Numerical Methods and Computation

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Lecture No. # 13

Solution of a System of Linear Algebraic Equations (Continued)

In our previous lecture we derived the Gauss elimination procedure for the solution of a system of linear algebraic equations. We also solved an example using the method but we have mentioned that the procedure may fail if any pivot power turns out to be a zero or a very small number. Now to to avoid such difficulties we shall use what is known as the pivoting procedure or pivoting strategies.

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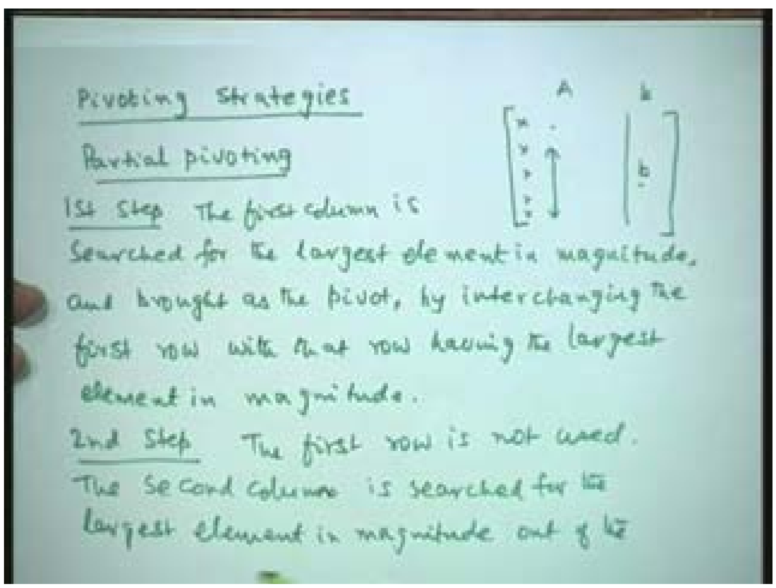

So let us discuss the pivoting strategies that we would like to use here. The first procedure that we shall use is known as the partial pivoting. What we do in this procedure is when we have our system of equations, the augmented matrix constructed out of it, as our A B, in the first step I take all the elements in the first column, find the largest element in magnitude and bring that as a pivot; that means we interchange that equation in the first equation with those equations which has the largest element in magnitude. So that will be the first step in the Gauss elimination procedure. Before we start elimination using the Gauss elimination we will do this procedure that means we will say in the first column, we searched for the largest element in magnitude and brought as the pivot by interchanging the first row with that row having the largest element in magnitude. Now you can note that if the first element is zero or very small then we are automatically avoiding that particular division by that particular pivot because that element which is zero or very small number is thrown somewhere to some other position in that particular column.

Now in the second step we repeat the same thing but leaving the first row as it is. So the first row is not used. The second column is now searched for the largest element in magnitude out of the n minus one element. We are leaving the first element over here and then we are using only these n minus one elements of the second column. So the second column is searched for the largest element in magnitude out of the n minus one term in the second column leaving the first elements leaving the first element.

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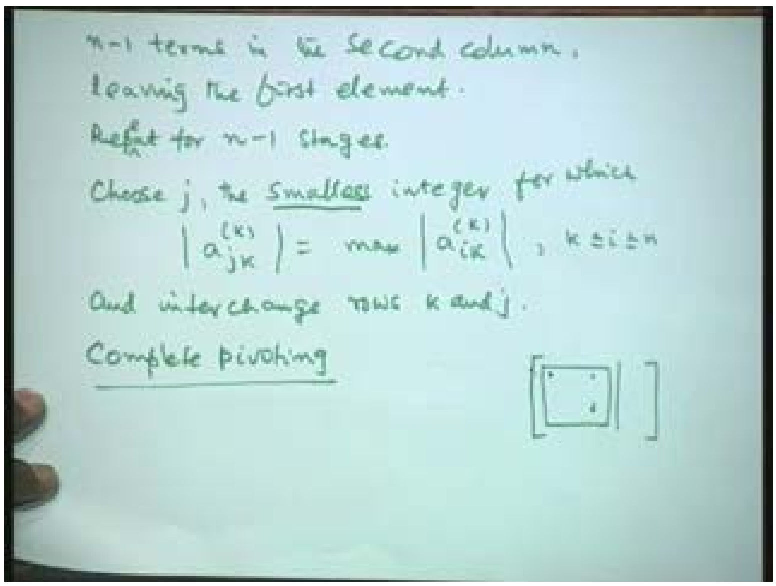

We repeat the procedure in n minus one stages. Now it it may be possible that when we are searching for the largest element in magnitude. There may be a number of elements which are equal. Say for example number of elements is 5, 5, 5. So then we will interchange with that one which comes first. That may be a smallest j that comes in will be taken as a row at which interchange the elements. So even if you have the number of elements which are equal, we do not face any problem. Mathematically this means; let us write down mathematically, we shall choose a particular number j that means that particular column. Choose j, the smallest integer for which magnitude of a_{jk} at any particular pivoting time is equal to maximum of a_{ik} of k for all k less than or equal to I, less than or equal to n. Interchange rows k and j, we are now at the position of kk. Now this searches all the elements between k and n, so we are going below the pivot and then bringing that particular mean, the smallest integer which is a nearest one. So we take the smallest integer for each this relation holds and then interchange rows k and j. So this how we program it when we are programming for finding the largest element in magnitude. Now this procedure is called the partial pivoting procedure the remaining part is a same as a Gauss elimination procedure that we have done here.

The second type of procedure is called the complete pivoting. This procedure is not used because it is expensive. What we will be doing here is that when I start with this augmented matrix here I am going to take this entire matrix, find the largest element in magnitude of the entire matrix, and bring that element. Let us suppose this element is largest here. I will bring that as the pivot here that means it requires interchange of equations and interchange of columns. So if I interchange the first row with this k^{th} row and then interchange the k^{th} row with the first row, so that means first k^{th} column with the first column. Therefore we are interchanging both rows and columns. In other words we are now shifting the variable xk to the first position that means it is also involving the solution vector in which we are exchanging x_1 with xk variable. So it involves both row and column interchanges.

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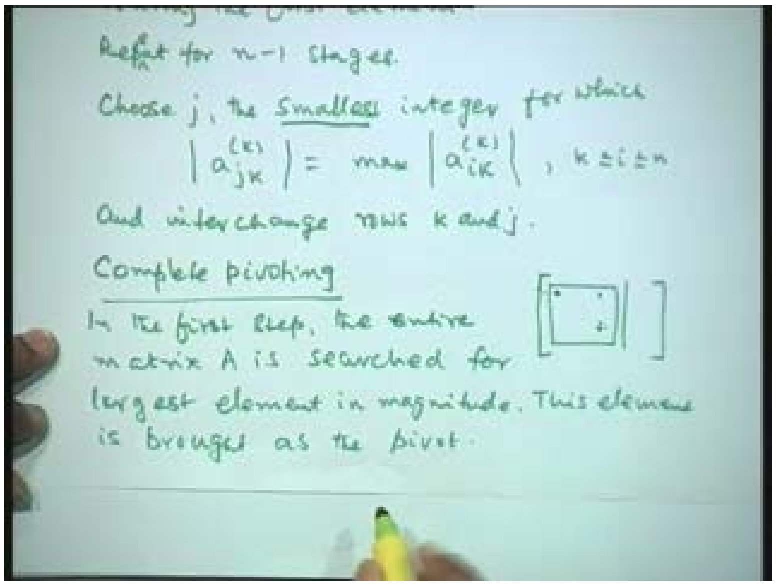

So let me just write down the first step. In the first step the entire matrix A is searched for largest element in magnitude. This element is brought as a pivot.

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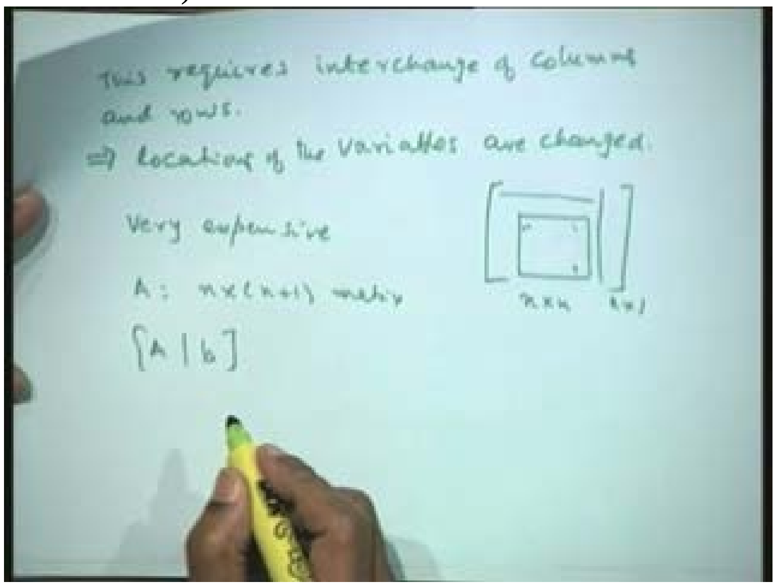

Now what does this imply. This implies or this requires interchange of columns and rows. This implies that the location of the variables is disturbed. So this means locations of the variables are changed. Now let us suppose we have done this and we have gone to the next stage. Now the first row is left as it is. Now I will be taking this element of this n minus one into n minus one matrix in the second stage again find the largest element in magnitude bring that as a second pivot. So again this requires interchange of the second row with that row and then this column with this second column. Now therefore an interchange that was made earlier that means let us say variable x_1 and x_{50} was interchanged earlier, again it is possible that the variable that is changed again gets disturbed. Therefore we need to keep a track of the location once the procedure is complete; where is the variable, in which position the variable is now. X_1 may not be in x_1 position, it might have gone to the hundred position, it might have shifted from hundred to fifty, then it might have been changed to this position. So we need to keep an index and then keep tracking where all the elements are. Therefore the procedure is very expensive because we will be comparing in the first stage n square elements, n minus one square element, n minus two square elements and so on.

Besides that to keep track of all these locations will be very expensive therefore this procedure is very expensive and they have actually computed how the partial pivoting works, viz a viz the complete pivoting and it is found that complete pivoting does not fare in any way better than the partial pivoting. So all the strategies that are adopted from the computer software would use only a partial pivoting. Then I mean if you are writing a computer program in this procedure what we normally do is we take this augmented matrix as n into n plus one matrix. So what I would take here is this is n into n and this is n into one so b is taken as the n plus 1^{th} column of A that means we don't have to write down another array for b. So you can declare A itself as n into n plus one matrix; so that you identify the n plus 1^{th} column as the right hand side vector and then proceed with the solution procedure and then the first n into n elements will be the required matrix U and then the right hand side vector is going to be z. So that is how we can write down the programs in a single loop .We can go and then do it. Now earlier we discussed about consistency and inconsistency with respect to this augmented matrix.

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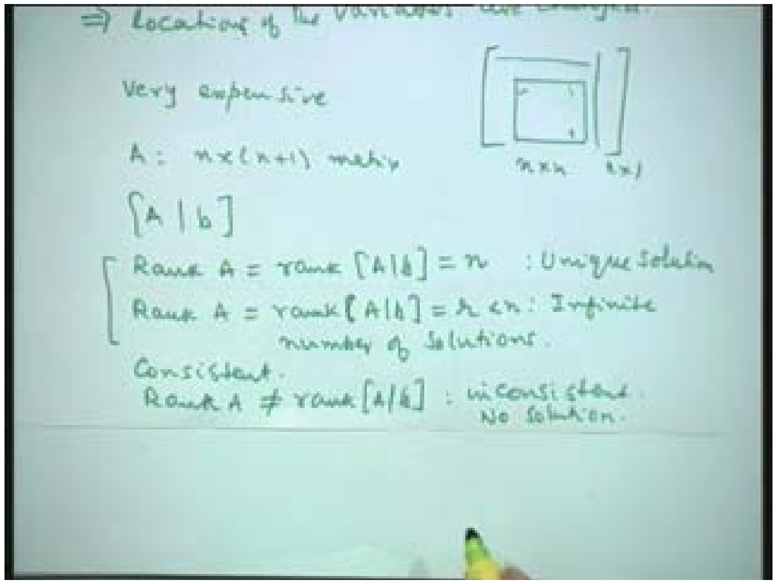

We said if rank of A is equal to rank of this augmented matrix and this is equal to n then we have a unique solution. Then we have if rank of A is equal to rank of A, the augmented matrix is equal to some r less than one then we have got infinite number of solutions. Now in both the cases we call this as a consistent system. So in both the cases it is a consistent system. Now lastly if rank of A is not equal to rank of Ab then it is inconsistent. So if rank of A is not equal to rank of Ab, then it is inconsistent that means it has no solution. Now these concepts that we have talked of should automatically be reflected in the Gauss elimination procedure. Let us see what you mean by the infinite number of solutions case. When the system has infinite number of solutions that would imply that there are number of rows which are dependent rows. Since these are dependent rows, number of variables can be transferred to the right hand side and then we can obtain the solution as one parameter family or two parameter family or in general; in this case n minus r parameter family of solutions. Therefore here this shows that there are n minus r rows which are linearly dependent. Now in our computation purpose this would mean that we'll have that many rows with zeros. Gauss elimination should automatically tell whether the system has unique solution or it has the infinite number of solution or inconsistence.

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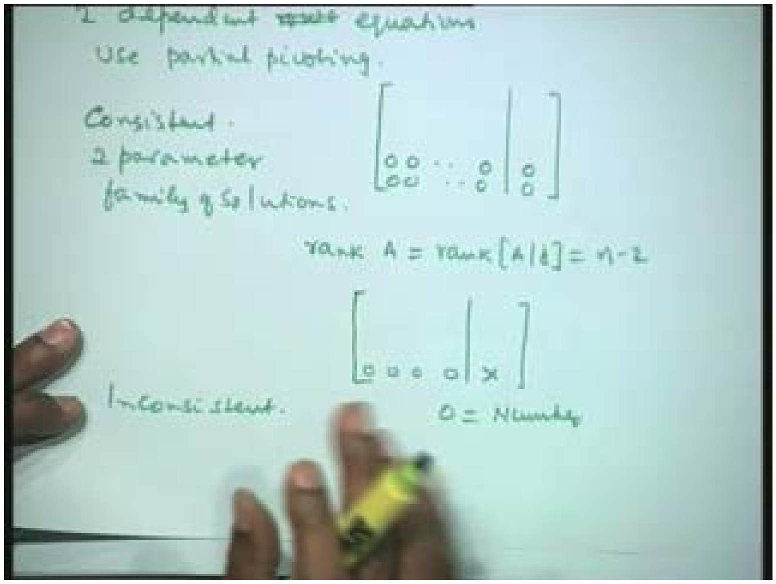

Let us take the case when we have two dependent equations. Now when I perform the Gauss elimination using the partial pivoting, let us say we use partial pivoting, because they are dependent equations, we are subtracting the equation from another equation. So zeros would be automatically produced. Therefore what happens is the last two rows will turn out to be 0, 0, 0, 0. So the computer can automatically check the number of rows that it is having zeros by your index and then these rows will then tell us that many number of parameter family of solutions. So this will then tell that the system is consistent as 1 or 2, number of rows in which zeros are produced as the as the order of the parameter family that we have. So in this case the system is consistent and it has got two parameter families of solutions. Now we can see in this case as we are saying rank of A is equal to rank of the augmented matrix Ab is equal to n minus two; rank of this will be n minus two and there are zeros. Therefore I have got n minus, n minus two that is two parameter family of infinite solution. Now suppose the system was inconsistent then what would have happened here is I would get here something like this 0, 0, 0, 0 and it gets eliminated and you have some non-zero number over here. Now we have inconsistency. We can see that when we multiply by the vector x this will be zero is equal to a number. So what we would be producing is equal to zero is equal to a number, that's what this implies. Therefore the system will be inconsistent. Therefore the computer the automatically looks at these values and then say that once you have an element on the right hand side this can be an inconsistent system.

Therefore this checking of the unique solution, infinite solution inconsistency is automatically applied in the computer software. Now let me take a problem corresponding to this, as to how it will be able to say that the given problem is inconsistent.

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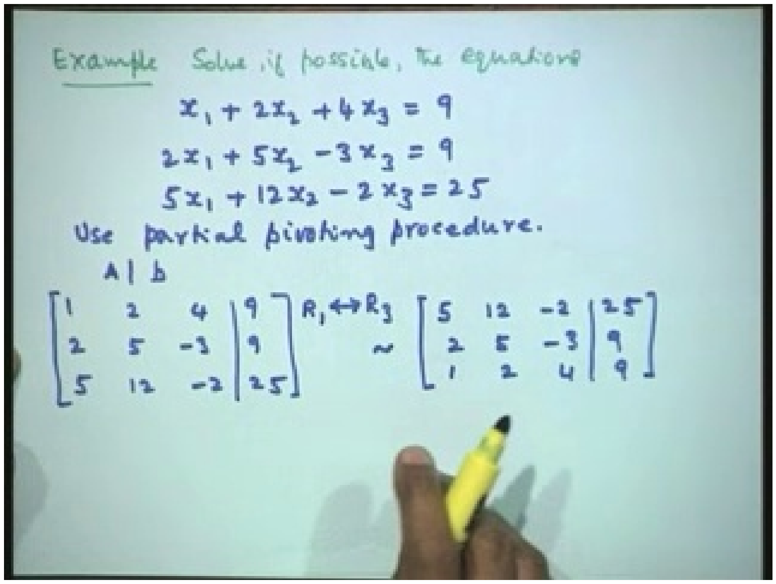

So let me take this as an example. Let me take this equations as x_1, $2x_2$, $4x_3$ is nine; $2x_1$, $5x_2$ minus $3x_3$ is nine; $5x_1$, $12x_2$ minus $2x_3$ is twenty five. Use partial pivoting procedure. Now let us write down our augmented matrix Ab. So we will have 1, 2, 4, 9; 2, 5, - 3, 9; 5, 12, - 2, 25. Now I search the first column for the largest element in magnitude; 5 is the largest element. So I will inter change row one and row three. So I would therefore have 5, 12, - 2, 25; 2, 5, - 3, 9; 1, 2, 4, 9. So this is an equivalent matrix.

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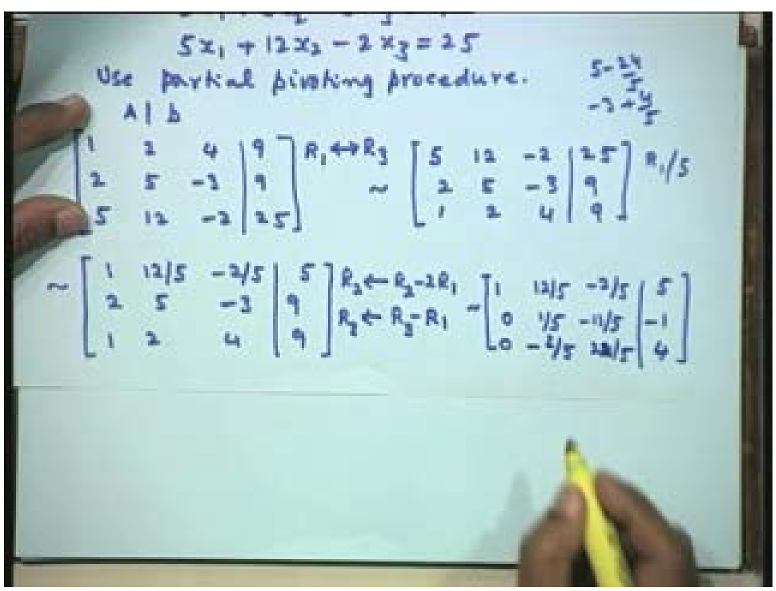

Moving on to the next step; as we have done in the previous example let us make this element as 1. So I would divide the first row by five, then this will be equivalent to this - I will have 1 here, I will have 12 by 5, - 2 by 5 and this is 5; these stay as it is. Now we make these two elements as zeros. So I need row two to be replaced by row two minus two times row one; row three replaced by row three minus row one. So I can multiply two times and subtract to get my new row and just subtract the first row from the third row; I get a zero over here. Then I will get the equivalent system. So 1, 12 by 5, - 2 by 5, 5; and two times subtracting this will give zero, so I can just put these values over here. We are multiplying two minus 24 by 5. So I will have here one by five; then we have got this as minus three and minus this plus four by five, so that is minus eleven upon five. Then we are multiplying two, subtracting, so I have - 1 here. Then row three minus row one; I have a zero here, I have got two minus this, therefore this is minus two by five; this is four plus four by five, two by five twenty-two by five; then this is equal to four.

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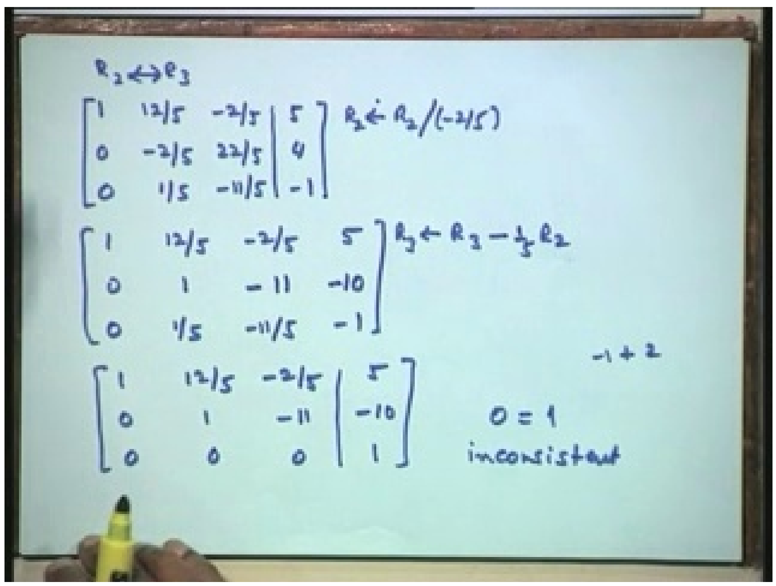

Then the first stage is over now. For the second pivot I will search these two elements for the largest element in magnitude. So I need now an interchange between row two and row three. So I would now perform the interchange of row two and row three. Therefore I will have 1, 12 by 5, - 2 by 5, 5 and the second row becomes – 2 by 5, 22 by 5, 4; 0, 1 by 5, - 11 by 5, - 1. Now let us again make the second element as one. So I would divide throughout the row two; row two divided by minus two by five. Then I will have 1, 12 by 5, - 2 by 5, 5; 0, 1, - 11, - 10 and then 0, 1 by 5, - 11 by 5, - 1. Now I make this zero, so now row three is replaced by row three minus one by five of row two. So I would produce 1, 12 by 5, - 2 by 5, 5; 0, 1, - 11, - 10; 0, 0, and you have minus one plus two, so I have a 2. Now you can see that the procedure has implicitly shown that the systems of equations are inconsistent. We have zeros here and one here, therefore what we are implying here is that zero is equal to one. So what we have produced is an inconsistent system of equations. This is how inconsistency can be observed. We would have also noted that since we are using partial pivoting, this particular behavior is thrown to the end of the last rows of the given augmented matrix.

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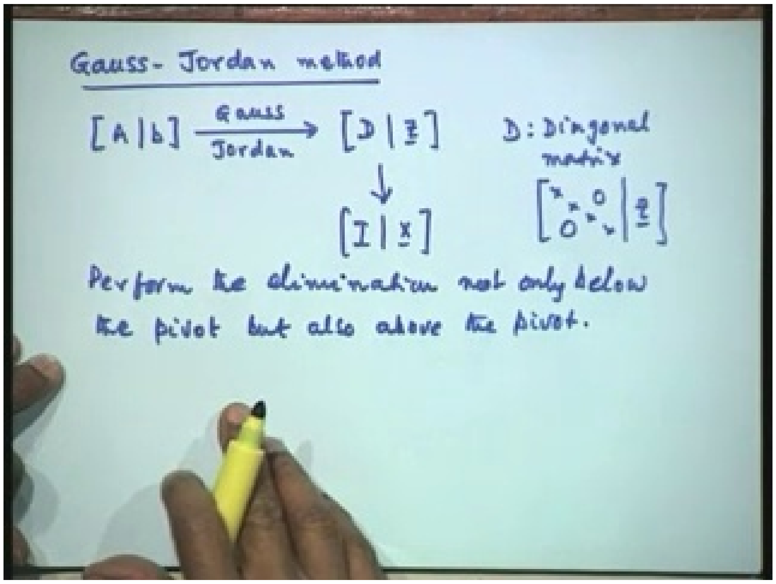

Now a variation to this Gauss elimination procedure is known as the Gauss Jordan procedure. So let us just describe what this variation, it is the Gauss Jordan method. We remember when we started the discussion that there are three matrices for which the solution was trivial. That is when A is in the form b or l or U. The Gauss elimination procedure has used the third one where A has been reduced to upper triangle matrix U; the Gauss Jordan procedure follows that A will be reduced to a diagonal matrix b. So we start with your augmented matrix Ab. We perform the same elementary transformations. So what we will be having is a Gauss Jordan procedure. This will now reduce to D of some z where D is a diagonal matrix. Now once I have D diagonal elements, this diagonal matrix is of the following form, some elements over here that we have produced zero zeros over here and we have produce z over here. So I can divide each of these equations by the diagonal elements and then I can produce here from here I and x. So once I divide by these diagonal elements I would produce 1, 1, 1 over here and that means this is I and this is the x. Therefore this is our solution vector automatically coming from this part.

Now if this is the method, the difference between the Gauss elimination and the Gauss Jordan is only that at any stage of elimination when we choose the pivot the elimination is done not only below but also above. So if I take a pivot over here, I have got the elements below and above also. So at the each stage I will eliminate elements below and above the pivot, then that will automatically reduce to the D form but if you are dividing by the element before the procedure, you will directly land up into I and x. Therefore this implies that perform the elimination not only below the pivot but also above the pivot. So we will perform the eliminations not only below the pivot but also above the pivot. This gives us the D, this gives z. Now divide by the diagonal elements to give I and x. As I mentioned earlier that before the elimination if you are making the diagonal pivot as one, then this will automatically land up into the final format I is equal to x.

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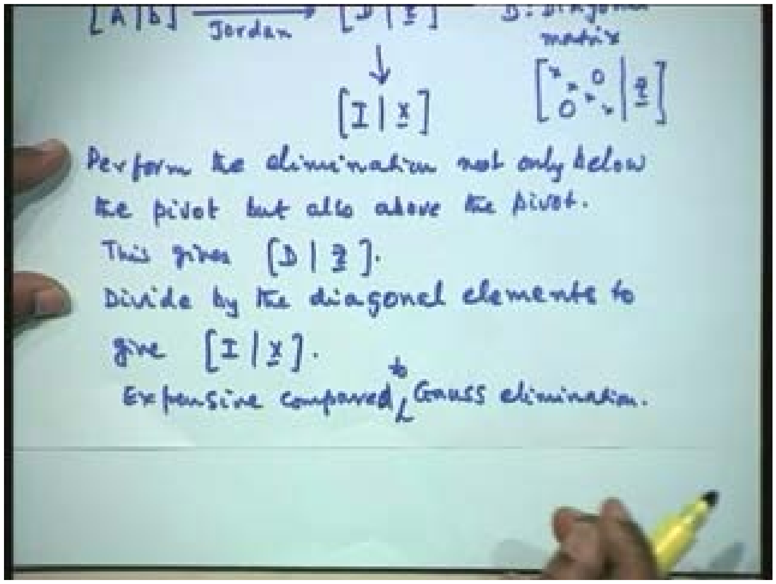

Now this procedure looks attractive but we will show later on that the Gauss Jordan procedure is much more expensive than Gauss elimination procedure. There would be substantial amount of difference in the amount of the operation count between the two methods but then why do we need this method. Now interestingly we need this method for another thing. In our lecture we gave a simple example of a system of equations wherein you would like to solve a system with different right hand sides. If you have to use a Gauss elimination procedure or Gauss Jordan procedure for those equations, each time when I have to solve I have to take the right hand side, then change it and then solve it; and again change it, and then solve it. However if I have the inverse of matrix then independent of the number of right hand side that we have we can use the same inverse matrix and find a inverse of b_1, a inverse of b_2, a inverse of b_3 and so on and find the solutions. So in that case we need a procedure which can give us an inverse of a matrix. The Gauss Jordan procedure we have discussed can be used to find out the inverse of a matrix.

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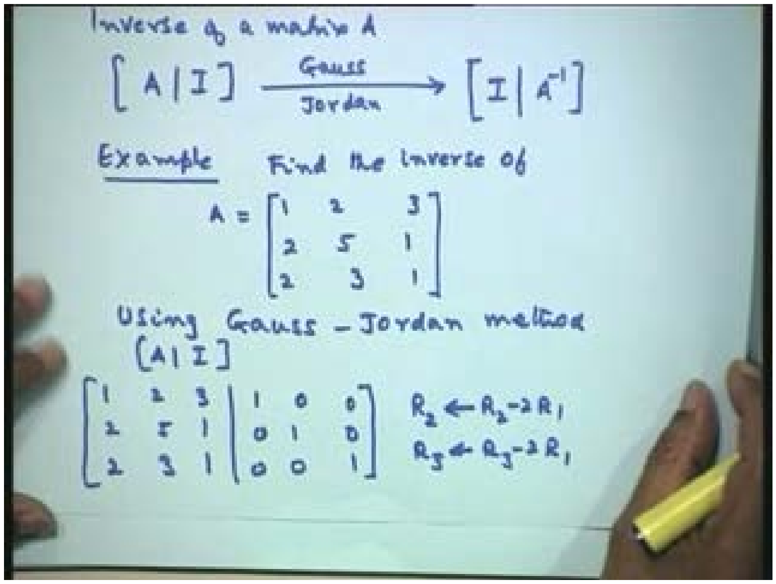

The simple thing that we do here is I start with the augmented matrix A and I. I start with the matrix A augmented with the identity matrix of the same order. So this will be the augmented matrix A and I. I would perform the Gauss Jordan procedure, then I is A is reduced as I and hence I must be reduced as A inverse; because A inverse is equal to I therefore this gives me A inverse. Therefore this Gauss Jordan procedure would give me the inverse of a matrix. Now let us take an example for this. Let us take an example for finding inverse of matrix. Let me take the matrix as 1, 2, 3; 2, 5, 1; 2, 3, 1 using Gauss Jordan method. Now let us write down our first step here which is A and I, the augmented matrix AI. So I will write down 1, 2, 3; 2, 5, 1; 2, 3, 1; 1, 0, 0; 0, 1, 0; 0, 0, 1. Now the R_2 is replaced by R_2 minus $2R_1$; row three, row three minus R_1. So we are making both of them as zeros.

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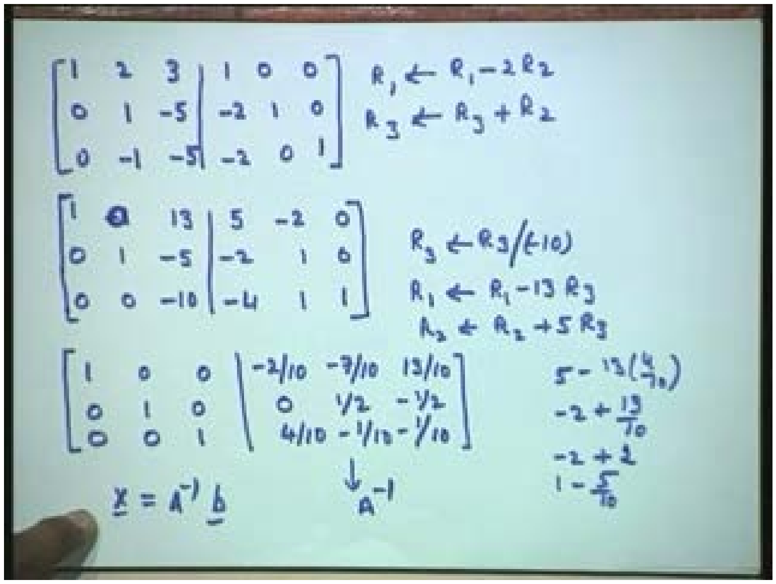

Then the first row stays as it is; the second row is 0, 5 - 4 that is 1; 1 - 6 that is minus five, this is – 2, 1, 0; so that is the second row. The third row is 0, 3 – 4 that is – 1, 2 – 6 that – 5; and we have multiplied this by two and this is – 2, 0, 1. Now here both are equal. So I have to perform in the second stage a zero here and zero here also. So what we do now is; row one minus two times row two which will bring a zero over here and then row three is row three plus row two, so that will bring me a zero here. So we are performing both these operations at the same time, so that have a zero here and zero over here.

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We are not doing partial pivoting. We are just doing the Gauss Jordan procedure. Now this will give us 1, subtract it by 2, that is equal to - 1; this is 13, this is 5, – 2, 0. So I have multiplied all of them by two and subtracting them. Then second row stays as it is. The third row is row three plus row two, so I will have – 10, -4, 1, 1. Now before we proceed I can make this element as one. So let us make this element as one; row three is equal to row three divide by minus ten. Let’s work backwards. Once we divide by 10 we have 1. So let us write down what will be the next step. Then we will perform row one is minus thirteen times row three (because we have made this as plus one) and then row two will be replaced by row two plus five times row three. So we have one here, so let’s first of all make this as one. So I will have this as 4 by 10, 1 by 10, and 1 by 10. So I have divided by minus ten throughout, so we will have here 4 by 10, - 1 by 10, - 1 by 10. I am now subtracting row one; row one minus thirteen row three. Now this is 1, this is 0, this is 0, and thirteen times this is zero. So then I have here 5 and – 13 times 4 by 10; that is equal to – 2 by 10.

I have the second element as – 2 and I am multiplying this by - 13. So that will be 13 by 10. So that will be – 7 by 10. Then I am multiplying this by 13 and subtracting, so I will have here 13 by 10. Then we go to the second row 0, 0, 1. Now let’s take this element; this was -2, five times of that are equal to zero. So this element is zero. Then I have 1 and five times of this which is 5 by

10 and that is equal to 1 by 2 and lastly we are multiplying by five, that is -1 by 2. Therefore this is the required A inverse of the matrix.

Now if I want to solve the number of systems, I would then take it as x is equal to A inverse of b as the required solution, if I am using it for solving a given system of equations. So here we have been asked to just find the inverse, but however we could use this as the finding the solution of the system of the matrix by finding the inverse of a matrix. So you could use this A inverse and then find x is equal to A inverse of b.

Now very often it is found that the Gauss elimination procedure is sometimes little bit prone for round of errors, the way in which the computations are done. We can have some alternative procedures which look more attractive than Gauss elimination for the way in which it is done but we shall also show that the method will be mathematically equivalent to Gauss elimination procedure. Now I will give a brief introduction of that.

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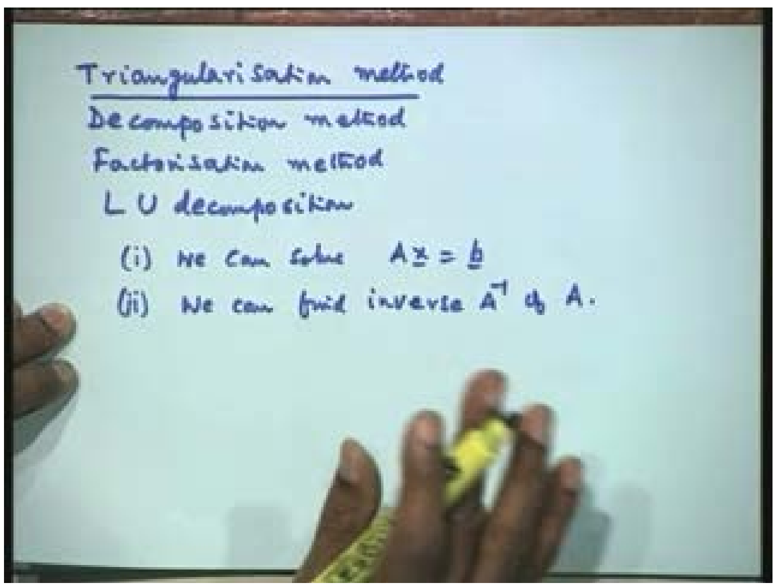

Those methods are called as the Triangularisation method. There are number of names for it. It is called the Triangularisation method and some books call it is as decomposition method. Some other books call it as factorization method and some other simply call it as L U decomposition. So all these names are for the same method that we shall be describing now. This method can be used for two purposes; one is we can solve a given system of equations Ax is equal to b. Secondly, we can also use this to find the inverse of a matrix; we can find the inverse A of A. So this method can be used for finding both the cases which are finding the solution as well as finding the inverse of a matrix.

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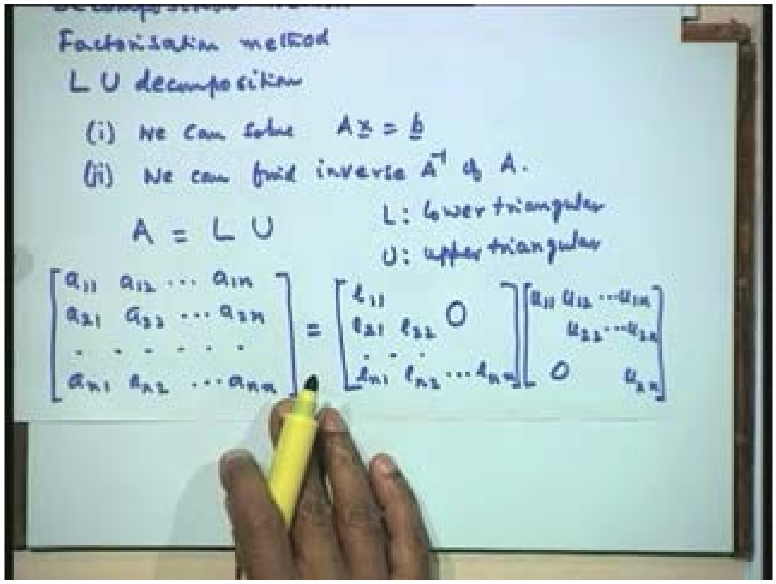

The method is based on the fact that we can decompose a given matrix as the product of two matrices; one is lower triangular and the other is upper triangular. So what we will write is, we will write down A is equal to L into U; so that is why you can see it is called the L U decomposition. So L is a lower triangular matrix; U is an upper triangular matrix. Now let us just write down how it looks. So let us write down a_{11}, a_{12}, a_{1n}; a_{21}, $a_{22,}$ $a_{2n;}$ a_{n1}, $a_{n2,}$ $a_{nn.}$ L is lower triangular. Therefore it will be l_{11}, $l_{21,}$ $l_{22,}$ l_{n1}, l_{n2}, l_{nn}; and 0. We have here u_{11}, u_{12}, u_{1n}, u_{nn}. The idea behind the procedure is simply that we multiply these two matrices on the right hand side, compare with these elements on the left hand side and then obtain the solution for this. Before we do actually multiply and then compare it. Let us just look at one aspect of the number of unknowns that we have here, so that some simplification can be made, so that this evaluation or computation of lijs and uijs are very simplified.

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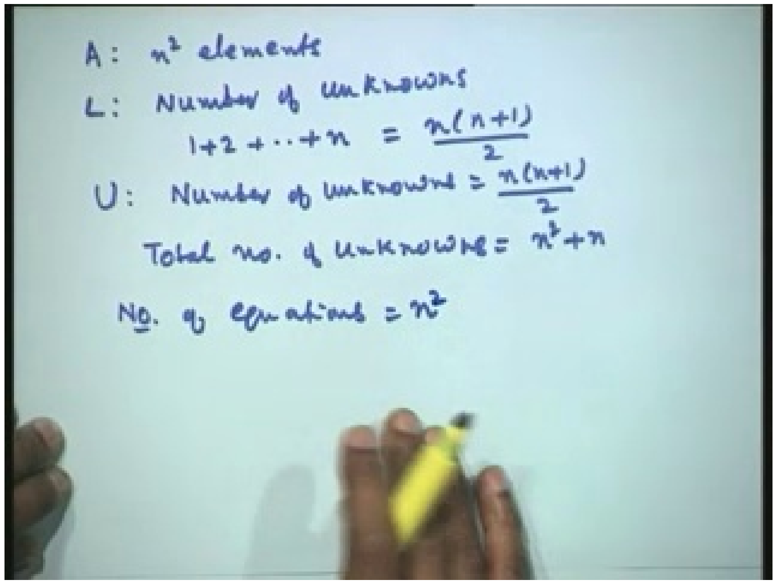

If you look at the number of elements on the left hand side, if you take A it has got n square elements. Now let's look at L. The number of unknowns (these are unknowns) are; let's add them up, so this will be 1 + 2 + 3…. N. So these are the number of elements, which is your n into n plus one by two. Similarly if I take U the numbers of unknowns here are the same. I can count from backwards; there is one in the last row, two in the second n minus 1[th] row, n in the first row. So the number of unknowns will be again be equal to n into n plus one by two. Therefore the total unknowns are n into n plus one n square plus n. When you multiply the right hand side and then compare it the number of equation will be equal to number of elements here. So the number of equations will be n only; number of equations is equal to n square.

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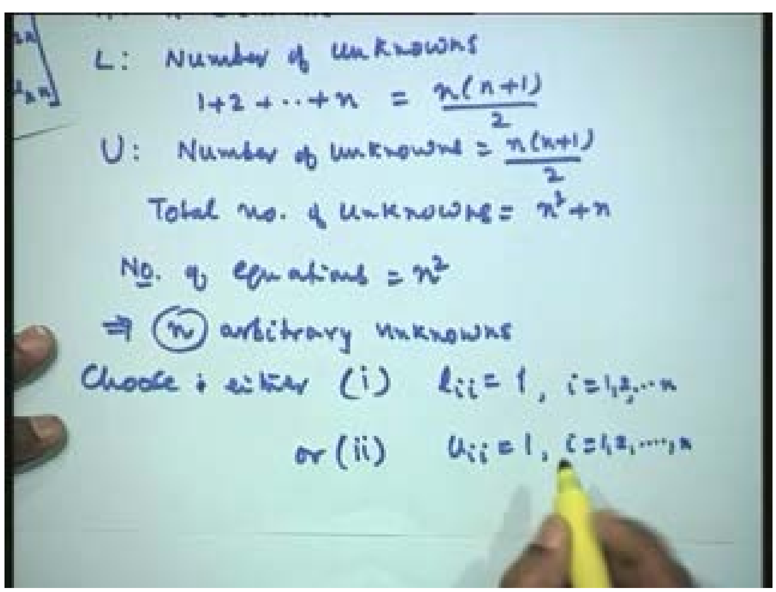

This implies that we have n arbitrary parameters because there are n square plus n. These are numbers of n square equations; therefore this implies that we have got n extra unknowns that are n arbitrary unknowns. Now this information is very valuable information that it has got n arbitrary unknowns and we shall choose the values of these n arbitrary unknowns, so that the computation of L and U becomes very simple for us. Now there are two sets of values that can be chosen, where this computation becomes simple; therefore we can choose either l_{ii} is equal to one or choose u_{ii} is equal to one where I is equal to 1, 2, 3 …n.

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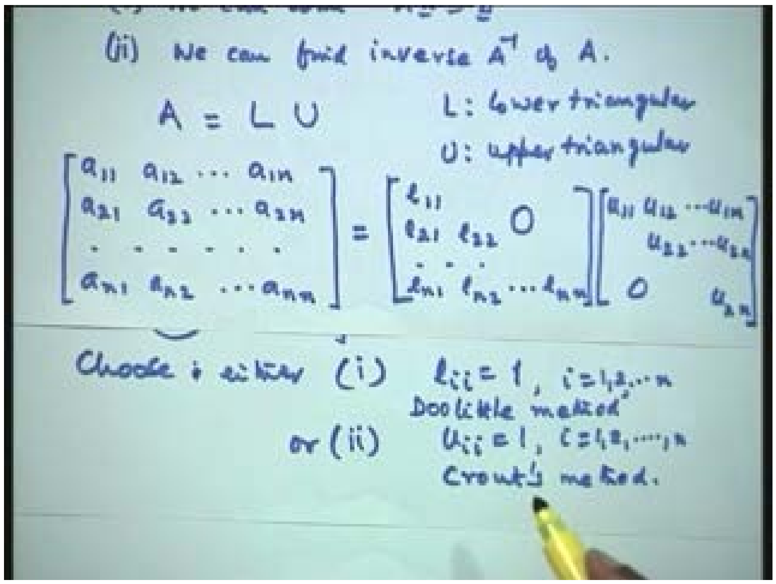

We are saying that I would like to choose all these diagonal elements of L as one or all the diagonal elements of U as one. Now this particular choice of these ones would simplify evaluation of lij's and uij's because again by simple forward substitution you can get all the elements of lij's and uij's. However there are other names. The first lii method is called Doolittle method; it is named after the author and uii one is called the Crout's method. Now both are useful; for computation purposes both are identical. There is no change in the solution or the number of operations or anything, everything will be identical except that it is our choice as to which one we would like to use.