Space Flight Mechanics

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Lecture No - 76
Orbit Determination (Contd.,)

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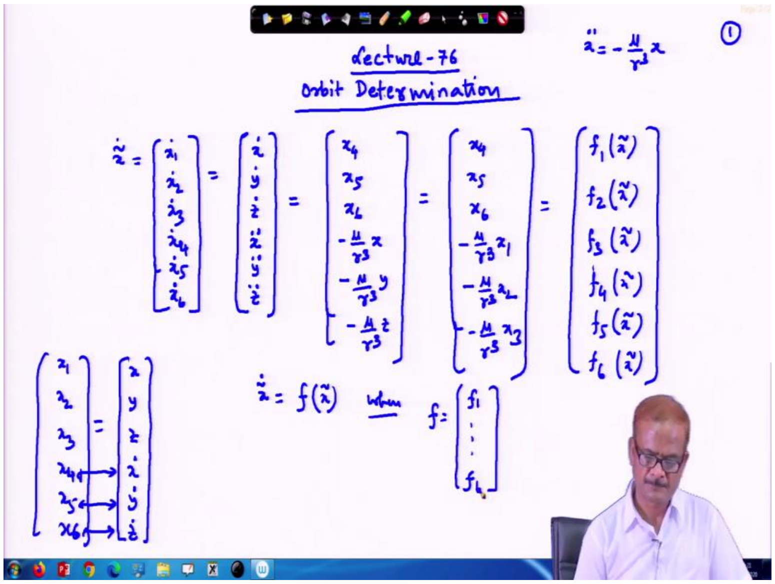

Welcome lecture 76 so we have started with orbit determination and will continue with that. If you remember the last time we have written the state equation. State equation it was defined as

$$\dot{\tilde{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ -\frac{\mu}{r^3}x \\ -\frac{\mu}{r^3}y \\ -\frac{\mu}{r^3}z \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ -\frac{\mu}{r^3}x_1 \\ -\frac{\mu}{r^3}x_2 \\ -\frac{\mu}{r^3}x_3 \end{bmatrix} = \begin{bmatrix} f_1(\tilde{x}) \\ f_2(\tilde{x}) \\ f_3(\tilde{x}) \\ f_4(\tilde{x}) \\ f_5(\tilde{x}) \\ f_6(\tilde{x}) \end{bmatrix}$$

and where we have defined? x_1, x_2, x_3, x_4, x_5, x_6, xyz, $\dot{x}$, $\dot{y}$, $\dot{z}$ according to this notation $\dot{x}_1$is equal to $\dot{x}$, $\dot{y}$, $\dot{z}$, $\ddot{x}$ $\ddot{y}$ $\ddot{z}$ and finally we wrote it in a format.

So what we see from this place it is x_4 equal to $\dot{x}$, x_5 equal to $\dot{y}$ and x_6 equal to $\dot{z}$. So here we write x_4, x_5, x_6 and $\ddot{x}$ basic equation we had $\ddot{x}$ equal to - $\frac{\mu}{r^3}$ x this was our basic equation so we replace this quantity here $\ddot{y}$ equal to $-\frac{\mu}{r^3}$ y and $\ddot{z}$ equal to $-\frac{\mu}{r^3}$ z. And finally everything we write in terms of x_1, y_1 and so on. X is the quantity here x_1 we write here x_1 similarly this becomes x_2.

And this we can write as f_1 f_2 x tilde f 3 x tilde f 4 x tilde f 5 f 6 and we can see that our state equation it can be written as f times $\tilde{x}$, where f is equal to f_1 to f_6.

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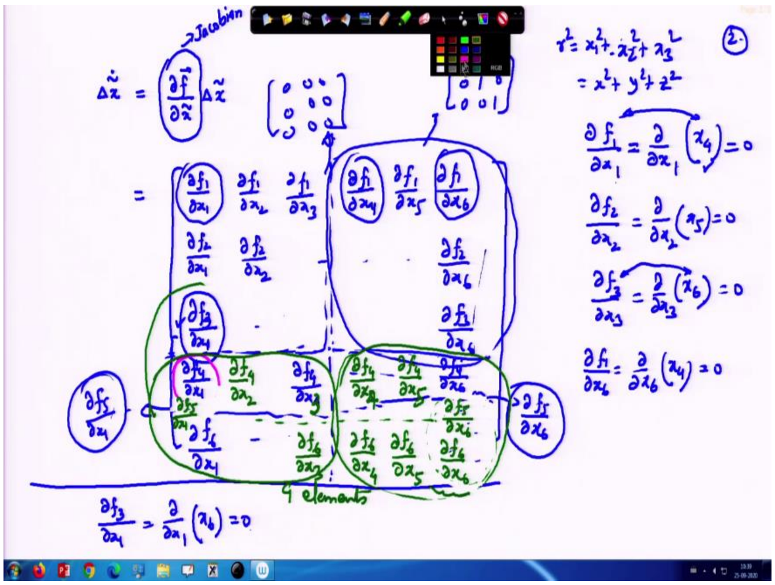

W e do one more operation here in this place. Also, we should note that the

$$r^2 = x_1^2 + x_2^2 + x_3^2$$

which from $x^2 + y^2 + z^2$. Once we have got this so without currently explaining what I am doing, I just want to linearize this equation. So this can be written as from your Taylor series expansion you can just look into that. It can be written as shown in below.

$$\Delta\dot{\tilde{x}} = \frac{\partial \vec{f}}{\partial \tilde{x}} \Delta\tilde{z}$$

Where f it is also a vector and x is also vector this forms a Jacobian. And this we can, write as $\frac{\partial f_1}{\partial x_1}$, $\frac{\partial f_1}{\partial x_2}$, $\frac{\partial f_1}{\partial x_3}$. Similarly the other items can be written $\frac{\partial f_2}{\partial x_1}$ and so on $\frac{\partial f_3}{\partial x_1}$ and the last one will be $\frac{\partial f_6}{\partial x_1}$,….and last one $\frac{\partial f_6}{\partial x_6}$ this forms the Jacobian Matrix so one by one all the 36 elements need to be evaluated. So let us do few of them I will evaluate and rest you can check yourself.

If we go back so f_1 is x_4 so $\frac{\partial f_1}{\partial x_4}$ this becomes and x_1 is x_4 this gives you by evaluating the first element we are evaluating. So this is one and these are all independent variables. Here if you look once we have casted it in the form of the status format. So x_1, x_2, x_3, x_4, x_5, x_6 they are all appearing as independent variable and therefore this quantity is 0. If you look this way so the quantity from this place to this places all of them will be 0.

You can check for another one letter say $\frac{\partial f_2}{\partial x_2}$ so $\frac{\partial}{\partial x_2}$ and f_2 is here is x_5 and therefore this also this turns out be 0. Any other element let us say that and in the same way you will have $\frac{\partial f_3}{\partial x_3}$, f_3 is x_6 so this quantity also vanishes. Let us evaluate this quantity and therefore we have on the right hand side we can write $\frac{\partial f_3}{\partial x_1}$ and f_3 is x_6. Again this is not a function of x_1 and therefore this is zero.

So, this quantity here the what I am showing from here to here all the elements will be these 9 elements will be 0. All of them they are 0. So this forms a; this is in this format all the 9 elements here are 0. It remains to determine the other total 9 we have determined rest 27 we have to further decide how much what will be the values of those elements.

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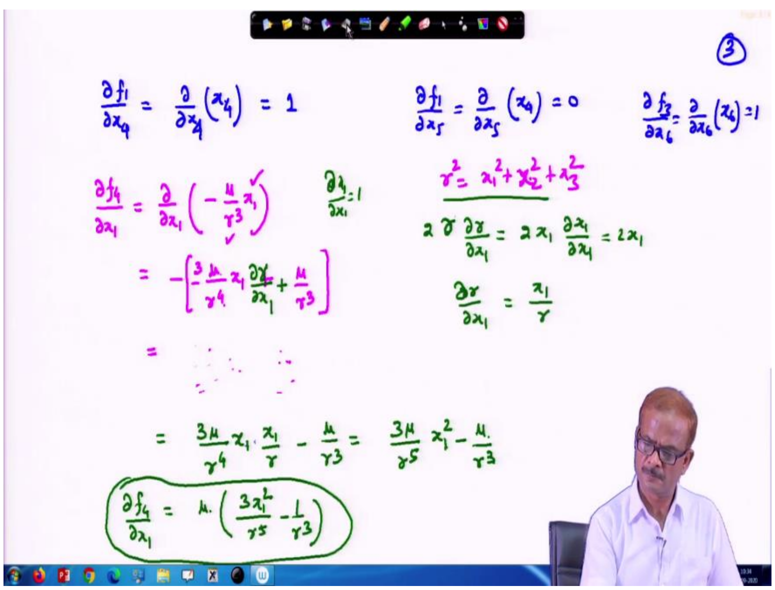

Let us first work out the $\frac{\partial f_3}{\partial x_4}$ and f_1 has we have written here f_1 is nothing but x_4. So this is 1. Next is $\frac{\partial f_1}{\partial x_5}$ so $\frac{\partial f_1}{\partial x_5}$ this quantity is 0. So what you will see if you do this exercise you will see that this quantity from here to here it comes here in this format 100 010. So you can check for $\frac{\partial f_3}{\partial x_6}$ and f_3 is from here this is x_6.

This quantity is also equal to 1 rest other quantities of the diagonal terms they are 0 let us say $\frac{\partial f_1}{\partial x_6}$ and f_1 is from this step we are getting this quantity is again 0. By these way all the quantities all the diagonal of the terms elements of this partition will be 0. Rest then we have from here we will have $\frac{\partial f_4}{\partial x_1}$ and so on. And we have $\frac{\partial f_4}{\partial x_3}$and this will going to $\frac{\partial f_4}{\partial x_6}$.

And of course, here then $\frac{\partial f_5}{\partial x_1}$it will start from this place and in this case you will have the element here this particular to the fifth row. It will; the last element. This part will be $\frac{\partial f_5}{\partial x_6}$. So the fifth I am not shown here. This is the first element of the fifth row and this is the last element of fifth row. The first we will check for this element $\frac{\partial f_4}{\partial x_1}$; and what is f_4? For that we need to go here.

Then see f_4 is this quantity $-\frac{\mu}{r^3}x_1$. But r is itself a function of $x_1^2 + x_2^2 + x_3^2$. We need to differentiate this and this both in taking the partial derivative. Once you differentiate it will appear like this -3 $\frac{\mu}{r^4}x_1 + \frac{\mu}{r^3}$ and we can rearrange them to writer as $3\frac{\mu}{r^4}$. One more term we are missing which we have to introduce here.

Once we are differentiating this r so also we have to write here $\frac{\partial r}{\partial x_1} + \frac{\mu}{r^4}x_1$ by $\frac{\partial f_1}{\partial x_1}$ that will be equal to 0. This quantity is missing that quantity will need to work out from this place. 2r $\frac{\partial r}{\partial x_1}$is equal to $2\frac{\partial f_1}{\partial x_1}$ this is $2x_1$. Therefore $\frac{\partial r}{\partial x_1}$ this gets reduced to x_1 by r. If you want to insert here in this place this line.

We directly write first here in this place and then there after being modified this is minus minus plus

$$3\frac{\mu}{r^4}x_1.\frac{\partial r}{\partial x_1}\frac{x_1}{r} - \frac{\mu}{r^4} = 3\,\frac{\mu}{r^5}\,x_1^2 - \frac{\mu}{r^3}$$

So this is what we get as the fourth row first element. In this way, we can find out all the elements of; all the elements which are involved from this place to this place. There are 9 elements here. All these 9 elements can be evaluated.

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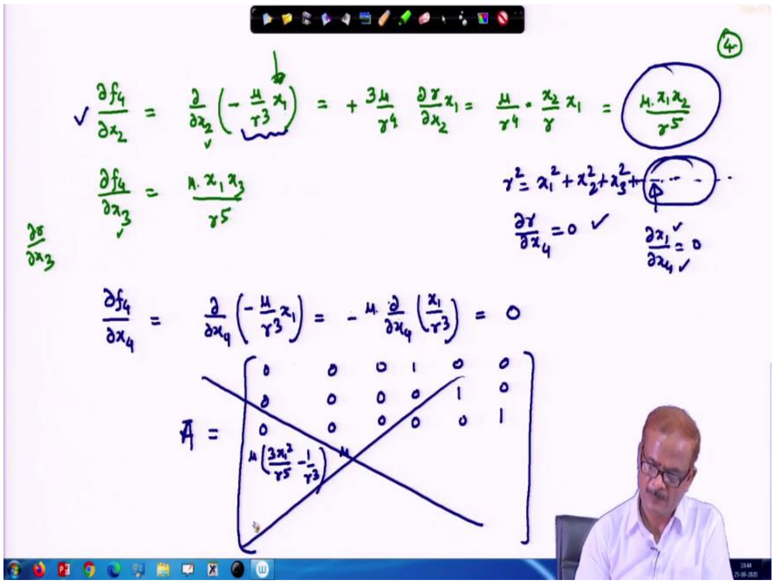

So the next page I will summarize this and if you calculate this part 2: $\frac{\partial f_6}{\partial x_6}$. So this also we need to check and there after we will come to this part ∂f_4here the fourth row second element it is $\frac{\partial f_4}{\partial x_2}$. So, $\frac{\partial f_4}{\partial x_2}$ and this quantity f_4 is - $\frac{\mu}{r^3}$ x_1 we can see that x_2 is here and x_1 is here, x_1 is not a function of x_2 therefore, we cannot differentiate the second part and will get here minus minus plus $\frac{\mu}{r^4}$ and $\frac{\partial r}{\partial x_2}$ and this will appear here.

$$\frac{\partial f_4}{\partial x_2} = \frac{\partial}{\partial x_2}\left(-\ \frac{\mu}{r^3}\ x_1\ \right) = \ +3\frac{\mu}{r^4}\frac{\partial r}{\partial x_2}x_1 = \frac{\mu}{r^4}\frac{x_2}{r}x_1 = \frac{\mu x_1 x_2}{r^5}$$

So this is $\frac{\mu}{r^4}$ x_1 we have to write here $\frac{\partial r}{\partial f_2}$ this quantity will be $\frac{x_2}{r}x_1$. So this is μ times x_1 x_2 divide by r^5 in the same way you will have $\frac{\partial f_4}{\partial x_3} = \frac{\mu x_1 x_3}{r^5}$. Therefore naturally contains x_1 will appear and because of this x_3 and differentiation of r with respect to f_3 will get your $x_3.r^5$. This way you will be able to complete all terms.

Now the only time that we are left with is here in this row $\frac{\partial f_4}{\partial x_5}$. This we have to work out this particular partition. Similarly in the 6th row the last row this will be $\frac{\partial f_6}{\partial x_1}$ no this is only up to 3 this is up to 3 here. Here if we starts with 4 and then $\frac{\partial f_4}{\partial x_5}$. Similarly you have $\frac{\partial f_6}{\partial x_4}$ so there will be $\frac{\partial f_6}{\partial x_2}$ $\frac{\partial f_6}{\partial x_5}$ last element $\frac{\partial f_6}{\partial x_2}$ I will write it little clearly.

Here you will have $\frac{\partial f_5}{\partial x_6}$ similarly here in this case $\frac{\partial f_5}{\partial x_1}$ and so on. So this row will continue like this. So we evaluate this quantity $\frac{\partial f_4}{\partial x_4}$. So we were working with this particular term $\frac{\partial f_4}{\partial x_4}$ is this quantity $\frac{\mu}{r^3}\ x_1$. So if you look into this, this will be the partial derivative so this gives us μ we can take it out $\frac{\mu}{r^3}\ x_1$ already we have looked into the r square this is function of $x_1^2 + x_2^2 + x_3^2$.

And therefore $\frac{\partial r}{\partial x_4}$ this quantity will be zero because nowhere here x_4 comes there is nowhere x_4 present in this term. Thereafter there is nothing you do not have anything here. So this quantity gets reduced to 0 and x_1 also $\frac{\partial x_1}{\partial x_4}$ this quantity is 0 because if one is not a function of x_4. So this way you will see that this quantity is equal to 0. So, all other terms in that matrix turns out to be 0.

So, finally we write this as the Jacobian matrix which you are writing in this matrix. Let us say this Jacobian matrix can be writing as this. This matrix can be written as A equal to 000 and other elements we have to enter here so $\frac{\partial f_4}{\partial x_1}$ by already we evaluated $\frac{\partial f_4}{\partial x_1}$ is a quantity we have written here. This $\left(\frac{3x_1^2}{r^5} - \frac{1}{r^3}\right)$ is the first element. So the second element $\frac{\partial f_4}{\partial x_2}\ \mu$. We would not be able to adjust here. Let me try. We will go on the next page.

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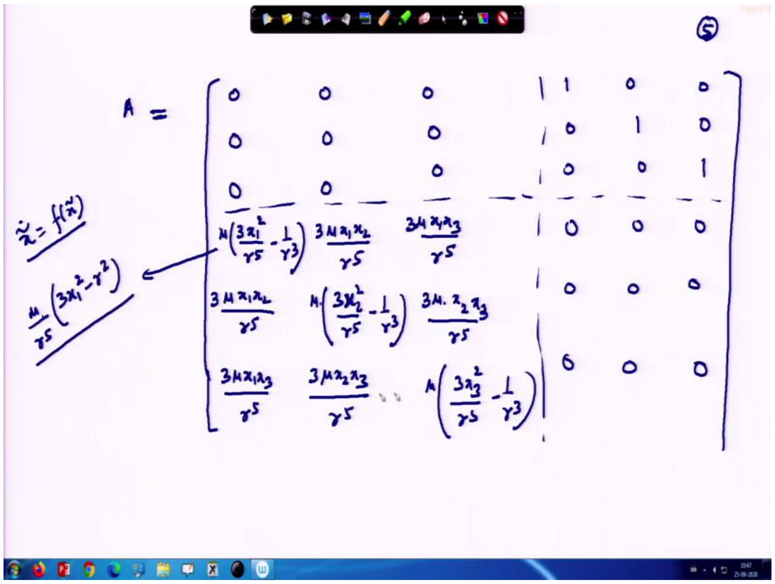

The first element we have $\mu\left(\frac{3x_1^2}{r^5} - \frac{1}{r^3}\right)$. The second element we are taking from this place this particular element, which is coming here $\frac{\partial f_4}{\partial x_2}$ this quantity is $\frac{3\mu x_1 x_2}{r^5}$. The next element we are written $\frac{3\mu x_1 x_3}{r^5}$ and if you further differentiate you will get the other terms also and it will appear like this $\frac{3x_2^2}{r^5}$ you can check the other items.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \mu\left(\frac{3x_1^2}{r^5} - \frac{1}{r^3}\right) & \frac{3\mu x_1 x_2}{r^5} & \frac{3\mu x_1 x_3}{r^5} & 0 & 0 & 0 \\ \frac{3\mu x_1 x_2}{r^5} & \mu\left(\frac{3x_2^2}{r^5} - \frac{1}{r^3}\right) & \frac{3\mu x_2 x_3}{r^5} & 0 & 0 & 0 \\ \frac{3\mu x_1 x_3}{r^5} & \frac{3\mu x_2 x_3}{r^5} & \mu\left(\frac{3x_3^2}{r^3} - \frac{1}{r^3}\right) & 0 & 0 & 0 \end{bmatrix}$$

So this 3 is missing here, so we need to place the 3. Similarly 3 is missing here in this place. So 3 is missing here replace 3 in all places. So 3μ now if this will be x_2, x_3 you can see this is called diagonal terms they are symmetry. So, this constitutes your 9 into 9, 18 elements and rest of the elements are 100 010 and here in this place. So this is your A is the Jacobian Matrix which we have got after linearizing the equation $\dot{\tilde{x}} = f(\tilde{x})$. This term briefly also it can be written as $\frac{\mu}{r^5}(3x_1^2 - r^2)$. So this way also you can write not a problem.

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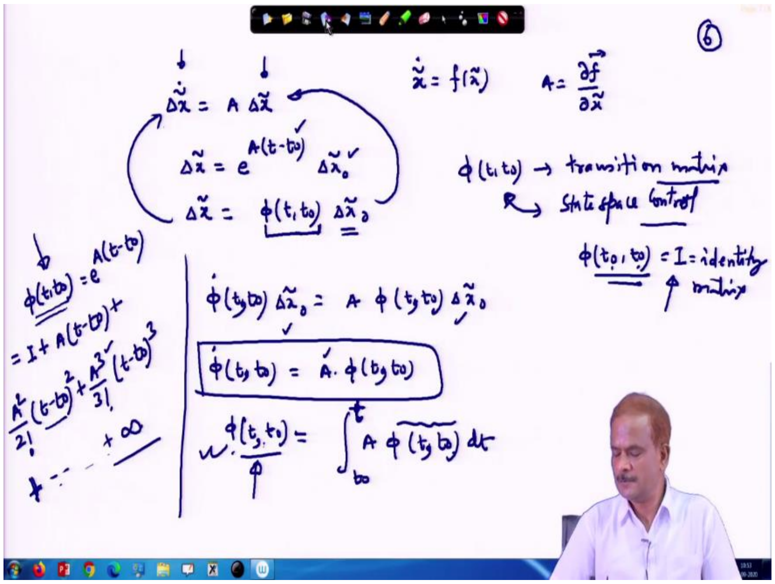

Once we have determined this Jacobian matrix $\Delta\dot{x}$ we can writer as $\Delta\dot{x} = A\Delta\tilde{x}$, which we have got after linearizing this equation $\dot{\tilde{x}} = f(\tilde{x})$. There A is nothing but $\frac{\partial\vec{f}}{\partial\tilde{x}}$ this is a vector itself and $\tilde{x}$ and this can be solved. So solution to this from your basic mathematics or those who have done the control course they must be aware off.

This is A times $t - t_0$ or from your differential equation this can be written this way. That t equal to t_0 that is initial time and $\Delta\tilde{x}$ is the initial state as appearing in this equation and this quantity is written as $\varphi(t, t_0)$ so if we insert here in this place. So we get $\dot{\varphi}(t, t_0)$ this is a constant. This is the initial condition and this part insert here on the right hand side. So this gets reduced to $A\varphi(t, t_0)\Delta\dot{\tilde{x}}$.

$\Delta\tilde{x}_0$ is not equal to 0 and therefore $\varphi(t, t_0)$ this quantity equal to $A\varphi(t, t_0)$, Where $\varphi(t, t_0)$, this is your transition matrix. So any controls you look into the status phase method, status phase control and there you will find what this transition Matrix properties are? So I will just stated that phi t t 0 this is equal to I this is I identity matrix. And therefore this $\varphi(t, t_0)$ at difference instant of time it can be evaluated by integrating this.

So this quantity will be a $A\varphi(t, t_0)dt$. So you are starting with this initial value you can go on and keep on integrating and step by step you can get this or either if the this transition matrix is also written as this is equal to $e^{A(t-t_0)}$ so this is

$$= I + A(t - t_0) + \frac{A^2}{2!}(t - t_0)^2 + \frac{A^3}{3!}(t - t_0)^3 + \cdots \infty$$

So this method of evaluation evaluating the transition matrix at different instant of time it is a costly because you have to keep multiplying the matrices one after the other. And more term you keep more accurate this will be. Here in this case once you have you are starting with the value of A and you have this $\varphi(t\ t_0)$, $\varphi(t_0, t_0)$ initially write in the beginning equal to identity Matrix. You can just keep integrating and you will get this result the transition Matrix at different instant of time and this is common not 1.

These are common so way the transition matrix is evaluated and why we are doing this it will be shortly visible to you. So we stop here and we will continue with this topic in the next lecture.