Introduction to Computational Fluid Dynamics

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Lecture - 20
Artificial Dissipation, Upwinding, Generating Schemes

Okay good morning. So what we were doing last time is, we were looking at the stability analysis for heat equations. We were doing the linearized stability analysis for heat equation. To remind you as to why we were doing that. He had suggested that FTCS was unstable because the second derivative term that showed up in the modified equation for FTCS at the wrong side was negative right. The coefficient for the second derivative was negative right. And I had suggested that we could add artificial dissipation of our own right.

We could just add the second derivative term and try to knock out that negative term possible make it even positive okay. That was the suggestion. And then I had sort of casually mentioned that you can add as much dissipation as you want. You can add as much artificial dissipation as you want okay that is fine right. So, what we were looking at now is what is the effect of this artificial dissipation. So, if you add this, you can add as much as you want right.

Then we had concluded that, the equation with that much artificial dissipation was almost like heat equation. You added an enormous amount of artificial dissipation. That was like heat equation ad was there a consequence to that. So, we are just looking at the stability analysis of heat equation. We had already done it right. I will just quickly repeat it.

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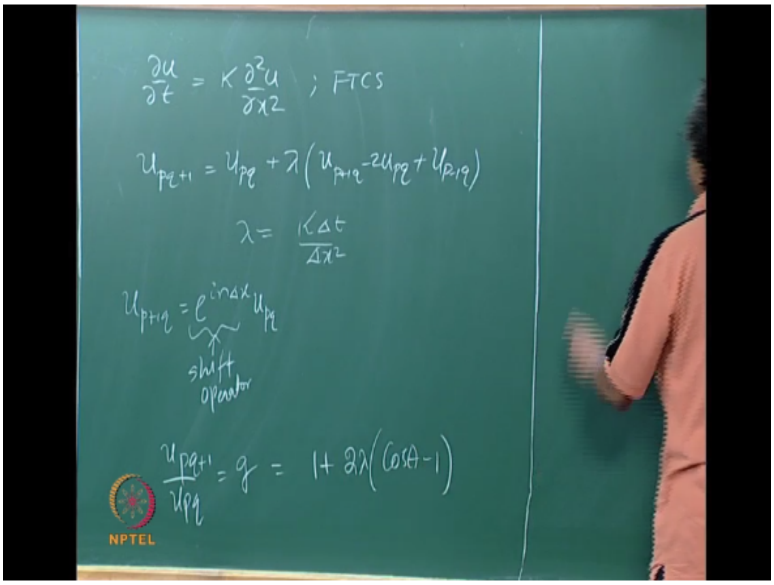

So, what did we have? We want the heat equation, looks like that. If you use FTCS. If you apply FTCS to this, again the problem by itself is not complete right. I mean you have to apply the boundary condition. I will tell you what are the boundary conditions, what is actual physical problem that we are solving. But, since this von Neumann stability analysis is only at a grid point right. We can just discretize this and ask the answer that question.

So, FTCS applied to this is upq+1 is upq+ I will say lambda times up+1q-2upq+up-1q where lambda is kappa delta t/delta x squared okay. And when we went through a same process of writing up+1q as e power i n delta x upq. Incidentally, in some books you will see this called the shift operator because it shifts us by 1 grid point okay. So, some books you may see this called the shift operator okay. So, doing this, what did we get? We got upq+1 divided by upq which is g, the gain.

What did this work out to be? 1+2 lambda times cos theta-1. If you want that the modulus of this to be < 1.

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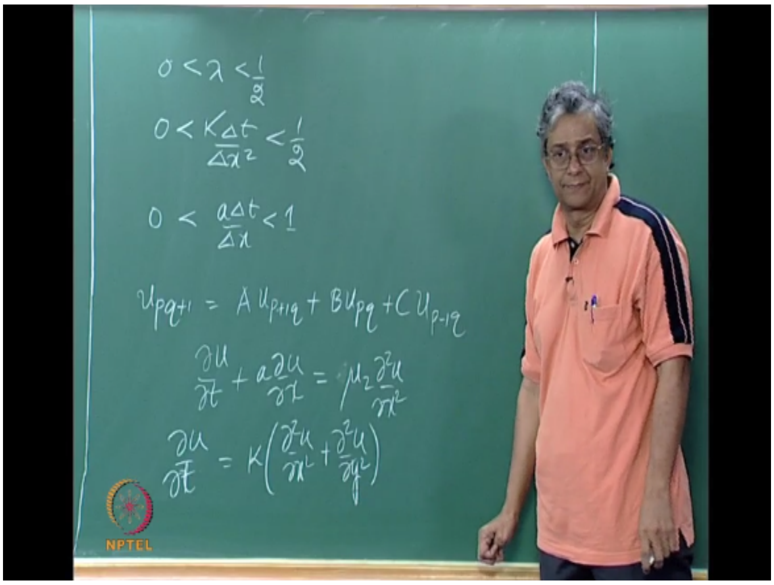

And therefore, we got the condition that, 0< lambda < 1 half is that fine. That is 0 < kappa delta t/delta x square<1 half. So, if kappa is very large so the question is what dominates? What was the CFL condition the regular condition that we call the CFL condition. That was 0<a delta t/delta x<1 okay. Depending on the relative magnitudes of a and kappa, you will end up applying the appropriate stability condition.

So, if you make your artificial dissipation, the heat equation that is kappa but if you add it as artificial dissipation, it would be mu 2, if you make the artificial dissipation too large, so large that mu 2 dominates for all practical purposes, it becomes like heat equation. Then this is really a stability condition that you need to be worried about okay. So, if a is of the order of 1 and kappa is of the order of 1 million then, this is really a stability condition that you have to be bothered about fine okay.

Of course, in general, you could say that it seems, that the general explicit scheme right using 3 points seems to be of the form upq+1 is some A times up+1q+B times upq + C times up-1q. Are we making sense, in general an explicit scheme involving the points p+1, p, p-1 would look something like this okay. May be what you can do is, you can try out this, try out see how you do the stability analysis for this. Just substitute go through the same process.

Just go through the same process and you will see that you can come up with the stability condition in terms of A, B and C okay. You can make a general statement, you can come up with a stability condition in terms of A, B and C. You can try it out, if you have any difficulties, get back to me. Is that right okay. So, this is as far as the 1D heat equation is concerned, so you can add artificial dissipation, you have to be very careful how much artificial dissipation that you had right.

You cannot add too much of it. The other thing that you want to remember is, the artificial dissipation term that you are going to add to the equation that you are solving. That is, if the equation, way back when u was the perturbation but, we will go back to situation where u is the variable that you are solving for. So, the equation that you are solving a dou u/dou t+a dou u/dou x and you want this to be =0 but, instead of 0, you are going to solve this. You are clearly not solving the equation that you are set out to solve okay.

You are clearly not. So, it is not though you are doing FTBS setting sigma=1 right. So, the solution is contaminated. So, from your point of view, you want to keep mu+ as small as possible. You want to keep mu 2 as small as possible. Is that fine okay. So, and we rationalize this. We justified this saying, anyway I am solving the modified equation right. So, I am just fixing the modified equation to my satisfaction. So, this term possibly can be picked exactly from the modified equations.

When I do a demonstration I will actually show you how to pick this right. How we would pick this term. We will try out a few things and see what it does to the modified equation okay. So, in the later class, I will actually do a demo of deriving the modified equation but in a automated fashion. Then we will see whether we can add different terms and see what that does okay right. In this conversation, I am going to do a little aside note.

We are going to take a step aside right because, I am with heat equation, I am going to continue with heat equation just for a brief period right. And then we will come back to this conversation right. I want to look at 2d heat equation. Simply because of the stability condition that I have got right, simply because of the nature of the stability condition that I have got, I am going to make a point I am going to use it to make a point.

I am going to look at 2d heat equation. 2d heat equation is dou u/dou t=kappa dou squared u dou x squared + dou squared u/dou y squared. It is an isotropic material. So, kappa, there is only one thermal conductivity right. Thermal conductivity is not changing based on orientation okay. So, if I did FTCS for this, what would I get?

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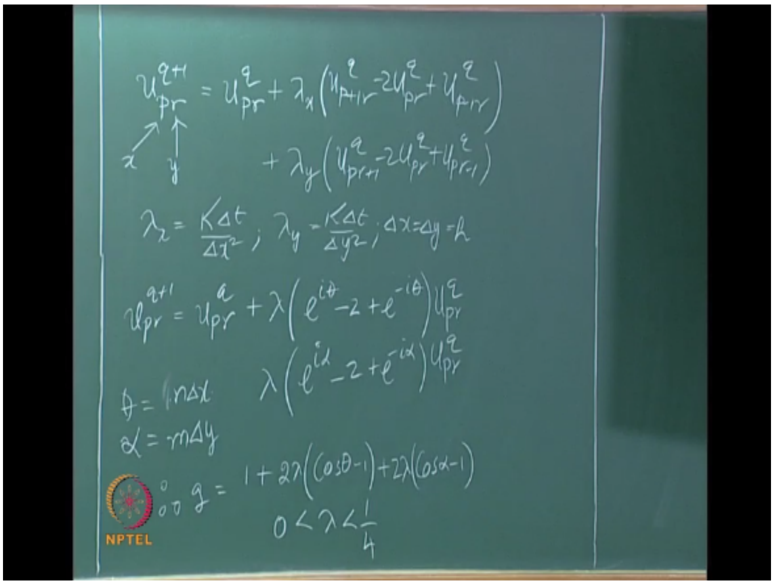

So, now I will use the superscript for time because, I will otherwise I will have too many subscripts in the denominator. I will use p and r for x and y and that is q+1 okay q is in time, this is along x, that is along y, uprq fine +lambda x up+1rq-2uprq+up-1rq+lambda sub y upr+1q-2uprq+upr-1q, where lambda x is kappa delta t/delta x squared, lambda y is kappa delta t/delta y squared. So, to keep life simple, I am going to make delta x=delta y=h okay.

So, I can then combine these. So, what does this give me? Uprq+1=all of these. I am going to now do the same thing that I did for, in the 2d case what I did for in the 1d case right. So, I do not know whether if I just write it down whether it is going to be a hassle for you. So, this is uprq may be I do not skip a step +lambda times what is this going to be? Remember, now delta x and delta y are the same right. So, i n delta x and i m delta y okay inh and imy.

So, that is going to give you 2 sets of theta's okay. May be I will just write it out and then I will explain what I am doing. So, this will give me a e power i theta-2+e power -i theta times uprq. And the other one will give me a lambda e power i alpha-2+e power -i alpha uprq, where theta is i theta is n delta x and alpha is m delta y. but, we have decided delta x=delta y but the wave numbers are still different okay. However, much we simplify, the x and y coordinates keep on falling apart.

It does not matter okay. So, what does it give me? Therefore, the gain when you take 1 times step. What is the gain? 1+, it is going to be very similar to what we had earlier. 2 lambda cos theta-1+2 lambda cos alpha-1 okay. And when is mod g<1? So, you can combine these 2 basically right when because you basically have to look at when these values are the largest. So, that will correspond, I will let you work. That will correspond to 0<lambda<1/4 okay.

Just like we did in last class, you can just work out mod j<1 and you will see that this is the condition that we get right. And we can sort of guess that if it was 3d Laplace equation, it will be 0<lambda<1/6 okay. You will just get one extra term for the z coordinate okay right. But, this is really what I was interested in. this condition, when I saw the 1 half, I said okay just let us take an aside and look at.

So, lambda=1/4, what happens here when lambda is 1/4? What happens to this equation when lambda is 1/4? When lambda is 1/4, you get a very familiar looking equation.

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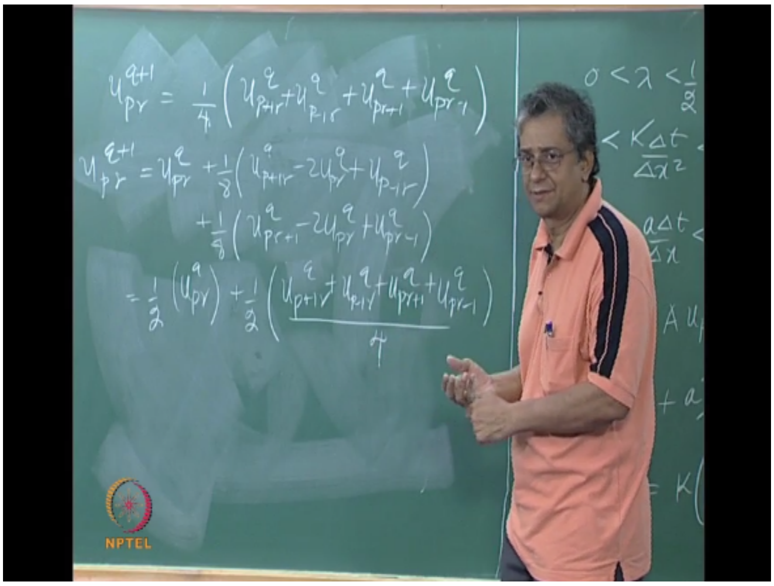

Uprq+1 is 1/4 up+1rq+up-1rq+upr+1q+upr-1q right, which was our iterator solution to Laplace's equation fine. We will revisit this. I could not resist taking this aside. We will revisit this later. But, what this basically says is, marching heat equation and time, this is same as sweeping Laplace's equation in space okay. So, now there are 2 different ways by which we can get solutions to Laplace equation.

Either you take Laplace's equation, add a time derivative and solve the unsteady equation and allow this solution to evolve in time. Or you do one iterative method Gauss-Seidel or whatever it is or Gauss-Jordan and if you are doing Jacobi iteration not Gauss-Jordan, if you are doing Jacobi iteration, it looks like all you are doing is solving heat equation. What if lambda instead of being 1/4 were 1/8 what would you get? Can you guess lambda instead of being 1/4, if I take lambda =1/8 what would you get?

That is like picking some kind of a relaxation parameter. Anyway if you want, we can have this. As I said if you want we can have this conversation later. But, that is like picking the relaxation parameter. So, you see you have uprq+1, if you picked it as 1/8 right, you would get a linear combination of upr at q right 1-omega times that, whichever way you want you will get a linear combination okay. The only constraint that you have is that, you have a stability condition here okay.

May be I can work it out. Since I have gone down this path. May be I can just work it out. What happens when lambda is 1/8 what do we get from the equation can you just tell me? Uprq+1=uprq lambda is 1/8 may be you can just read it out that is +1/8 up+1rq-2uprq+up-1rq+1/8 upr+1q-2uprq+upr-1q okay. There are 4 of these therefore, it becomes 1/2. So, that becomes 1/2 uprq+1/2 up+1rq+up-1rq+upr+1q+upr-1q divided by 4. So, the 1/8 I have written it as an average +1/2. This is like taking omega=1/2 okay.

I do not know how many of you tried when I said why do not you try SOR with Jacobi. I do not know if you have tried it. Did you try SOR with Jacobi anyone? Well, if you have tried it, you would have found that, for omega>1, it would not have worked right. Because for Jacobi iteration, you have a stability condition that says lambda has been <1/4. That is omega can only be at the most 1 fine. Gauss-Seidel let can go up to 2, Jacobi it cannot. In Jacobi, there is a stability condition that says that lambda <1/4 or which corresponds to omega=1 okay.

If lambda >1/4 you would not get the average right and it would not work fine okay right. This sort of connects. I wanted to connect what we were doing with Laplace's equation right with all this evolving in time, I want you to understand that, so marching in time to a steady state solution is the same as sweeping in space. So, there is no sense getting worked up saying oh you are going evolving in time, I am just doing sweeping in space I am not, you have an extra coordinate, I do not have that coordinate.

No. they both basically are the same right. What it does is, it gives you a different perspective. The same algorithm, it gives you a different way of looking at it. So, as long as you keep that in mind that, whether I am marching in time or sweeping in space okay that, there is an equivalence okay that we are fine. Is that okay right. So, that is the end of the aside that, we will get back to where we were.

What we were talking about now is, how much dissipation can we add? We saw that, if you add way too much dissipation, the stability condition changes right and the stability condition gets actually worse, the time steps that you have to take will get much smaller. So, you have to be very careful whether how much dissipation you may be tempted to add a lot of artificial dissipation. That does not quite work.

The second thing is, if you add a lot of artificial dissipation right now, the way we are adding it, you could say we are adding it explicitly right we are adding it at the current time level. The way you are adding the artificial dissipation actually contaminates the solution right. So, if you say Hey wait a minute, I am not actually solving the equation that I set out to solve. Anyway I am solving the modified equation. I am going to add artificial dissipation. That is one argument.

The other thing is, look you know you are solving the modified equation, instead of improving it, so that the modified equation gets closer to the actual equation, why are you making it worse? Right that is a counter argument that you can get right. So, we have to be a bit careful how you handle this. But you have an awareness that whether you like it or not, when you solve the problem, there is numerical dissipation that showing up because, the modified equation is not the same as the original equation.

So, you have to have an awareness to what it is right. The second thing is, if you are going to add something to it, you have to add it carefully right okay. Let me try to come to this problem in a different direction okay.

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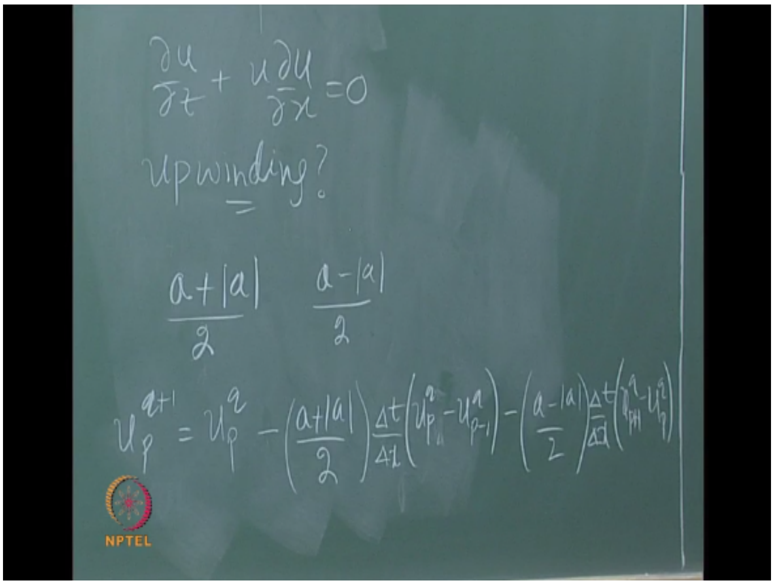

What if you add dou u/dou t+ a dou u/dou x=0 and I did not add I have been writing this and I did not do that or what if you add instead of a dou/dou x, you add either a as a function or you add something else you add some u right. The kind of equation we are used to in fluid mechanics, would be something like this. Where u is the solution and therefore you do not know what it is right what it is beforehand. A prior you do not know what is the value of u okay. So, what if you had the situation.

How do you ensure that you are doing FTBS right? Look at this equation in greater detailer little later. But, right now I am using it just for motivation. How do you ensure that you are doing FTBS or I am sorry how do you make sure that you are using upwinding right? It is not so much FTBS but, how do you make sure that you are doing upwinding right. If you do not know what the sign is, you have a scheme you do not know what the sign is, how would we make sure that we are doing FTBS.

So, there are different ways by which you could do it. Of course, you could have if then kind of a discretization right. So, the algorithm then becomes, it is a true algorithm. You turn around and say, if a>0, use backward space, if a<0 right use forward space. That is possibility a=0 I do not know right. A=0 of course does something to this particular equation so life becomes easier. A=0 can be a headache right. We will see what that headache is at a later time.

So, the other possibility is that, you can ask the question is there a way for us to create a switch, an automatic mechanism so that I do not have this conditional statement right.

Possibility. So, the question is, what is that quantity and what is this quantity? So, if I divide this by 2 right. So, if a is positive, this would be 0 right and that will be a. if a is negative, this is going to be 0 and this would be a right fine. So, now, we have something that looks like a switch.

Something that 0,1. So, you could then turn around and say that upq+1, the objective here is, I want you to see where we started off with these. We started off with Laplace equation, central differences, we tried it out, we tried out various things, some things worked, some things did not work, how do you develop these algorithms. What is the way by which, we are grouping around but, as we get along you get better at it right? And there are lots of little, little tools that you can use to construct algorithms.

You have to get an idea as how these things happen. So, this is upq, I seem to have automatically shifted to superscript, it does not matter right -a+mod a/2 delta t/delta x* what do you want here? Upq-up-1q thank you -a-mod a up+1q-upq. Is that fine? So, this would do it. This would automatically switch. This is one way to do it. So, automatically switch. You do not have the conditional statement but, then you are doing a lot of work right.

You eliminated the conditional statements but you are doing a lot of work. You are going to evaluate these terms independent of whether that 0 or not right. You are going to add them all up and throw them away. And you are going to add them all up and it may turn out that what you have fortunately here, you are not going to end up with round of because, they are identical right. But, these kind of algorithms, you have to be a bit careful. So, you are going to calculate all of these and just check it because that happens to be 0.

What is the other possibility? In the last class we saw something. What was the difference between central difference and forward difference or backward difference? Do you remember?

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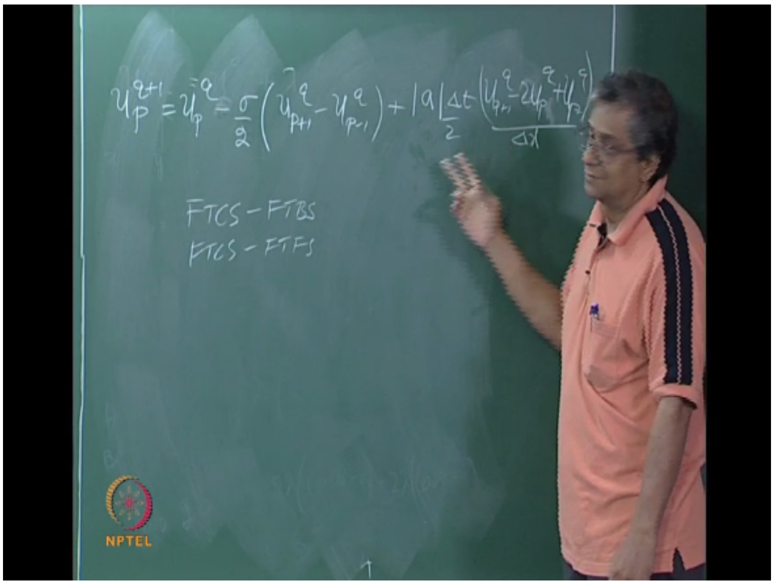

I want to add something to this, what was the difference between central difference and backward space or forward space. Do you remember? Second derivative but the coefficient is very important okay. So, it is something like, I will add mod a because we do not know the sign of a so I will add mod a right delta x/2 dou squared that is, I do not want the continuous thing, I want the up+1q-2upq you can go and check this from your last class +up-1q divided by delta x.

Is that okay, is that fine? Why did I do mod a here? I do not know the sign I want you to check that both FTCS and FTBS what is the difference between FTCS. FTCS-FTBS, FTCS-FTFS okay and see what you get. So, if you add this quantity, what is it going to do? It is going to convert the central difference to backward space. You understand? And by taking this sign out of the game, I have essentially made sure that whether it is forward difference, whether this is positive or negative, that is going to cancel out.

You can just try it out and see okay. And it will always be upwinding. That is another way to do it. You add the right kind of artificial dissipation right. Of course, from a stability point f view, what we are talking about earlier, I cannot afford to have this a to be negative, that is pretty obvious. That is mu 2 negative right. Does that make sense? If a is negative, that is mu 2 is negative, it is going to diverge right. So, it is pretty clear that, it has to be modulus away.

See, there are different clues that we have is to why we are doing what we did. Either you can do it from here to see what is the correction term that I have to do in order to change the central difference scheme to a upwinding scheme. Or you can look at it from the modified equation that we have got in. Say, oh the coefficient has to be positive. Is that fine? Everyone? So, there is a way, there are clear cut ways by which you can determine what happens when you add a specific term.

But, the addition of this term does not eliminate the second derivative term, the addition of this term only converts the central difference scheme to a one sided difference, one sided first order scheme. So, we have lost the order of the scheme okay. Are there any questions? So, while we are at it. So, this is, so, we have seen that, you can do FTCS. We have FTCS, FTBS, FTFS right. We have to use either FTBS or FTFS depending on which way the stream is flowing.

So, it is better to do FTCS, a centered scheme. See, this is one way to look at it, one argument and just add something to it so that it becomes upwind biased or the other thing is, to say that you do upwinding directly right, you can get. So, whether you are doing upwinding, I would say, if you are doing this, you are also doing upwinding. It is very clear, if you are doing this, you are also doing upwinding right. So, there is no sense getting into an argument just to whether you are doing central differences or up.

It is just a matter of, in order to decap, I am going to have dissipation and in order to get the dissipation, I need to do something okay. So, there is no sense getting into an argument just to whether you are doing central differences or. But, the minute you say it is FTCS + the correction term then, we can ask the question, can I eliminate the higher order terms and get a more accurate scheme. That is the different story right. We will look at that as I said when I do the demo okay.

Now, that we have done this. I know, in the beginning of the class, I said I am not going to do a survey of lot of schemes and so on. But, I am going to do a few schemes now, just to show you, just to go on with the philosophy of how do these schemes evolve. How do we develop these schemes right? And I will get you to a point where you should be if you really want right, if you go out look at all the schemes that are out there and say, no, I do not like these, I have a better idea.

You should be able to come out with something on your own. Is that fine? Okay. So, we come here, what we will do is, in all of these as I said, we have clues for these things for what we have done so far. So, earlier when we derived modified equation, what had I suggested? What did we do? We expanded using Taylor's series and we substituted for individual terms in Taylor's series okay. Then you can ask the question, why we did not develop a scheme using Taylor series? This is a classical technique using Taylor series to solve differential equation.

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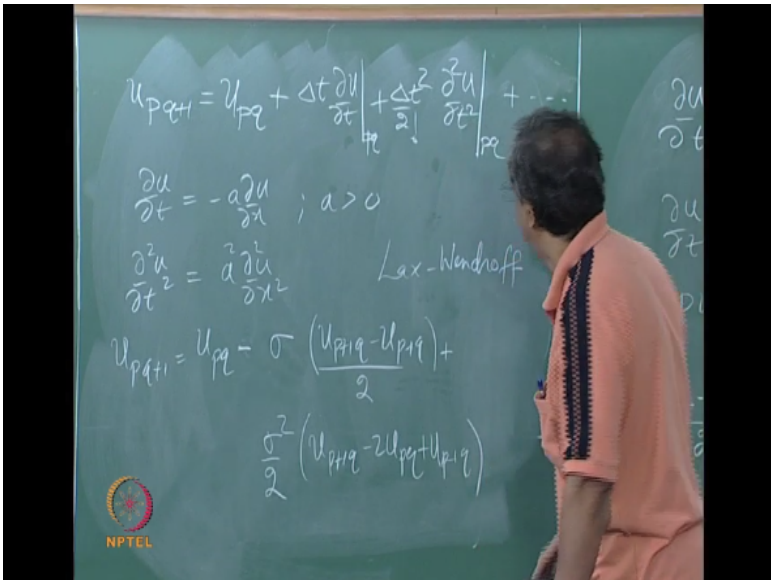

That is, if you have upq+1 I guess maybe I will stick to the subscript is upq+ let us do Taylor series delta t dou u/dou t at the point pq+delta t squared/2 factorial dou squared u/dou t squared at the point pq and so on fine. When we did the modified equation, of course we wrote the right hand side. But, here I am going to stop at this point. As a wait a minute. Why go there at all? You already have a trick.

What is dou u/dou t? dou u/dou t is -a dou u/dou x again I am back to a>0 that situation where a>0 and what is dou squared u/ dou t squared? A squared dou squared u/dou x squared. Substitute back upq+1 is upq+delta t a dou u/dou x is up+1q-up-1q/2 delta x - sign I do not know; I always forget that - sign okay fine +a squared delta t squared/2 delta x squared. I will do the open bracket here but, I am going to write it here or maybe I will write it in the bottom.

A squared delta t squared/2 delta x squared up+1q-2upq+up-1q close that. We just cooked up a scheme. This is called the Lax-Wendroff scheme right. All you just basically do is, you just go through do a Taylor series expansion. Classically it is solution to differential equations using Taylors series, that was normally done with ODEs in this application tool. PDEs, we get the credit for it okay. So, here you have it FTCS there so, this would be sigma and that would be sigma squared/2 fine.

And the scheme comes with its dissipation add right. You can try out; you can go through do these stability analysis for this. I am not going to do this anymore right. These I leave as exercise. You can try out the stability analysis for it. You can find out the modified equation and see what is the nature of the modified equation right. You can try to find out what is the nature of the modified equation. Is that fine everyone? Okay.

What else I am just going to know in a freewheeling fashion, connect all the bits and pieces that we have done to see whether we can come up with other schemes that is all right. I am just going to do a few of these before we sort of end this whole thing of linear wave equation right okay. So, I am going just sort of try around, something called 2 step Lax-Wendroff method, they do the following.

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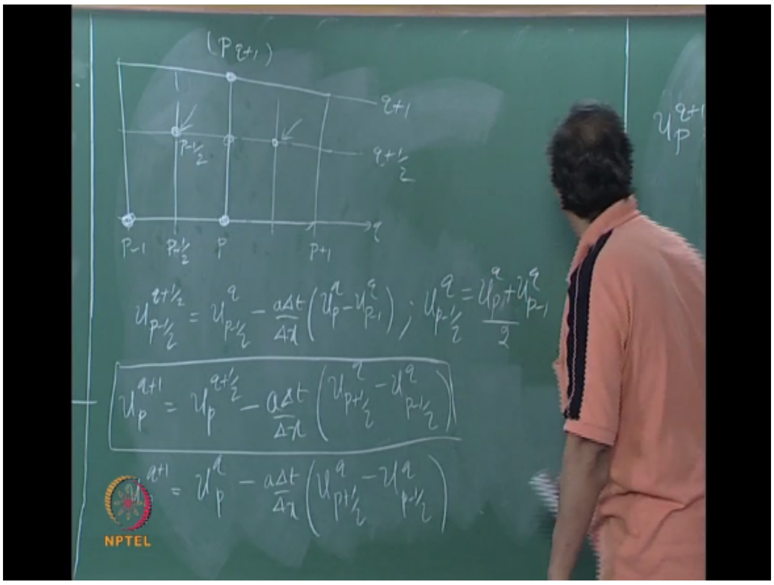

I will draw the stencil in this case simply because, I am actually drawing grid lines simply because it is easier to understand with the grid lines. So, this is p, p-1, p+1 at time level q, that is q+1. I seem to have drawn a line in between. I will also draw another line in between here, lots of extra lines right. So, that is, this is a point pq+1 basically that is what we want. I am going to do this in 2 steps, 2 different ways right.

I will tell you what is the final 2 steps scheme. So, what we can do is, you have the value here, you have the value here, if you had this in between value, you could do FTCS and find that. So, humor me, I will call that p+1/2 right I know I mean it is new accounting, it is integer but, we will call it p-1/2 good. Even with counting I have to make sure I get the sign right. Today is my day for negative sign.

Anyway p-1/2 so, then this would be p-1/2 well, you had expected that would be q+1/2 right. So, you can see that, if there was a way by which I could use FTCS to find that and a way by which I could use FTCS to find that, then I could repeat the process and use FTCS between these 2 to find that. Is that fine? Okay. 2 step process, the process what I just described now is not the 2 step Lax-Wendroff process, it is a 2 step process.

So, let me write that of first and then I will tell you what is the improvement. So, you can say up-1/2 may be I will go back to superscripts here q+1/2=up-1/2q-a delta t/2 delta x up-up-1 these are all at q. Is that fine everyone? How do I find that up-1/2q? give me a suggestion. Take the average. So, I can take the average up-1/2q is up+up-1 divided by 2 fine. So, in a similar fashion, up+1/2q well, it would be the same. So, obviously you are going to find this value using these 2 right.

For each of these intervals you do that. So, I will go directly to upq+1 so, here is the first suggestion upq+1/2-a delta t/delta x. We are assuming delta x are equal everywhere up+1/2q-up-1/2q. Does that make sense? And any time to find this intermediate value if you do not do it, take the average of the neighbor's okay everyone. It is fine. So, do you expect this to be stable using FTCS? Or unstable? Before you start the stability analysis, you first predict what you expect it to be and see whether you get it.

Right. And in order to do the stability analysis, you have upq+1 remember you have to get this in terms of right hand side has all upq's, which basically means that, you substitute for the p-1/2 in terms of p's and p+1 do you understand. Eliminate all the 1/2 that is the easiest way to do it. Just make the substitutions, eliminate all the 1/2. So, that you finally get upq+1 in terms of upq's. Take the ratio and you can do the stability analysis. Fine. This is a 2 step method, the 2 step Lax-Wendroff method.

You stop at this point and you say, wait a minute there is something here, why bother with taking FTCS here again? I had the value here. If I take a time derivative across this, I am taking a central difference at this point. I do FTCS to get that. I do FTCS to get this point, FTCS to get that point. Here I can do central time. I can actually do central time. I can get a higher order accuracy and time. I can do a central time. Is that fine? I can do CTCS.

That means, this second step instead of using this, in the 2 step Lax-Wendroff method, you would write upq+1 is upq-the rest of the stuff. Central difference in time, central difference in space okay. One of the reasons why I do not do this is, it can get dreary right. It can get tiresome; I am just putting up this is out there. I can already see it on your faces. It can vary out. This is a 2 step Lax-Wendroff method.

It is done in 2 steps. First, you get the q+1/2 and then you do the full thing, full delta t fine okay. I did this, so that you are aware that, you do not have to do it in one step. You can do it in multiple steps. The time integration can be done in multiple steps. In our old class, Runge kutta schemes and so on, the time integration is actually done in multiple steps. So, there are multi step methods right.

So that you are aware of it. A third thing that we did right back at the beginning when we were doing finite differences, what have we done so far?

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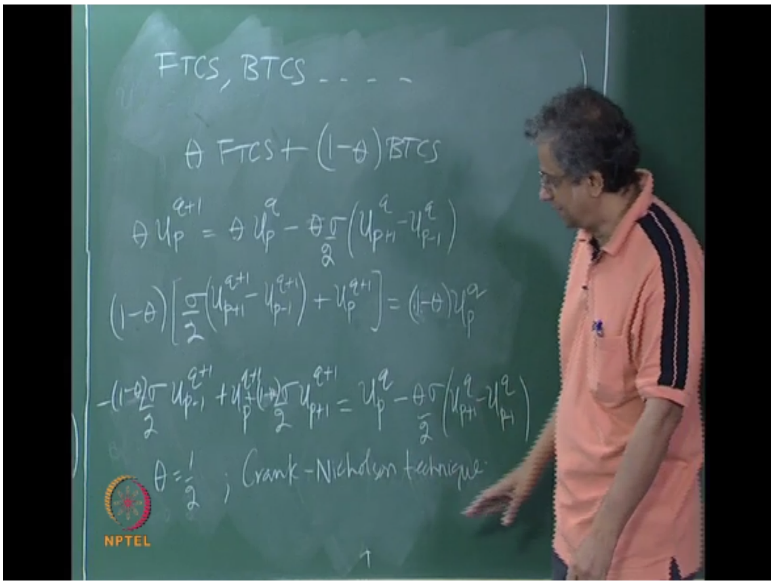

Amongst the various things, we have done FTCS, we have done BTCS and of course we have done lot of other stuff but, I am more interested in FTCS and BTCS. Before we derived finite differences using Taylor series, we did for forward difference and backward difference. Do you remember the first time I introduce central difference, what I did? I took the average of the forward difference and backward difference right.

I looked at the truncation of a term for forward difference and backward difference and said, wait a minute, these are the same magnitude but opposite sign. Why do not I take the average. I have forward times central space, backward time central space, why do not I take the average? Why do not I take the combination of these 2? You understand? Right. Or better still I can take a weighted average.

Why just average? Now, we are into things like SOR and you know things like. Why not just a weighted average, why not a theta times FTCS+1-theta times BTCS. That sounds wag right. Theta=1/2 wub be an average. How do we do this? Theta times upq+1=theta times upq-theta sigma/2 up+1-up-1. What is BTCS? 1-theta times sigma/2 up+1 q+1-up-1 q+1+upq+1=upq. Is everybody with me? Is that fine? Okay. So, at the hope we will see what we get? You add them up what do you get?

There is a 1-theta upq+1 just to point out, this always happens theta upq+1. So, theta will go away right. So, you get from here, a -sigma/2up-1 q+1+upq+1+sigma/2 up+1 q+1=again as I pointed out, the theta and -theta will cancel. Upq-theta sigma/2 up+1 q-up-1 q. I lost 1-theta somewhere. It should be here. I hope that is not too messy fine. And theta=1/2 you get a very famous technique called the Crank–Nicolson technique.

But, you have to solve a system of equations. But, clearly you can take various values of theta right. I would suggest that you try to do the stability analysis for this, you try to do the stability for the 2 step Lax-Wendroff method. All of these schemes p, p+1, p-1, q, q+1 both of these schemes represent the differential equation approximate the differential equation at the midpoint fine okay. So, I will see you in the next class. Thank you.