Space Flight Mechanics
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Lecture 4
Central Force Motion

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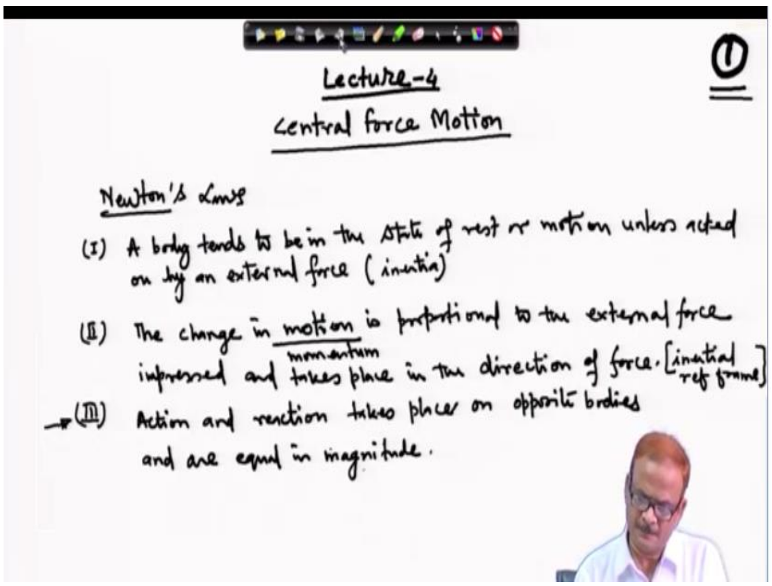

Welcome to the lecture number 4. Today, we are going to discuss about the central force motion. So central force motion, it is relevant in the case of the planetary motion. So you might have heard about the Kepler's law. So Kepler's law by gathering the data and looking into that data, the Kepler define certain rules, okay, regarding the motion of the planets around the sun and those rules are called the Kepler's law and this Kepler's law is basically the kinematic equation.

Whatever Kepler has written; it is in the geometrical aspect of the motion. It does not deal with the force anywhere, but Newton’s law as you are aware of, it deals with the force on a body and that results in a certain type of motion, whatever it may be. So your Kepler's law or the rules that Kepler formulated, they can be derived using the Newton’s law. So today, we will start with the basic definition of the Newton’s law and thereafter we will go to Kepler's law and then the central force motion.

We shall be working in using the vector rules. So let us start. Newton's law, we have the first law, the second law and the third law. As it is told that it is Newton's law, but what is the problem is, we will write it Newton's laws, because we have three here. So only the third one was the contribution of Newton. That is two were existing already, but over a period of time, it is known that all the three laws, they are termed as the Newton's law.

The first law, it says that a body tends to be in the state of rest or either in motion until unless acted on by an external force and this is also called the law of inertia. I am not going into too much of details of all these things, because already I have discussed all these issues in quite detail in other course related to this in MOOC's, which is on Satellite Attitude Dynamics.

The second law, it tells that the change in motion is proportional, what you call this is the rate of change of momentum, is proportional to the external force impressed and takes place in the direction of force. So second law of motion, this defines initial reference frame. So again I will tell you to refer to the MOOC's lecture on Satellite Attitude Dynamics and second law of motion, you can apply only in the initial frame not in any other frame.

The third law is action and reaction takes place on opposite bodies. I am writing roughly these definitions and are equal in magnitude. So quite details have been given already in another MOOC's lecture and therefore I am not discussing much. Now these three laws can be used to get the Kepler's planetary laws. So we go to the next page.
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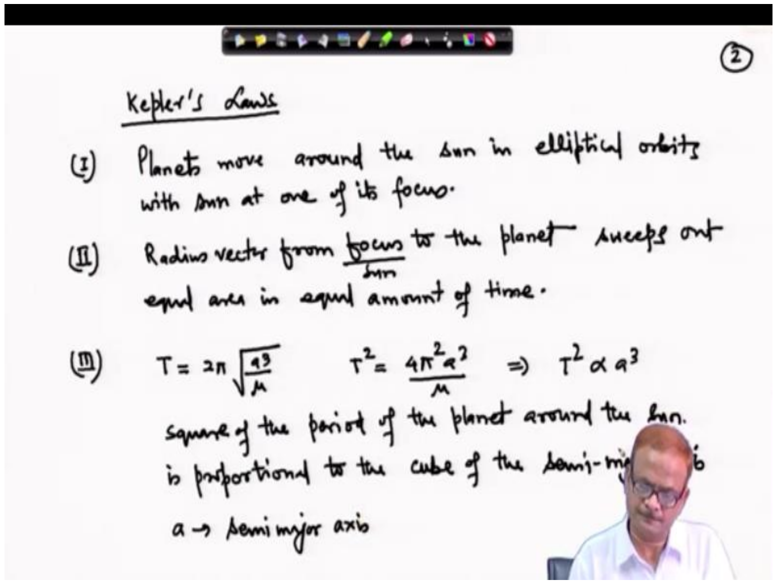

Planets move around the sun in elliptical orbit with sun at one of its focus. This is the first law. This is purely geometrical that the sun is going in elliptical orbit. The second law, this states radius vector from focus to the planet, here focus means, this is the sun, from sun to the planet sweeps out equal area in equal amount of time. Other way you can say that the rate of sweep of area is a constant.

The third law is related to the period of the planet around the sun. So this is directly written as say the period of the planet around the sun, this can be written as

$$T = 2\pi \sqrt{\frac{a^3}{\mu}}$$

So if we square both sides, so this can be written as

$$T^2 = 4\pi^2 \frac{a^3}{\mu}$$

So this implies that T^2 is proportional to a^3. So simply this is stated as a square of the period of the planet around the sun is proportional to the cube of the semi major axis, where a is the semi major axis.

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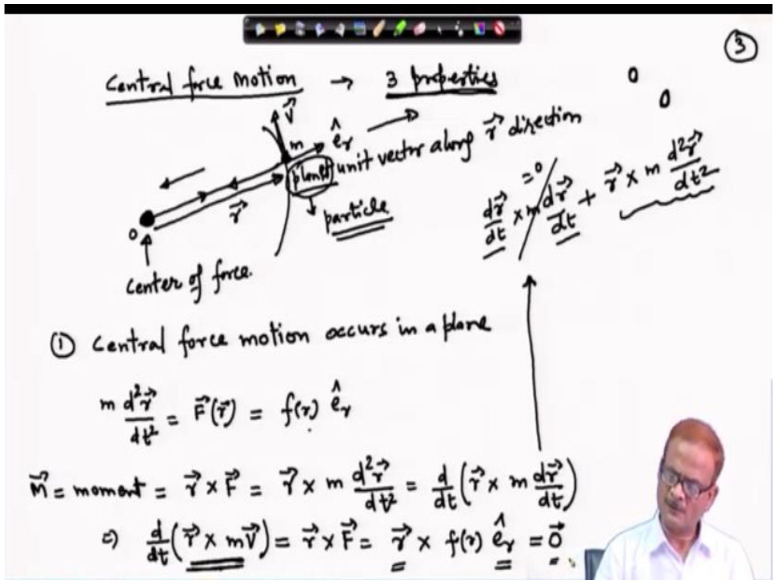

So we have got the Newton’s laws of motion and Kepler’s laws. Now we go into the central force motion. So central force motion, this has got three properties and as the name implies the central force means always directed towards the center. So if any planet is, say here some planet is there and there is a force acting either here in this direction or it may be in this direction, in either direction.

So your $\vec{r}$ is in this direction, so we will write here $\hat{e_r}$ the unit vector along r direction, $\vec{r}$ will be always measured from this place to this place. So this is your $\vec{r}$ and here this is planet, say this is a planet or it so happens that this is actually a particle. We assume this to be a point mass. So this is a particle and the center of force is located here. So this is your o, this is the center of force.

If we apply Newton’s law of motion to this particle, which is moving in this orbit, so velocity will be tangent to this orbit at any instant of time. So we can get all the three properties that I am going to work out. So the first one is central force motion occurs in a plane and we have to prove this. So this is the first property of the central force motion. So say m is the mass of this particle, so we can write $m\frac{d^2\,\vec{r}}{dt^2}$. This is equal to the force acting on this particle.

So here I am not telling that whether the force is directed along this direction or the force is directed along this direction. Irrespective of that, we are deriving it. Whether it is towards the center or either it is away from the center, but this will be valid. So the right hand side, this is a function of

r, because this is the central force motion, so we can write this as f(r) times $\hat{e_r}$, because this is directed towards the orbit.

So f(r) can be having either negative sign, so that it goes toward the center or it will be totally a positive quantity, so that it is out from the center toward the planet. Now if we work on this, now let us write this as m equal to moment as we know, this is $\vec{r} \times \vec{F}$. So from here this place, we can write this as

$$\vec{r} \times \vec{F} = \vec{r} \times m \frac{d^2 \vec{r}}{dt^2}$$

We can check this if we work out here, expand it, we will see that this

$$\frac{d\vec{r}}{dt} \times m \frac{d\vec{r}}{dt} + \vec{r} \times m \frac{d^2 \vec{r}}{dt^2}$$

So these two vectors are parallel to each other. Therefore, this drops out and this is the original we recover here. So therefore, this implies

$$\frac{d}{dt}\left(\vec{r} \times m \frac{d\vec{r}}{dt} \right)$$

we write this as $\vec{v}$, the velocity vector. So this is equal to

$$\vec{r} \times \vec{F} = \vec{r} \times f(r)\, \hat{e_r}$$

So these two vectors are parallel to each other. So the right hand side this becomes $\overrightarrow{0}$. Remember that this is a 0 vector. This is not a scalar, because here on the left hand side $\vec{r} \times m\, \vec{v}$, this is a vector.

You can look into this. So this is a vector and therefore, this is bound to be a vector. This is not just a scalar, but quite often it is customary just to write it 0. In the books, you will find it written in many places by a bold 0. So even if I do not put an arrow overhead, so assume that if the left hand side is a vector, so right hand side will also be a vector.

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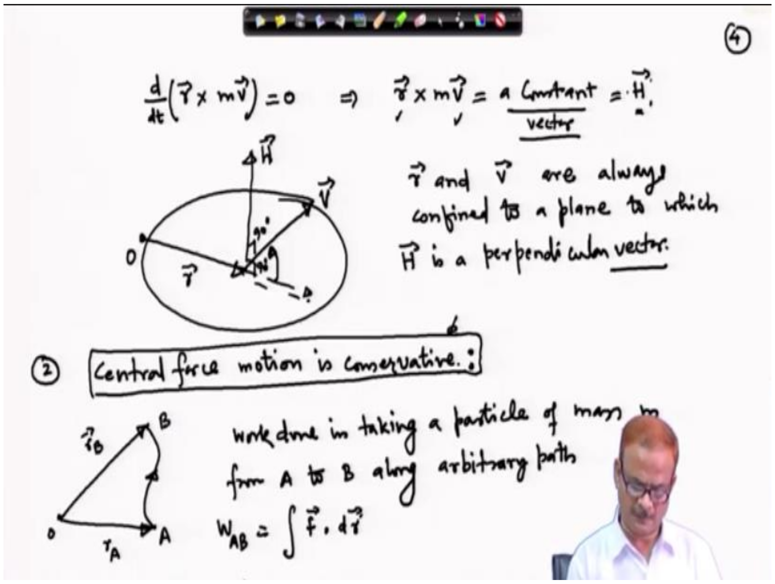

So therefore, from here we get

$$\frac{d}{dt}(\vec{r} \times m\,\vec{v}) = 0$$

this equal to 0 and this implies $\vec{r} \times m\,\vec{v}$ this equal to a constant, this we write as, we will use the notation $\vec{H}$. So this is a constant vector. So what it implies that each vector is always perpendicular to the $\vec{r}$ and $\vec{v}$. So if this is $\vec{r}$ and this is $\vec{v}$, this is the center. So it will be always perpendicular to both of them. That means this is the r direction here.

From $\vec{r}$ to $\vec{v}$, if you take the right hand rule, so it will be directed perpendicular to both of them. This is perpendicular to both of them. This is 90 degree, this is 90 degree and because here this is a constant vector, this is fixed in a space and it is bound to be perpendicular to $\vec{r}$ and $\vec{v}$ all the time, so this implies that $\vec{r}$ and $\vec{v}$ are always confined to a plane to which $\vec{H}$ is a perpendicular vector. So this proves that your central force motion, it takes place in a plane.

Now we have the next property, which is basically a conservation law. Central force motion is conservative. From your physics, you may be knowing that in the central force, if motion is conservative, then the total energy of the system remains conserved or it is a constant. So this is what we are going to work out. So work done in taking a particle of mass m from A to B along arbitrary path. So conservative force, it does not depend on the path. This is path independent.

So work done in taking a particle of mass m from A to B along arbitrary path, we write this as W_{AB} and this we can write as $\vec{f}$. dr. So this is the basic definition for your work done on any particle.

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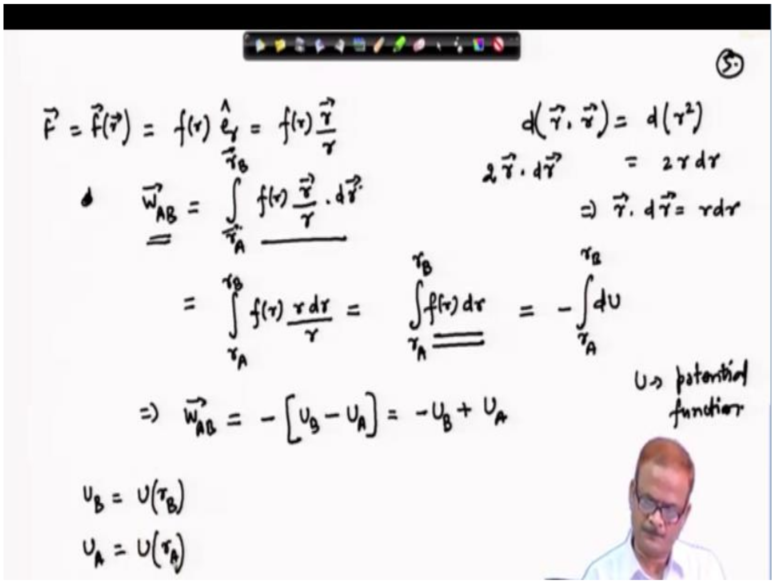

So this is basically a function of r, so as usual we have written, we can write it in this way f® times $\hat{e_r}$ or this can also be written as $\frac{\vec{r}}{r}$. If we use this now, in this equation we insert it $r_A$to r_B and f(r) dr. Now we are going to evaluate this integral, so for that we will look into this particular derivative. We write this, the right hand side will be d(r^2), which you can write as 2 r × dr.

The left hand side, if we take the derivative, this appears as, because it is a dot product, we can write it like this. So this implies $\dot{r}$, sorry here we are not taking derivative with respect to time. So we will write it like this, $\dot{r}$dr. So $\dot{r}$dr then becomes rdr. So using this property, we can simplify here in this place. This is f(r) and $\dot{r}$dr, this becomes rdr/r and because now all the scalars are involved here, so this is just r_A to r_B, so f(r) dr.

You can see that this is completely a function of r. It does not depend on path. As I have shown here, this is a curvilinear path, the particle we are taking from this place to this place. So it does not matter along which path we are going. The work done, it never depends on the shape of the path, the way we are taking from one point to another point. This is the specialty of the central force motion and this quantity is quite often written as, f(r) dr is written as –dU where U is a potential function.

These are from your basic physics. So this implies W_{AB} as $U_B - U_A$. Now for, we do little bit modification in this equation in this one and write in another way to get the kinetic energy expression and thereafter we will see that how good thing emerges. So here U_B implies $U(r_B)$ and similarly U_A $U(r_A)$.

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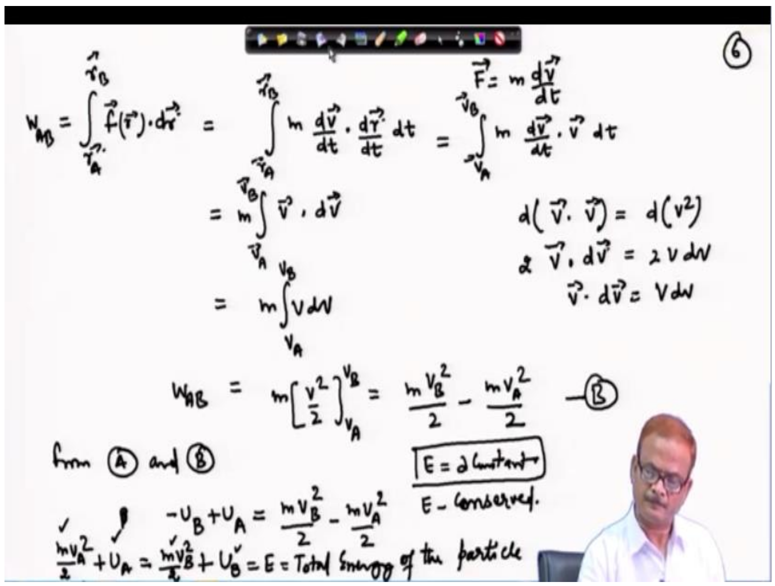

So now we have f(r). Now this, we use Newton's law, so we know that

$$f = m\, d\vec{v}/dt$$

So we use this and we write this as $m\, d\vec{v}/dt$ .dr and we will reformulate it. So if we put here dt in this place, so this gets little simplified in the way that we can write this as $m\, d\vec{v}/dt$ and this can be written as m $\vec{v}$. d$\vec{v}$. So once we are converting here in terms of V_A V_B, so we will write in terms of $V_A$and V_B. So we change the limits of the integration.

So earlier it was r_A to r_B, we are changing the limits of integration. So here this becomes V_A to V_B. So as per our earlier derivation, what we have looked that this quantity is nothing but $dv^2$and this can be written as 2 $\vec{v}$.dv . So $\vec{v}$.dv is equal to $\vec{v}$dv. So we use this to replace here. Now the scalar form comes. So using this equation, let us say this is A and this is B. So from A and B, we are here in this place. On this side, we have

$$-U_B + U_A = \frac{mV_B^2}{2} - \frac{mV_A^2}{2}$$

If we rearrange, this becomes

$$\frac{mV_A^2}{2} + U_A = \frac{mV_B^2}{2} + U_B$$

So this is the kinetic energy at A. This is the potential energy at A. This is the kinetic energy at B, this is the potential energy at B and this is nothing but your total energy of the particle.

Hence, the energy remains constant in going from, in the central force motion, going from one point to another point, if we are going from here to here, irrespective of the path, I can go along this path, I can go along this path, or I can take even this kind of path, so irrespective of the path. The work done is given by the change in the potential energy and as far as the total energy is concerned, so the total energy remains conserved. So we call this as the E, this is a constant. E is conserved.

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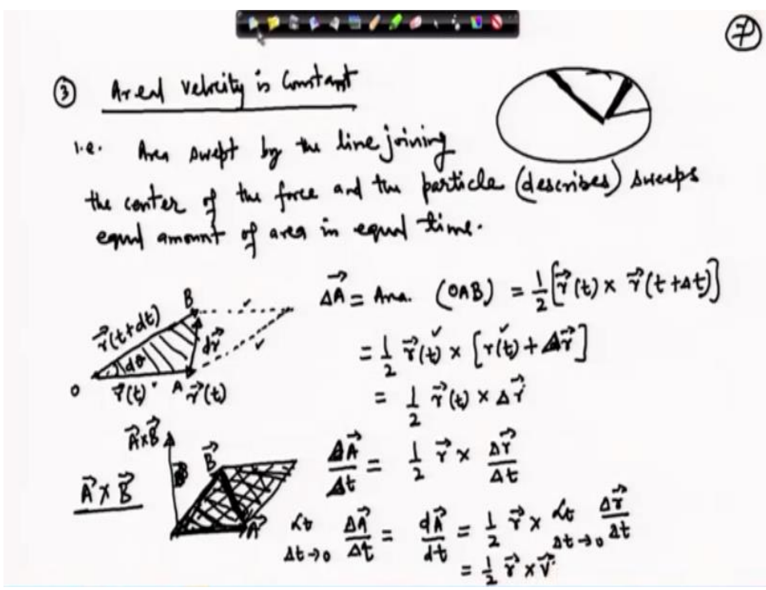

Now, we go to the third one. So the third property of the central force motion is areal velocity is constant. In the Kepler second law, we have written this as the rate of sweep off area or the vector directed, say here if I have an ellipse and if I say that the areal velocity is constant, so in the unit time if it is sweeping this much of area or this area or later on here in this place, this area in unit time, they will be the same. So the areal velocity is constant.

That is area swept by the line joining the center of the force and the particle describes or either sweeps equal to and this derivation is a very straight forward one involving just a vector. So let us say this is the center O, the center of the force. This is your $\vec{r}$at time t, this is $\vec{r}$, maybe a little longer at time $t + dt$. This angle we will write as d theta. So this vector we will indicate as dr.

So if we construct a parallelogram, drawing a line parallel to this, here in this place and drawing parallel to this here in this place. So you can see that our required this area, the area it is sweeping out, which I will hash here. So this area will be just half of the area of the parallelogram. So let us write this as A and B. So area OAB, this will be written as rt cross. Parallelogram area, how do we write it, this whole thing?

If we have two vectors, A× B, so A×B, so this vector is A here and vector B is here in this place. So A ×B, this indicates this whole area and the vector A×B is perpendicular to this. So that implies that the area vector is perpendicular to the plane containing A and B and the hashed area is the magnitude of A×B. Using this then, we can write here $r_t + \Delta t$. This is what we are doing here. This is vector B here and here this vector is A cross B, so half of this.

Half of this means, from here to here, this is half on this side. This is the half one. So using the same principle, we apply here and then we can write. So this is the area vector, because this is an elementary area, so we write this as dA over arrow. So this is r_t cross, now this vector r_t + dt is nothing but this r_t + dr. So r_t + dr and half of this we have to introduce, because this is the area of the parallelogram, so 1/2 we put here.

1/2 this and this, this is the cross product, so this vanishes and we get

$$= \frac{1}{2} r_t \times \Delta\vec{r}$$

So if we divide both sides by dt, here we are writing here in terms of Δt, so let us change it a little bit and write here in terms of ΔA. So this is ΔA. ΔA/Δt, then this becomes 1/2 and here also we will write in terms of Δ, this also in terms of Δ. So $\vec{r} \times \frac{\Delta\vec{r}}{\Delta t}$, here also we make it Δ and in the limit, Δt tends to 0 ΔA/Δt. This we can write as

$$\frac{d\vec{A}}{dt} = \frac{1}{2}\vec{r} \times \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}$$

this is nothing but.

$$= \frac{1}{2}\vec{r} \times \vec{v}$$

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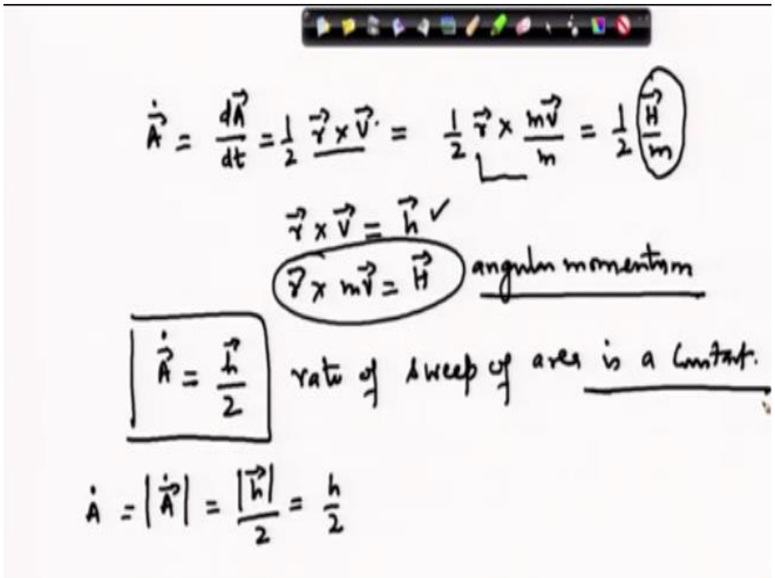

So what we see that dA/dt, this quantity is nothing but $\frac{1}{2}\vec{r} \times \vec{v}$ and from earlier derivation, the first property of the central force motion, we introduce m in the numerator and denominator here. So immediately we can see that this quantity is nothing but H/m. So what the notation we adopt, we will write

$$\vec{r} \times \vec{v} = \vec{h}$$

while

$$\vec{r} \times m\vec{v} = \vec{H}$$

Small h is obtained by dividing H by m.

So you can say this quantity is nothing but the angular momentum per unit mass. So this quantity, this is your angular momentum from your basic 12th physics you are aware of. So what we are observing here that $\dot{A}$, this is coming out to be

$$\dot{A} = \frac{h}{2}$$

So in the magnitude wise, if we write, if we take the magnitude, so the rate of sweep off area is a constant. So these are the three properties of the central force motion. So we stop here and in the next lecture we will continue with another topic. Thank you very much.