Satellite Attitude Dynamics and Control
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Lecture – 33
Gravity Gradient Satellite (Contd.)

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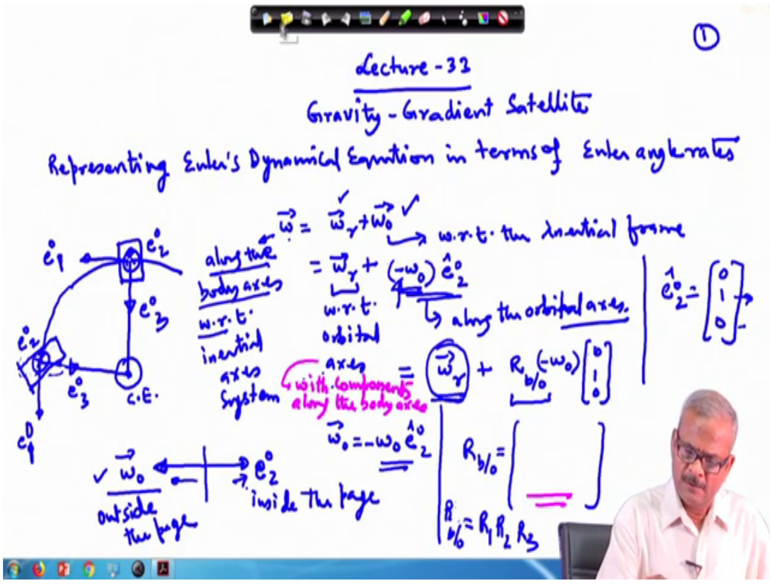

Welcome to the 33rd lecture. So, we have been discussing about the Gravity Gradient Satellite, we will continue with that. So, if you remember that if a satellite is moving in an orbit, and we have indicated the orbital axis as e 0 1, in this direction as e 0 2, e 0 3, and here in this place as e 0 2.

So, with respect to this frame, your satellite is rotating. And this frame itself it is rotating, as I have told you earlier. So, this is the centre of the earth ok. And satellite after some time, it will come here in this place, it may be oriented like this or it may have some different orientation, but the centre of mass is located here. So, let us say this is oriented like this.

So, if anybody axis is attached to this satellite, and here is your e 0, e 1, um, e 0 1, and this is e 0 3, and similarly e 0 2 is here. So, this the orbital axis system itself it has rotated and the body is rotating with respect to that. So, this we have earlier written as omega equal to omega r plus omega 0 ok. And this omega r is with respect to the body with respect to the orbital axis orbital axis. And this omega 0, this is with to the inertial frame. So, this is in a different frame, this is in a different frame.

And here, this we are looking from along the body axis, we look for that this should be along the body axis with respect to the inertial axis system. So, for this we need certain conversion, which we have to do here ok. So, this omega 0, if you remember that omega 0 is perpendicular to this paper and coming out of the page ok, because this is going here in the anti-clockwise direction. So, this way we can write here omega 0 as minus omega 0 times e cap o 2 why, because e cap o 2 is just opposite to the angular velocity of the orbital axis e o 2 is going inside the paper. While, this omega 0 is coming out of the paper and therefore we have put a minus sign here in this place.

So, it is a something like this, your omega 0 is say directed along this direction, which is the orbital angular velocity or the angular velocity of the orbital axis, while your e o 2 is directed like this. So, this is inside the page inside the page, and this is outside the page. So, therefore if omega 0 needs to be expressed in terms of the e o 2, so what we need to do? So, simply we will write this as we will take the magnitude of this, and multiply with e cap o 2 which is the unit vector along the e o 2 direction. And then put a minus sign that completes it that this is the vector along this direction.

So, once we have got it, but still this is along the orbital axis. We need to convert this along the body axis. So, for that we write this as the rotation from the orbital to the body frame times omega 0 with minus sign ok, and this is e o 2 cap I we will put further, right now let us write it like this.

So, what is the fact that this e o 2, this is a unit vector. So, this vector is like this. In the orbital frame, this vector appears like this. Only this component is existing this is 0, this is 0, the first and the third component. So, instead of writing this we can equally write this part ok, so 0 1 0, and then do the conversion.

This part we have to write, and this part we have already worked out R b o. If you remember this is the conversion, we have written a big matrix by giving rotation about first about the third axis R 3, then R 2 and R 1, so this is your R b o. So, here this is what it enters. And this part is our angular velocity of the satellite with respect to the orbital reference frame this one ok, this is with respect to the already we have written orbital reference frame with components along the body axis ok. Because, finally we have need the components along the body axis. So, with components here we can write in some different color with components along the body axis, this is this part ok.

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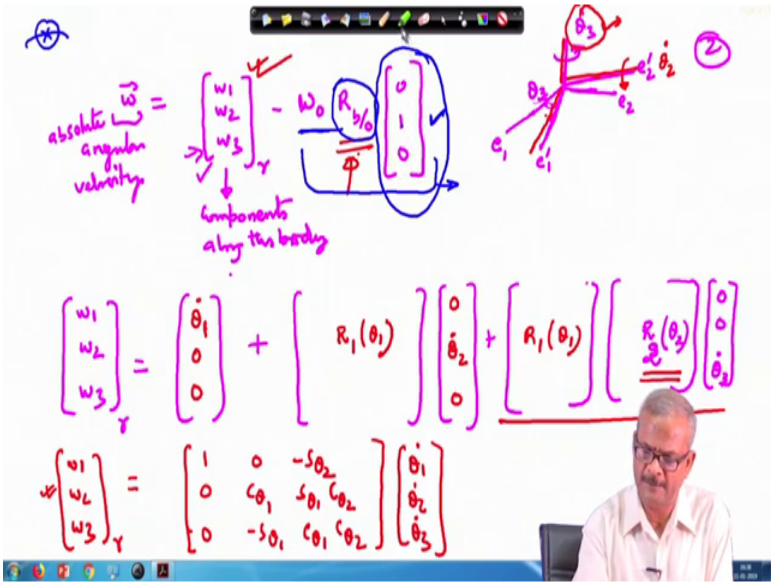

So, we need to complete this, therefore our omega the absolute angular velocity absolute so this is omega 1, omega 2, omega 3 r, but still this is in terms of the these are components along the body axis, but we need to convert them along in terms of the Euler angles. So, we need a proper change here also, and this we have already done, I will come back to that again. So, here there is omega 0, and then you have R b o, and times 0 1 0.

So, let us take first this part omega 1, omega 2, omega 3 this r ok. So, we need the conversion of this. So, here we have the first rotation is about the third axis. So, you have theta 3 dot here in this place, this part was 0 0. And this is your rotation about the third axis to convert this into the body frame; we need the rotation by theta 2. Similarly, the second rotation sorry this is about the first rotation; we have given about the third access.

Like here once we are rotating, so this is your theta 3 dot. But, if you see that once it has rotated, and from this frame, then again this is your theta 3 here, this angle this angle is theta 3. So, this angle then or either this remove it ok. So, this angle is theta 3. So, what we are doing, we are in this frame. So, if we have I do not remember the notation, I have used earlier. But, let us say this is e 1, then we have represent we may represent it by e 1 prime. This is e 2, then we can represent it by e 2 prime.

So, this theta 3 dot, it can be assumed to be in the frame which I am showing by orange line ok. So, all these are perpendicular to each other. So, from there finally, we have to go to the body axis system. So, how many rotations are required only two rotation. One rotation about the second axis, because the next rotation I am giving about the second axis. And the last rotation, I will give about the first axis. So, one more rotation here which is theta 1.

Similarly, here if we look further, so this is your theta 3 dot which is get converted into along the body axis. Then we have the theta 2 dot, so theta 2 dot will lie along this direction ok. And this we need to convert, so this I have done earlier you refer back to the earlier lecture theta 2 dot will lie here ok. And only one rotation is enough at that time this is R 1 theta 1. And here finally, you are giving the third rotation about the first axis, which is the body axis first body axis itself. So, there we have written this as the theta 1 dot this part 0 0. And from there we have got the matrix as 1 0 minus s theta 2, then 0 c theta 1, this part we have done in great details. It takes a lot of time to reproduce the same thing.

So, I am just avoiding few details, if you require you just refer back to the earlier lectures, this theta 1 dot, theta 2 dot, theta 3 dot. And here we have omega 1, omega 2, omega 3. This is omega 1, omega 2, omega 3 r. So, this is how your Euler angular rates are connected with the body rates ok. So, this part is done here in this place, what is remaining here this part.

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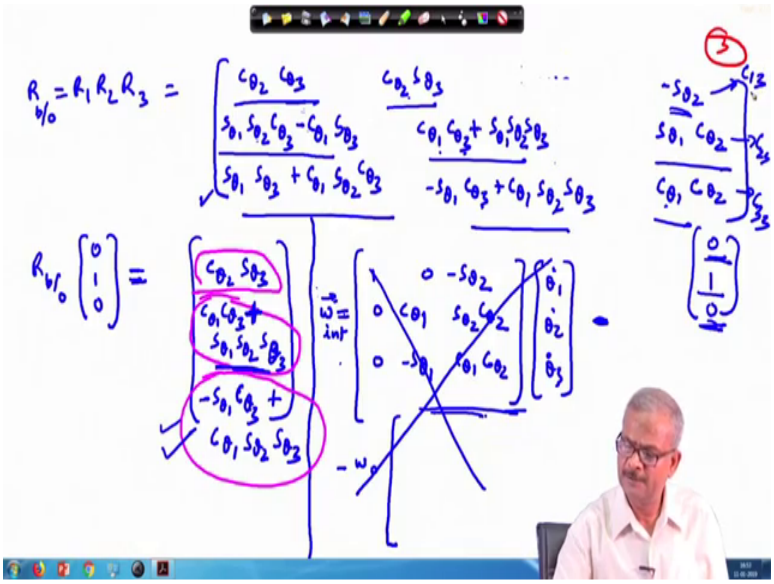

And for the third part, you remember that this rotation R 3 R 2 R 1, this equal to R b to o, this we have written as c theta 2 c theta 3 s theta 3, then minus s theta 2 ok. Refer back to the earlier lectures s theta 1 writing in a little smaller font, so that we can adjust here. And say s theta 3 so ok, then this is minus s theta 1 c theta 3, and the last column is minus s theta 2, and s theta 1 c theta 2. Let us verify it again this c theta 2 c theta 3 this is s theta 1 s theta 2 c theta 3 s theta 3 c theta 1 s theta 2 c theta 3 s theta 3 s theta 1 s theta 2 s theta (Refer Time: 15:07) minus.

So, this matrix will convert from the going back to this part ok. So, this is this vector is along the this particular vector, it is along the orbital axis which direction this is the second direction means, it is a going into the paper as in the original picture, I have shown this going into the paper ok. So, this will get converted along the body axis. So, what will be the component of that vector along the body axis, and then you multiply by this omega 0, so using this transformation.

So, now we have to pick up this R b 0, and operate on the 0 1 0 this vector. So, if I operate on the vector here 0 1 0, so what we are going to get? See if you check on this, so here on left hand side, we can write it like this, and there 0 1 0, this will then become equal to see the first one is 0 multiplied here. If you multiply this with this, so this will be 0, this will stand here, so which is c theta 2 times s theta 3 ok. And the last one this will be again this part, and this part they will be 0.

So, this way what we see that if we proceed like this, so again this one multiplied by this, this one multiplied with this, and this one multiplied by this ok. If you do this, so only this will stand which is again we will write in two lines c theta 3 plus s theta 1 s theta 2 s theta 3 s theta 3. Similarly, we keep this last line ok, so this will be giving you minus s theta 1, then c theta 3 plus c theta 1 s theta 2 times s theta 3 ok. And together with this, then you have this to be multiplied by this minus omega 0 ok. So, we go on the next page.

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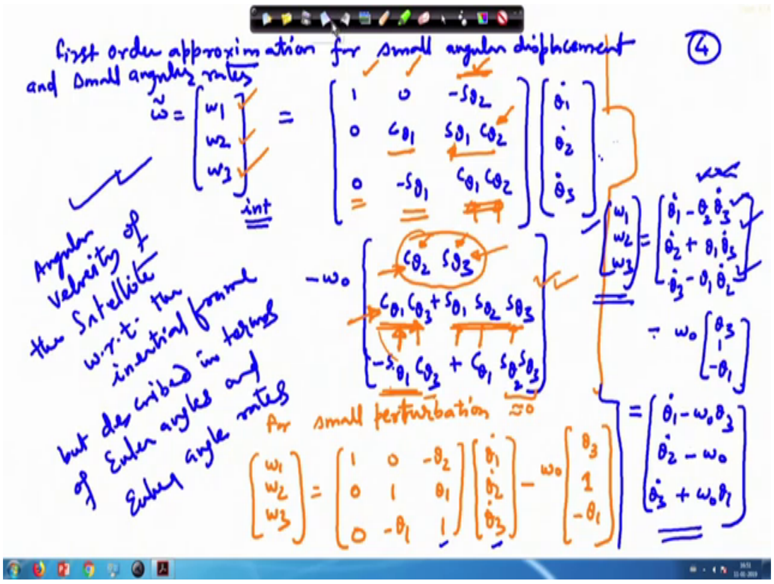

So, our omega then the omega tilde which is nothing but equal to omega 1, omega 2, omega 3, this is with respect to the inertial frame inertial frame I can write this as int. So, this becomes equal to the first part which is your here ok, this part we have to copy in that place minus s theta 2 0 c theta 1, then s theta 2 times c theta 2 theta 1 dot theta 2 dot theta 3 dot ok.

And then the other term, which is I will write the other term here in this place minus omega 0 times pick up this term ok. Here we have to write it separately, this equal to this we have already converted here in this place. So, this line is separate here ok. So, we are now writing here ok, this is omega inertial. I do not know this page perhaps, we have written back here in this place.

But, what was required that we pick up this part omega inertial this part, and minus omega 0, which is this part we have to write together. So, maybe we will write on the next page, this I will cancel it here. And write in the next page 1 0 minus s theta 2 0 c theta 1 s theta 1 c theta 2 s theta 1 dot theta 2 dot theta 3 dot, and then minus omega 0 times copy the other terms, which we have got here this terms.

These are the separate terms, this is one term here, this is another term here ok, and this is another term here, this plus sign is here, so this plus sign. So, we need to copy all these terms we are in this place. So, the first term is c theta 2 times s theta 3 c theta 2 times s theta 3, then the second term c theta 1 c theta 3 plus s theta 1 s theta 2 times s theta 3 c theta 1 c theta 3, and s theta 1 s theta 2 s theta 3, s theta 1 s theta 2 this part ok.

Similarly, picking up the other part this part. So, here we have minus s theta 1 c theta 3 plus c theta 1 s theta 2 s theta 3. So, minus s theta 1 c theta 3 c theta 1 s theta 2 s theta 3 ok. So, this gives you the angular velocity of the satellite of the satellite with respect to the inertial frame, but described in terms of Euler angles and Euler angle rates ok. If we have for a small [perturben/perturbation] perturbation for a small perturbation, we can approximate this angles. So, this angles can be approximated and then we can get this omega 1, omega 2, omega 3 in terms of the Euler angles, and Euler angle rates, once this angles involved are small. Therefore, this omega 1 omega 2 omega 3, this can be reduced to the first 1 remains 1, thus the here this 0 remains 0, and for a small angle this sign theta 2, this will get reduce to minus theta 2 ok.

Similarly, the others then this is 0, the c theta 1 gets reduce to here this part to 1 ok. And this part here sign theta s theta 1, this will get reduced to theta 1, and c theta 2 will be 1, so this becomes theta 1. Here this remains this part remains 0, and this part is minus theta 1. And here this is one and one both of them c theta 1 c theta 2, so this gets reduced to 1 ok. And this is multiplied by theta 1 dot theta 2 dot theta 3 dot. And then we have to subtract this omega 0 times something, now here this quantity. This c theta 2 times s theta 3 to a small angle approximation c theta 2 will be 1 c theta 2 will be 1, and this quantity this will be equal to theta 3. So, we write here simply as theta 3.

Similarly, this part this will get reduced to 1. And while this part will be 0, because here this is of first order, first order, first order. If we approximated, so three terms of first order multiplied together that becomes of third order. So, we can neglect as compare to the this one one, we are approximating c theta 1 is one, this has one, so as compared to this will be equal to this will be small. So, there we are writing this as 1.

And similarly, here s theta 2 s theta 3, this will be approximately equal to 0. And only thing that will remain here in this place c theta 3 will be equal to 1, so this gets reduced to minus s theta minus theta 1. So, writing on this part, then see here this minus sign is along with this or either we remove this minus sign here from this place and put here in this place this is with minus sign. So, this is continuation of this term and this term, they are adding together with this minus sign here.

So, finally we can wind it up here in this place, by writing omega 1 omega 2 omega 3, this equal to adding them. So, this is theta 1 dot minus theta 2 times theta 3 dot, this is the first term here. Then the second term will be theta 2 dot plus theta 1 times theta 3 dot. And the third term will be theta 3 dot this part, and then minus theta 1 times theta 2 dot ok. And then this part we have to add it up here, so this is omega 0 times theta 3 1 and minus theta 1.

So, this can be finally written as if we do further approximation to this, so the second order terms here like the theta 2 times theta 3 dot, these are a small angles and small angler rates. So, this type of terms can be neglected. So, all this terms this term, this term, and this term, all this three can be ignored. And therefore, then this can be written as minus omega theta 3, and theta 2 plus omega 0, and theta 3 minus the first term is minus here.

In the second term will also remain as a minus, here this is with a minus sign, this is with minus sign and here it comes with a plus sign, so that comes with a plus sign omega 0 times theta 1. So, omega 1 omega 2 omega 3 that gets reduced to this quantity and so what we are doing that by doing so we have done the first order approximation for a small angular displacement, and the small angular rates. So, this way omega 1 omega 2 omega 3 we have got.

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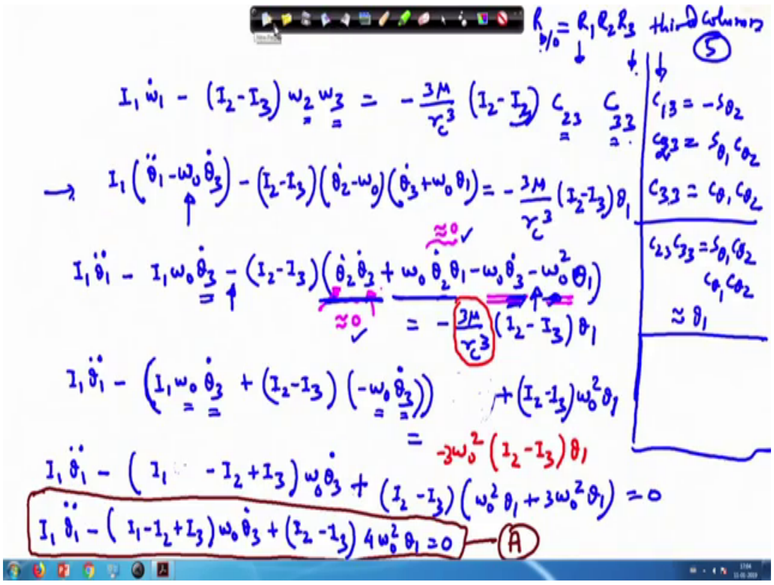

Now, we can write the Euler's dynamical equation. So, I 1 times omega 1 dot minus I 2 I t 3 I 3 times omega 2 times omega 3, this equal to minus on the right hand side we have the torque equation, because the torque term because of the gravity gradient term, which is minus 3 mu by r c whole cube ok, this we have done earlier times I 2 minus I 3.

And then the direction cosine terms, which appear from the rotation matrix c 2 3 and c 3 3. The usual notation is whatever the 2 and 3 appearing here this 2 and 3 we are writing, and there after we are just putting this 3 3. And this is coming as explained earlier, this is coming because of the gravitational force is acting along the third or vital access because of that it has appeared here.

So, we need to insert this omega 1 omega 2, so omega 1 dot therefore we can pick up from the previous place. And before that what we will do, we will also write for c 2 c 3. So, c 2 3 and c 3 3; going back c 2 3 and c 3 3. So, this is the term c 2 3, and this is the term c 3 3, and this is the term c 1 3. So, we need to pick up this and put, there in that place. So, c 2 3 so inserting all these values here. So, the first one is minus s theta 2 the c 3 3 is s theta 1 times c theta 2 and sorry this one we have written as s 1 3. So, this is c 1 3 first we will complete this part c 1 3, then this is c 2 3, and c 3 3 the last part of the your R matrix that matrix we have written as R 1, R 2, R 3 from or vital to the body frame ok. So, this is a third column. We are picking up the third column this is the third column, we are picking third column. And the last one is c theta 1 times c theta 2.

So, if we do this c 2 3 times c 3 3, this will be equal to multiplying this two terms s theta 1 c theta 2 times c theta 1 c theta 2. And then do the approximation, so s theta 1 will be equal to theta 1, and c theta 2 will be equal to 1, c theta 2 here 1 and c theta 1, this also equal to 1. So, this becomes equal to 1. Similarly, you have the other terms which will appear in this place, so we can insert one by one. So, the next let us complete that, we will write here first and then we will take up the other terms.

So, here I 1 times omega 1, omega 1 now we have to pick up from this place, this is our omega 1. This is the inertial angular velocity along the. Now, we have converted the angular velocity omega 1 this is writing for omega 1 ok. This is your omega 1 ok, this part. But now it is written in terms of the Euler angle rates, so theta 1 dot minus omega 0 theta 3, theta 1 dot minus omega 0 theta 3. And we have to take the derivative with respect to time. So, this will be dot.

And because the satellite is moving in circular orbit, so omega 0 this remains constant only thing we need to take a derivative of this. So, I 2 minus I 3 times omega 2, omega 2 as and we go back and look here in this place this is your omega 2 theta 2 dot minus omega 0. So, this is theta 2 dot minus omega 0. And omega 3 dot with this quantity theta 3 dot plus omega 0 time theta 1. So, theta 3 dot plus omega 0 time theta 1. So, this is the part here ok. So, this way we need to develop the all the terms ok.

On the right hand side then finally we have 3 mu by r c whole cube, this is I 2 minus I 3, I 2 minus I 3 and multiplied by this theta 1 ok. So, this is the equation we are getting, we next need to work it out further. We have to basically reduce it, theta 1 dot minus I 1 times omega 0 time theta 3 dot. This part we expand, so this is theta 2 dot theta 3 dot then plus omega 0 times theta 2 dot times theta 1 3 dot minus omega 0 times omega 0 is square this becomes theta 1. This equal to minus 3 mu by r c whole cube I 2 minus I 3 times theta 1.

What we see that the terms which appear this is second order term, this gets reduced to 0. Similarly this term here, this is a second order term, so this will also be equal to 0, this term remains. So, what we are doing that the terms which are of second order in terms of Euler angles that is in terms of theta 2, theta 3 as such or its rate. So, those terms we are ignoring, sorry, these two terms after ignoring them we are left with this time and these terms. So, here we can write this as ok. So, these two terms get deleted. This particular part and this part, it will be getting deleted, because these are almost zeroes and the other terms which are remaining are here in this place.

So, we pick up this and here if you look this is theta 3 dot and here also the theta 3 dot terms is there. So, we can combine them together. So, if we write it like this, and this minus sign we keep it outside. So, this is theta 3 dot. And from here this place this becomes plus i 2 minus i 3 times minus omega 0 theta 3 dot minus omega 0 a square theta 1 then this term will be remaining this part, this part will be remaining we have to write that part also. So, this is minus that gets this minus, and this minus sign that gets reduces to plus I 2 minus I 3 times omega 0 square theta 1.

And on the right hand side, the quantity which is present here this part, this we will write as omega 0 square. Why we are writing like the omega 0 square I will come to that later on, this is I 2 minus I 3 times theta 1 and this 3 part is missing. So, we put the 3 part here. This is the bracket is closed here. So, this becomes I 1 times theta 1 double dot my theta 3 dot ok. Once we take this part here if we write it separately, so we have to eliminate from this place. So, rather than keeping it here we have kept it outside. So, this part we are removing otherwise the same term written twice.

And if we combine them together, what happens then, so I 1 omega 0 minus I 2 plus I 3, and omega 0, omega 0 we will take it outside the bracket, so that it simplifies in just in one step. Omega 0 is common here; omega 0 is common here. Theta 3 dot, theta 3 dot is common, theta 3 dot, this is omega 0. And this will be equal to finally we have this term here.

So, I 2 minus I 3, we take it outside here, this is the plus sign this particular term we are picking up this will bring from this side to this side. So, omega 0 a square theta 1 plus 3 omega 0 a square theta 1 this will be equal to 0. So, the finally, we get term like this I 1 minus I 2 plus I 3 times omega 0 times theta 3 dot plus I 2 minus I 3 which we have already written here omega 0 square theta 1 3 plus this. So, this becomes 4 omega 0 square theta 1 equal to 0. Now, let us name this as A. So, this is your equation A. The same way, if we go in the go for the second Euler's equation, so there we will find terms like this ok.

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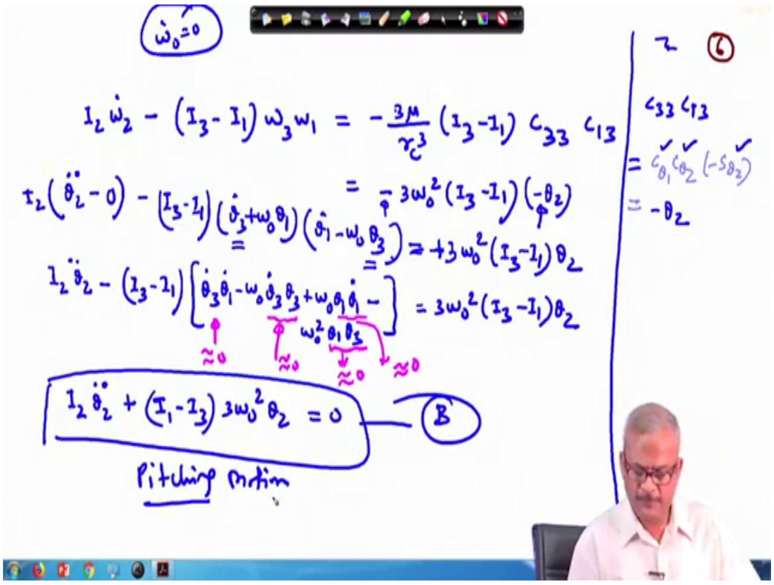

Let us go in the next page and then workout. So, we have I 2 time omega 2 dot minus I 3 minus I 1 times this is the earlier I have told how to write this equation. You do not have to remember in this one this is just a in a circular order. So, we can write it that way only omega 3 omega 1 minus 3 mu by r c cube times I 3 minus I 1 and c 3 c 1 and then write 3 3.

And this part we are writing as 3 omega 0 square I 3 minus I 1 and c 3 3 times c 1 3. How much this will be c 3 3 is c theta 1 c theta 2. Therefore, this is c theta 1 c theta 2, and then c 1 3 is minus s theta 2. So, this is minus s theta 2. So, if we do the first order approximation, this will be equal to 1, this will be equal to 1, and this is minus theta 2 for that gets reduced to minus theta 2. So, just put here minus theta 2.

On the left hand side, we have to insert omega 2. So, omega time then take the derivative of this. Omega 2, now coming to this place this is your 2 this part the theta 2 dot minus omega 0. So, theta 2 dot means the derivative of that will be theta double dot minus omega 0, so that will be simply 0. Omega 0 dot this equal to 0. So, we just put that. So, this is I 3 minus I 1 times omega 3 and omega 1. So, omega 3 again we have to go back and pick up from this place, theta 3 dot omega 0 omega 1; theta 3 dot plus omega 0 theta 1; theta 3 dot omega 0 plus theta 1. And we required then omega 1. So, omega 1 term is this 1 theta 1 dot minus omega 0 theta 3, theta 1 dot minus omega 0 theta 3.

On the right hand, this term as we have to rewrite theta 2. And this minus sign, this minus sign that makes it plus. So, therefore, we put it plus sign here in this place. Now, expand this particular part. If we expand it, so what we see that this is theta 3 dot theta 1 dot then minus omega 0 times theta 3 dot times theta 3, theta 3 dot times theta 3, then plus omega 0 time theta 1 time theta 1 dot, and finally, this minus omega 0 square theta 1 times theta 3.

So, what we observe from this place that this term is 0 being second order, this is also because of this, this is 0. This part also is approximately 0; this is being of the second order all these are of second order. So, this is also 0. So, as a whole this term gets eliminated. On the right hand side, then we are left with 3 omega 0 a square I 3 minus I 1 times theta 2. So, this can be written as I 2 time theta double dot plus I 1 minus I 3, we are taking this on this the left hand side 3 omega 0 square theta 2 and this equal to 0. So, this is your equation number B, which is the equation of the pitching motion. We will discuss about all this things ok.

So, we will stop here. And then, the third equation, we will discuss in the next lecture.