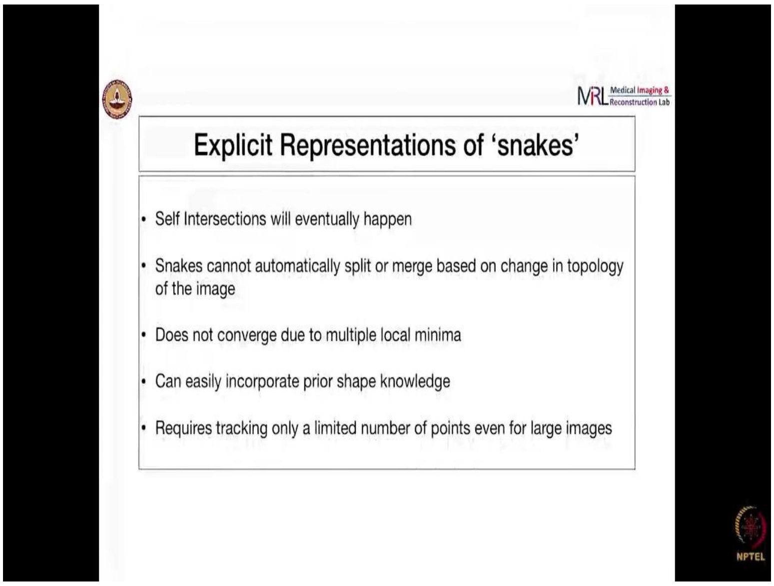

So, what is the problem with the snakes algorithms that we saw earlier, because that self-intersections will eventually happen. Also, when we initialize the contour, it does not automatically split into different parts of the multiple objects in the image. So, a priori, if let us say, if you have 5, 6 objects in the image that you want to segment, then we need to initialize 5 or 6 independent contours.

So, automatically splitting and merging of contours does not happen. And in general, convergence to local minima will occur because of the gradient term, there are going to be quite a few noisy pixels in an image. And eventually those will contribute to the local minimum. So, that problem is always there. But advantage is that you can incorporate prior shape knowledge, we can look at this in the next class, next week's lectures.

So, next week lectures, and you also have the advantage that you only have to track a limited number of points, even for very large images, so that is a advantage. However, the other biggest problem is the convergence, local minima, as well as are difficult to program in terms of keeping track of the contours so, that they does not intersect with itself. And also if you want to split and merge, if there are multiple objects in the image.

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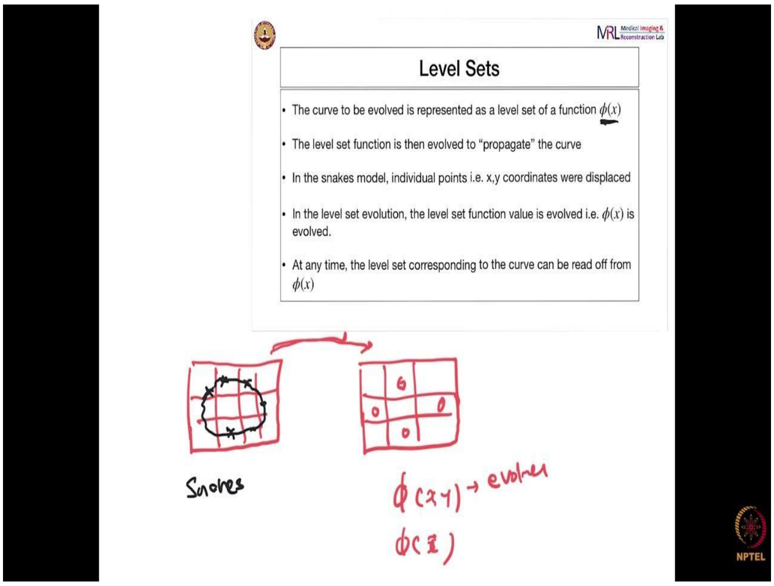

So, this is where the level sets representation comes in. So, what we do here is that we embed the curve that we are trying to propagate into this level set function. Typically, are denoted by phi. And we actually evolve the level set function that is, the function is evolved to propagate the curve. But how is it different, here we are actually changing if you think of the phi as a function, then we are changing the value of the function at every point.

But in this next model, we are actually displacing the point itself. So, which is that is a big difference between the two. So, we will look at what that means, the next few slides, but in general, what we do is we will embed the curve in the function as phi. So, how do we embed the curve in phi, we will may we will embed as a 0 level set, or ending from typically the 0th level set.

So, phi is like, think of it as a function, which has values everywhere in the sense, if you discretize it, you can think of it as a 2D grid, same size of the image, and has equal number of pixels as the image and every pixel has a value. And we embed the curve in that function by setting those pixel values to 0 that correspond to the curve. That is typically how is it, so, for instance, so typically, I will just give you a very small demonstration here.

So, for instance, if you have, let us say original image size, so and let us, I am just going to make this into a small grid. So, if you have a curve which is something like this. So, typically, you would in an image, you would keep track of individual discretize this curve and keep track of these individual points and then move them and then you displace those points. And those points as you saw earlier, would be displaced or so as to make sure that the curve overall is smooth and also that it gets closer to the edge of the image.

So, it will be driven by the image forces so, in the case of this is your typical this is your snakes representation. Now, when you talk about, let us say, level set representations, let us redo this very simple. So, what we will do is set to 0 so, this values so, this is the function phi of x, actually phi of x comma y, but we will just use phi of x as a vector something.

So, you will set the values to 0 of this function wherever there is a curve so, approximately that what we see here and you would evolve phi of x comma y. This is evolving. So, the value of this function will change. So, then where are we how we keep track of the curve is by figuring out where the function is 0, where phi of x comma y is 0, and then connecting those points will give you the curve in turn.

So, going from here to there, this formulation helps in many ways, and we will see how that helps. Because all the problems we saw earlier with snakes that is the self-intersection inability to split into multiple or merge, et cetera. Those are easily solved in this context.

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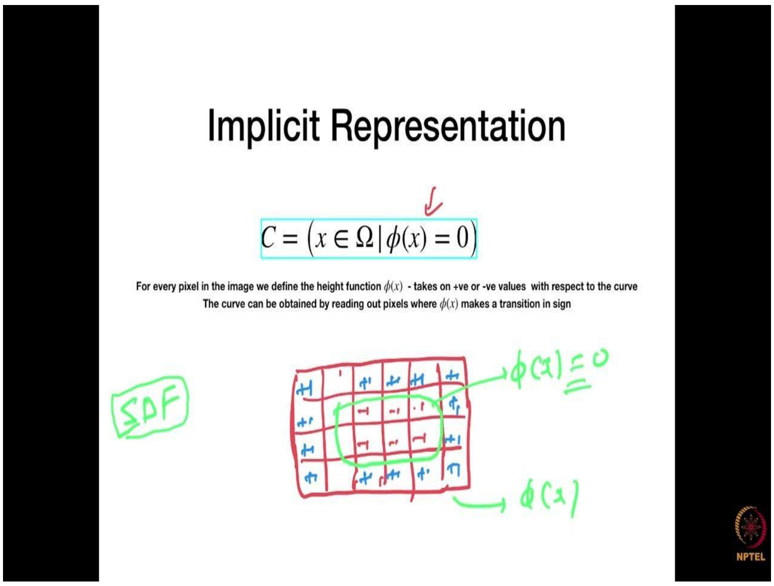

So, how do you represent the curve? So, it is implicitly represented a sense that we have to solve for phi of x equal to 0, that is what this means. So, for every pixel in the image, we define the height function phi of x, so that is why the phi of x will be the same size as the image once we discretize this of course, here we are always talking in the continuous picture. But when you are in the real world, when you solve the problem, the image discretize so, phi of x will also be discretize.

And every for every x phi you will take on a value discretized value of x with 5 will take on a value which we will call the height function, it can easily take on positive or negative values with respect to the curve. So, and we can read out the curve by reading out the pixels where phi of x makes a transition inside. So, what this means is typically you this is typically referred to as a sign distance function, but I want to show you a slightly easier version like.

So let us say I just have to draw a few more ticker grades so, as to this easier so, if you think of it, so let us say the object so, everything inside, you would put minus 1, I would say inside outside what I mean by that. And everything outside I am putting out same value, but typically it is depends on the distance from the curve, but you would do something like this that is just one area where it is minus 1.

So, your curve would be something of this curve, this would actually be a curve because that is where the transitions happen. So, and this would be the value phi of x so, this whole image or this grid is phi of x. So, for every pixel in the image, you will define a position x and phi of x will have a value and it will be positive or negative depending on whether it is inside or outside the curve. And this is usually referred to as the signed distance function.

So, how we do that is if you define all the zeros, we define this. This is your initial contour, let us say you initialize it, then all the pixels were phi of x equals 0 and then you calculate the distance or the distance of each one of the pixels outside and inside from this curve and that we or the shortest distance from this curve and that is usually the sign distance function. (())(8:49) is abbreviated as SDF.

So, this is the typical representations, I have given you a much simpler one wherein everything inside the curve is negative, outside the curve is positive and the transition regions will tell you where actually the phi of x equals 0 always and that will be the segmentation that you are seeking.

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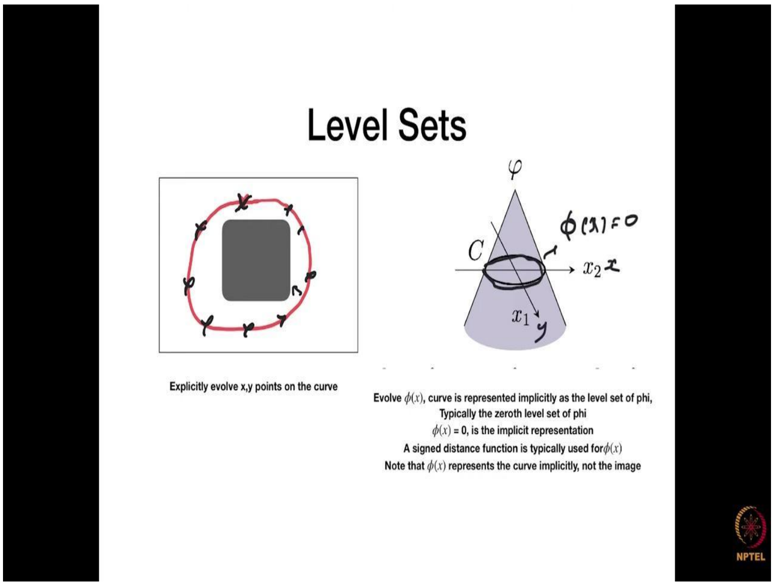

So, here just to give you again once again reiterate this point, so, if you are doing the usual snakes algorithm, then you have the discretizations will use a different color, the continuous curve, but you have a discretization of this curve. And you would move this curve towards the edges of the image, this is the edge let us say this is the object you are looking at. However, in the case of this level sets, the curve is represented as a implicitly as phi of x equal to 0. All the points where phi of x equals 0, you will connect them to get the curve.

So, in this diagram, so if you see, what I have done is how do we extract the curve? Basically, here is you can think of this as a x and y, which in 2D, so, and at this plane, this particular as a circle here is this corresponds to phi of x equal to 0. So, we just have to cut at that point to get the curve and the shape is automatically represented. So, phi is you think of phi as a height function. And, when you all you have to do is figure out where phi equal to 0, and that will give you the shape of the curve at any time that you want. We will look at further.

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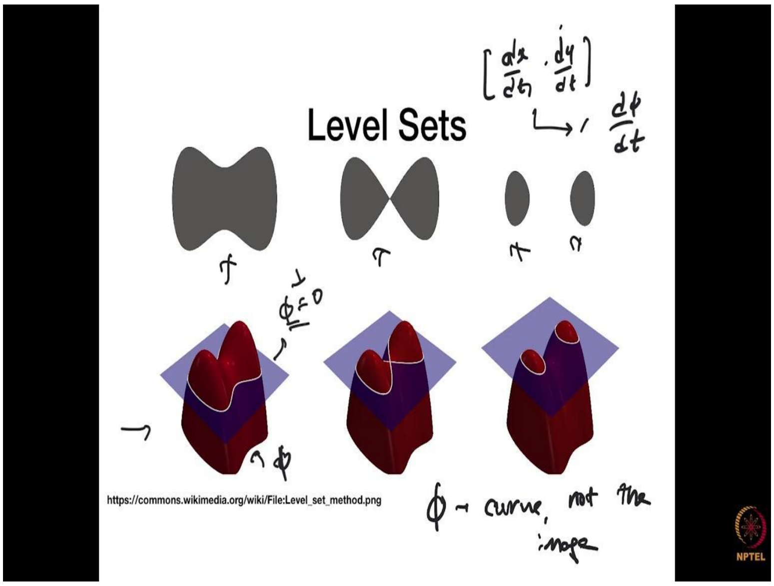

So, for instance, like, these are some of the ways of how you can get curves at different, like for example, segmentations that are that connect multiple objects, et cetera. So, for instance, this red is the phi, it is here, it is a shown as a height function. And then this plane where you are cutting corresponds to phi equal to 0. So, depending on where the plane cuts through, you can get different types of shapes.

And in fact, if you have disjoint objects, those are also particularly correctly segmented. So, just by tracking phi equal to 0, we can just wish we will be we do not have to worry about self-intersections, whether we can track multiple objects, et cetera. But remember, that 5 actually tracks the curve and not the image. So, you keep updating phi, based on whatever formulation you have, variational formulation you have, but in the end, extracting the curve of just makes it figuring out where phi equal to 0.

So, then we do not have to keep track of individual points in the curve, we do not have to treat no beforehand the number of objects et cetera. So, that is the level set framework. But there are some idea about calculations that you have to do, some more derivations are have to be done, because you are going from your time, you are going from dx by dt, dy by dt, that you are calculating, you are tracking x and y points on the curve.

Now, you have to go to d phi by dt, you have to figured out d phi by dt changes so, that then you keep updating phi. And then you just have to figure out where phi equal to 0. So, this relationship is from here to there, what is the relationship that is what we have to (())(12:35).

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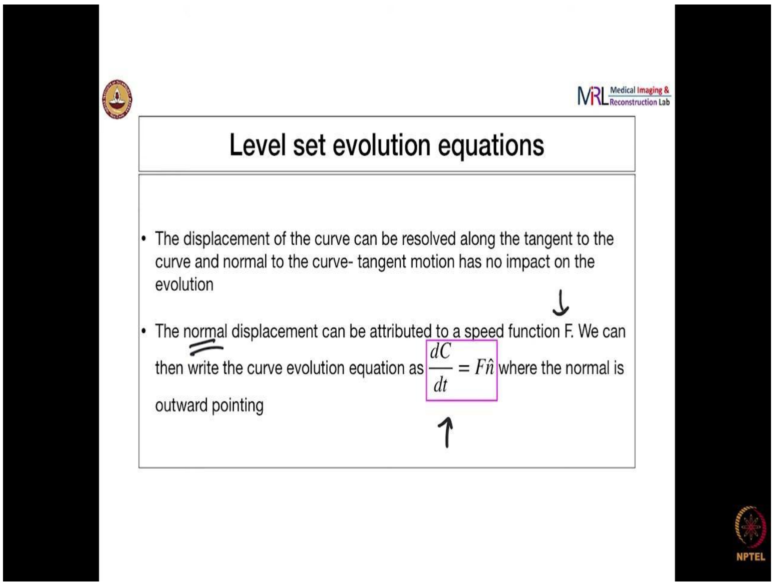

So, the level set evolution equation, so we are looking at so far, the next algorithm purely looks at evolution of x comma y on the curve. So, now we are going to look at evolution of phi, how does phi change with time. So, before that, let us look at some of the developments that led to this level set formulation. So, the displacement of the curve typically can be resolved along the tangent to the curve and normal to the tangent.

Say, tangent motion has no impact on evolution. Because if you are moving along the tangent then the curve stays in place. Curve does not shrink or expand or deform in any way, curve just remains as it is. So, the normal is typically what the displacements normal to the curve or what actually causes the deformation of the curve and also for the curve to move. So, the normal displacement, we will typically attribute it to some speed function f.

And we can actually write the curve evolution equation as the $dC/dt = Fn_{cap}$, Fn_{cap} where n is the normal which is outward pointing or some case inward pointing normal.

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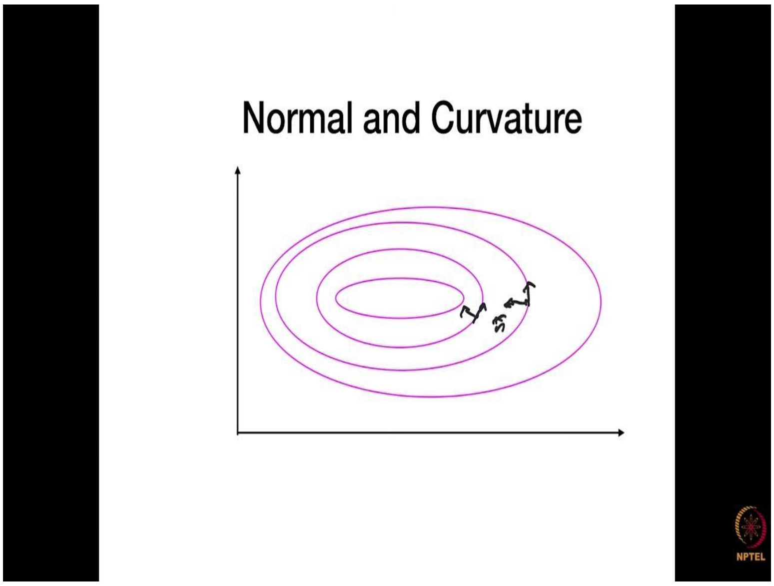

So, this is what I was talking about. So, if you look at this set of contours. The normal let us say at any point will, let us say point in this direction. And the tangents will be there. So, let us set a point by point so, you can look at it at different points. So, along the tangent to these curves, there is no motion only along the normal you have you have some kind of component of the motion.

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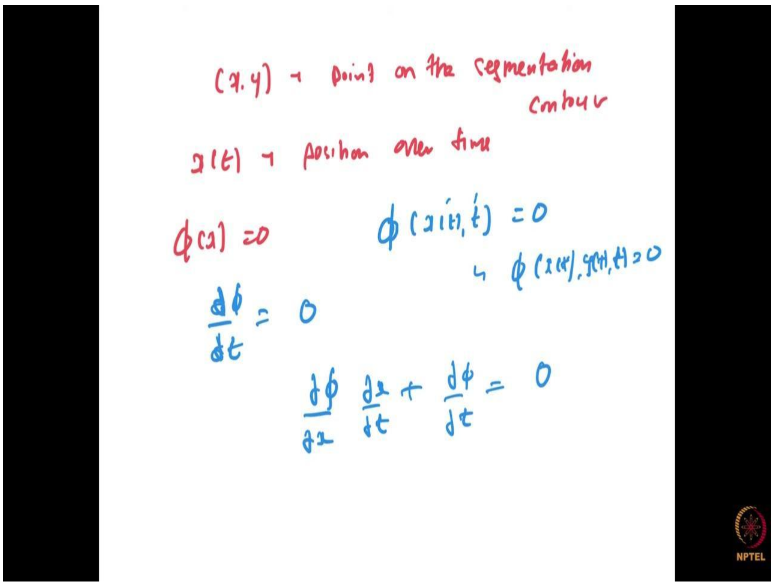

Then let us try to relate how this phi changes corresponding to how x comma y move? So, if we consider a point x comma y, we are only going to do 2D so, we do not have to worry about the dimensions, but it can also be done in 3D. So, we can say x comma y point on that curve, point on the segmentation contour. Tried to see how evolves and related to and related to how phi will evolve.

Now we know that we are already stated that $\phi(x)$ equal to 0 is what determines the contour. So, wherever the segmentation contour is present phi at that point or those points will be 0. So, if x comma x of t, so we know that it is (())(15:24) evolve this in time. So, x of t is position over time, this position over time. Here, this means that we are evolving the contour under the influences of several forces, we saw that there is one due to the gradient, others due to the regularization terms.

So, this transfer $\phi(x)$ comma equal to 0 is our segmentation contour. So, that will directly translate to phi of. This is true because this is segmentation, that is what we want. So, this is the zeroth level set, and it always gives us the contour, it does not matter what shape it is, we just want to make sure that this we when we extract phi equal to 0, it is the contour.

So, if we know phi at t equal to 0, which is possible because we always initialize the contour, then at any time t we know that d phi by dt equal to 0, I make I am excluding the explicitly a dependence on x x, here, when I say x comma x of t, I went x comma y, I can also write $\phi(x_t, y_t)$.

So, if $d\phi(x)/dt$ whereas across the dependence on x of t and t, then using chain rule we can write this as so, we will write this as delta phi delta x in delta x t plus delta phi t. So, it is just a d phi by dt equal to 0. Now, this actually translates to what, this will actually translates to we can write it in the following form. Let me go to the next page that it is easier to write.

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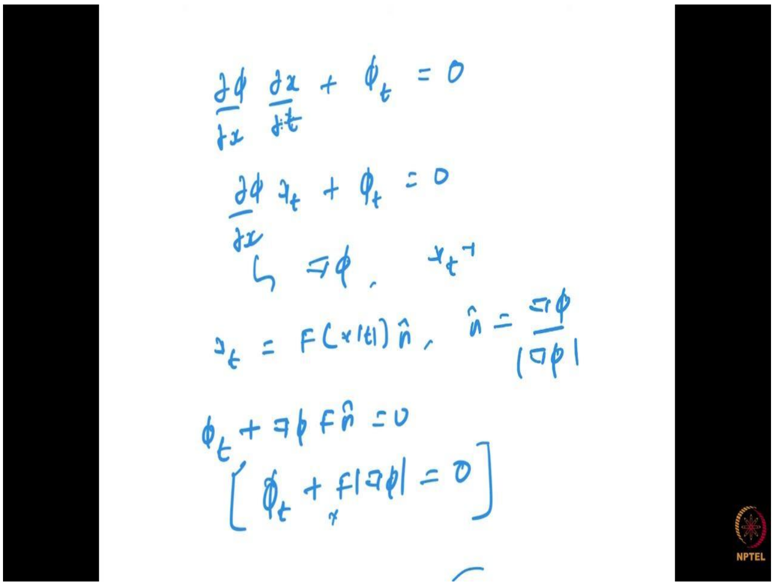

So, we have delta phi by delta x plus let me rewrite this let us call this phi t this is partial derivative with respect to t x equals 0 so, then we can again rewrite this, so, we get delta phi by delta x is x t plus phi of t 0, so, this is nothing but gradient of 5. And x t, x t is just a speed because we want to speed the sense the speed at which x moves. And we know that this speed is always going to be along the normal to the surface or normal to the contour.

So, we can always say then say x t equal to some speed function f. And N hat, where now H hat, it is very easy to see that N hat will be proportional to this, this is the normal. Why would this work because phi is nothing but a phi equal to 0, if you look at the level sets of phi, the gradient is always going to be perpendicular to the level sets at every point. So, that is why gradient of 5 will give you a normal so, we can then write instead of x t we will replace it with some speed function.

So, phi t plus gradient of phi some speed function F n cap equal to 0 or if you after some rewriting, we can always come to this form phi t plus F gradient of phi. So, this actually gives the equation of motion of phi given that x has though if x moves, what is the corresponding change in phi, this is what is basically this represents at some level, because F here tells you the speed function which corresponds to x where the motion of x and y points so, that is what this is the relationship.

So, we will use this is what is typically used to rewrite everything so, there are other quantities also involved.

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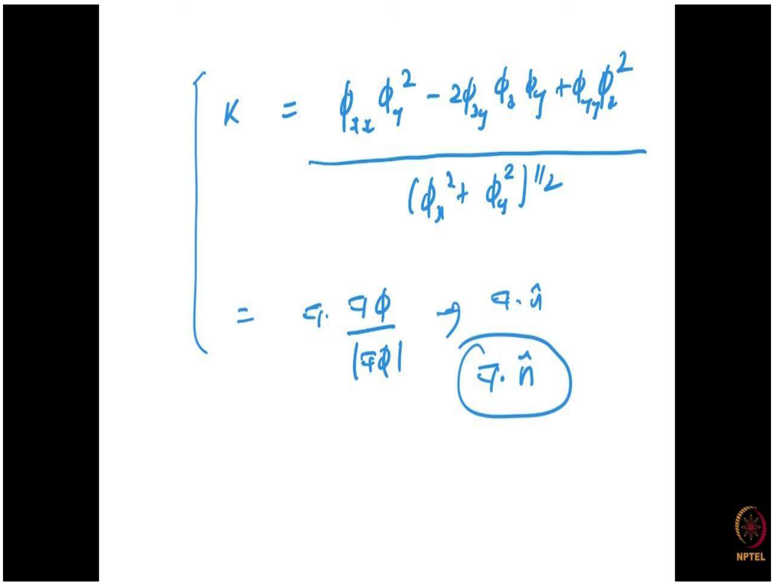

So, we can define curvature. So, curvature turns out called kappa is given by the following expression phi x x minus 2 phi x y again these represent partial derivative with respect to x and y. So, it is the notation typically x and y plus, so, it can also rewrite it as del dot phi so, this is nothing but this is basically del dot n normal gradient divergence with respect of their normal.

So, this again can be derived. So, we will not go into the derivation on how we come to this. This you can take this as the definition of the curvature del dot rewrite this is not clearly written del dot. So, (())(21:58) quantities that we would run into when we are using level set formulations for active contours.

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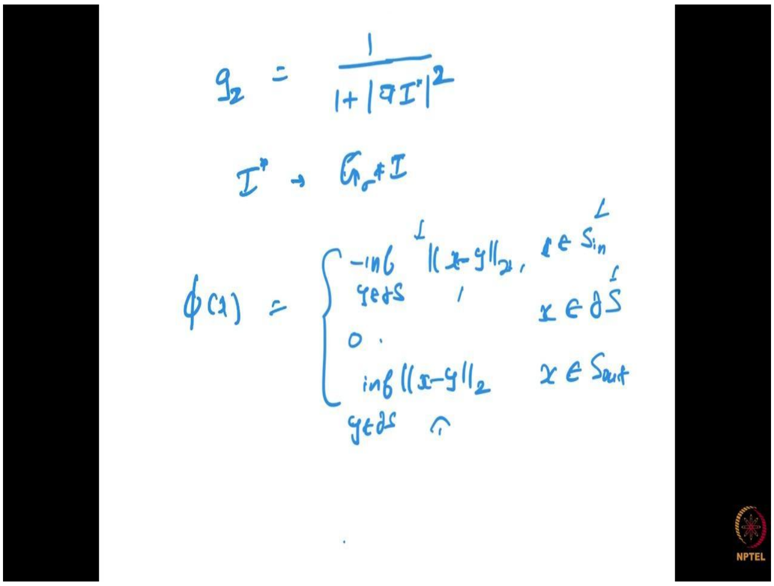

We can now look at the geodesic active contour. So, which is basically the improvement on the snakes algorithm that we saw earlier. So, first we begin with the definition of an image edge function, image edge indicator function generally denoted by this can use 1 plus gradient of smooth version, I am just going to I will be some star is I star is nothing but a smooth version so, you will have some Gaussian filter involved with I so, I have just not indicated it there so. It is a edge function.

So, its value the edge is very high when the gradient is very high this function will have a low value and if it is when the image is smooth in those regions this will have a higher value and the smoothing is done to reduce noise because sometimes use a simple Gaussian filter we can get rid of some of the noise and so, in general this function will take lower values at image boundaries compared with other regions.

So, we will also define the level set function phi in terms of the signed distance function so, how do you do that? We do phi of x equal to the can think of this as the minimum distance y belongs to S so, if it is inside there is a negative sign, it is 0 if it is on the boundary. So, delta this delta is belong denotes the boundary, this is inside the boundary.

And if it is outside again this is the squared Euclidean distance you can think of norm to norm or just the Euclidean distance typically , not the squared Euclidean distance I think I must go that was a mistake, there is just a two norm. So, just a straight distance. So, now we have this x where x belongs to S out which means you are outside the boundary.

So, if you are let us say doing some organ segmentation so, everything inside the organ will get a negative sign everything outside you are going to get a positive sign and on the boundary of the organ you get 0. So, an x this is the this denotes the Euclidean distance between x and y. So, this is called the signed distance function and one like we mentioned the idea is that phi of x equal to 0 will give you the segmentation contour.

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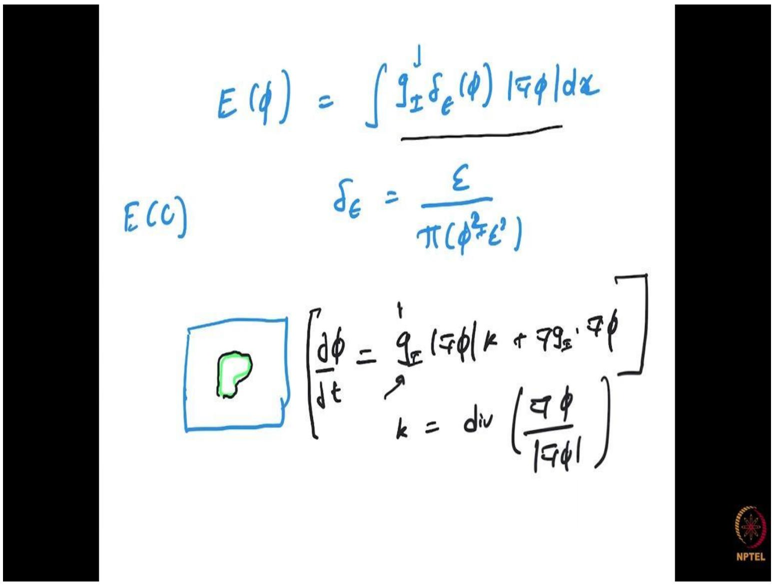

So, we also want to look at the national energy or the energy functional. What is the energy functional? So, for this problem the energy functional written in terms of phi so, typically we would have written E terms of c the contour so, now because now we are embedding the contract in phi we will write it in terms of c we will in terms of the phi itself as a function of phi so, over the image.

So, here this delta function is basically this is just some approximation of the delta function the continuous (()(25:56) delta function or differentiable is a continuous differentiable approximation this is called the smooth direct delta function. Again gradient is the delta is the gradient operator. So, what does this integral do? All it does that it does is it calculates the line integral of the edge function, calculates the line integral of the edge integrated function along the zeroth level set of phi.

So, this is so, this basically this they used to call the geodesic active counter because the geometric perspective what the energy function does is to find a minimum length curve, is called a geodesic curve of image in the image I and that but then is just not any distance it is weighted by the gradient you can think of it that, a gradient weighted so, why does this work because let us say you have a curve which is on the edges of the gradient of the image.

Let us say you have such I will just draw the usual box and let us say you have an organ (())(27:14) let us say you have this organ. So, if you draw a curve which sits right on this edge of this organ let us say, now here the gradient will be very high. So, if we calculate a gradient weighted distance which is what this is doing, we will do a gradient weighted distance along the image edge then that would corresponds to a certain distance which will be either maximum we can create a maximum or minimum.

So, you want that because you want your curve to be on the edge of the object. So, this is a basic kind of a metric which is derived from the gradient of the image. So, it is also referred to as the geodesic active contour some distance metric that we get based on the edges of the image. So, if you write down the Euler LaGrange equations for this particular functional then we can write in terms of the level set formulation.

So, we can write this as delta phi over delta t is equal to that we saw the g I then gradient of phi there is a kappa plus gradient of g I dot . So, this kappa is nothing but call the curvature and we can write it as the divergence of this particular. So, we have started of the from the level set formulation to because I have not walked you through the actual derivation but this better of there it is given in the publication and for the reference out in the forums.

So, you can practice on that. There is also another way you can work with the way work this the way the snakes derivation is done, you can work with the contour itself, write down the functional intervals of the contour and then get the Euler LaGrange equations and from there they will substitute the now I embed the contour in level set function, you will get the corresponding level set formulation.

So, this particular so this is your geodesic active contour and by doing following for this phi over time till convergence, we can actually propagate the curve. So, in this formalism, the advantage also, again is that you have the level set. So, you can have multiple objects that you can segment simultaneously, you do not have to keep track of individual points or worry about self-intersections, et cetera, all of them are taken into account.

So, but still the point is that it still depends on the gradient, there is some curvature, dependent terms that gives it some speed, the smoothness aspect et cetera, but still depends on the gradient for propagation, not the gradient of phi, but I am more talking about the g the term g. This actually is a function of the gradient, so of the gradient of the image.

So, still it has problems with local minima et cetera. Because if you have a particularly noisy image, even if we smooth it, that is going to be a issue. So, just in case you were wondering, why would we do this, I mean, you still have the same issues cropping up again and again, the local minima problem.

See in the context of let us say medical image segmentation, especially if you are going to have some users sitting there and doing it, this is a better way of doing image segmentation rather than drawing by hand. So, there so, you can always initialize the contour close to the boundary that can be but it can be rough. And then this particular formalism can be used to converge to the boundary.

So, if there is a user expert user involved, especially in the context of medical instrumentation, so very useful tool to have if it is been programmed properly. So, the other technique, which is again widely used and very popular is called the canvas segmentation or active contours without edged. Now, it is variational formulation is from something called the Mumford-Shah model so, we will take a look at that also, briefly.