Space Flight Mechanics

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Model No. 01

Lecture No. 05

Two Body Problem

Before starting with the two body problem, which is a part of the central force motion, so, let us look into the, how the central force motion or the two body problem evolved. So, John Kepler who was a student of Tycho Brahe, enunciated three laws, of which, first two appeared in Astronomia Nova in 1609. Later, he published Harmonics Mundi, Harmonics Mundi Libri 5, means Harmony of the World, and the third law appeared here, along with other 13 theorems. So, if, and these three laws are popularly known as Kepler's law.

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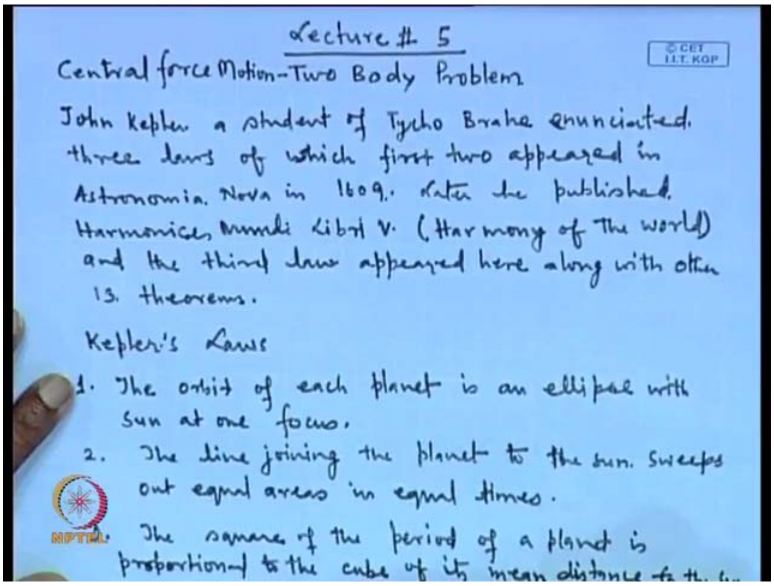

So, Kepler's, Kepler's laws, which are three in number, can be written as, the first one, the orbit of each planet is an ellipse, with sun at one focus. The line joining the planet to the sun, to the sun, sweeps out equal areasin equal times. The third law it states that, the square of the period of a planet is proportional to the cube of its mean distance to the sun.

So, Kepler's law, it captured the kinematics of the motion. So, Kepler's law, it captured, captured the kinematics of motion of heavenly bodies, which were inferred from observations made over a long period of time. But the dynamics of the motion remain unresolved, but the dynamics of the motions remained unresolved, until Issac Newton unlocked them. So, Newton enunciated three principles, or the three laws, which are popularly known as the Newton'sthree laws and he enunciated this laws in his book Principia. So, Newton enunciated three principles in 1687, in his book Principia.

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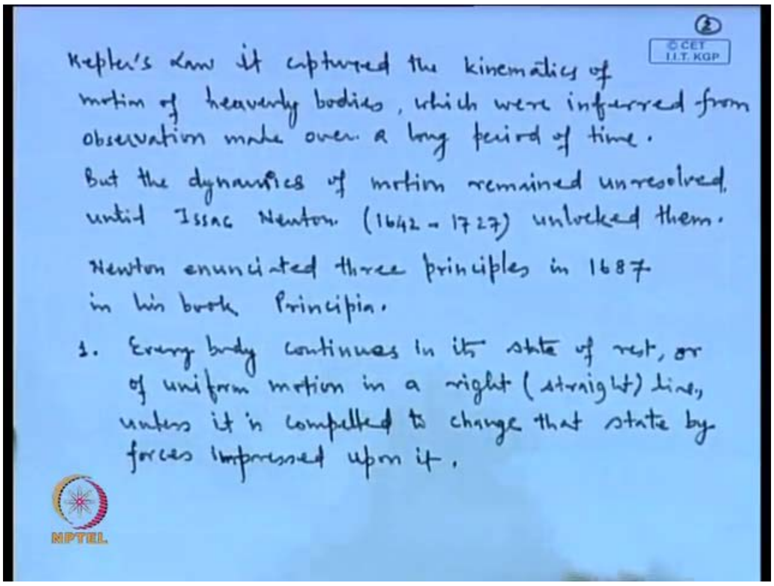

So, this three laws, so, out of which, the first law, it states that, everybody continues in a state law, in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state, by forces impressed upon it. So, we will discuss further, about implication of these laws. So, the first law is, everybody continues in its state of rest, or of uniform motion in a right, or the straight line, unless it is compelled to change that state, by forces impressed upon it. The second law it states that, the change of motion, the change of motion is proportional to the motive force impressed, and is made in direction of the straight line, in which the force is impressed. The third law it states that, to every action, to every action, there is always an opposite, opposite and equal reaction. For alternately, we may state that, or the mutual actions of two bodies upon each other, are always equal and directed to contrary parts. Now, the Newton's second law, you will see that, this leads to the first law; but the question arises, was Newton wrong, or he did some mistake in enunciating the first law, if the first law can be derived from the second law,then, what is the need of the second law. So, let us look into the second law; we need some mathematical formulation.

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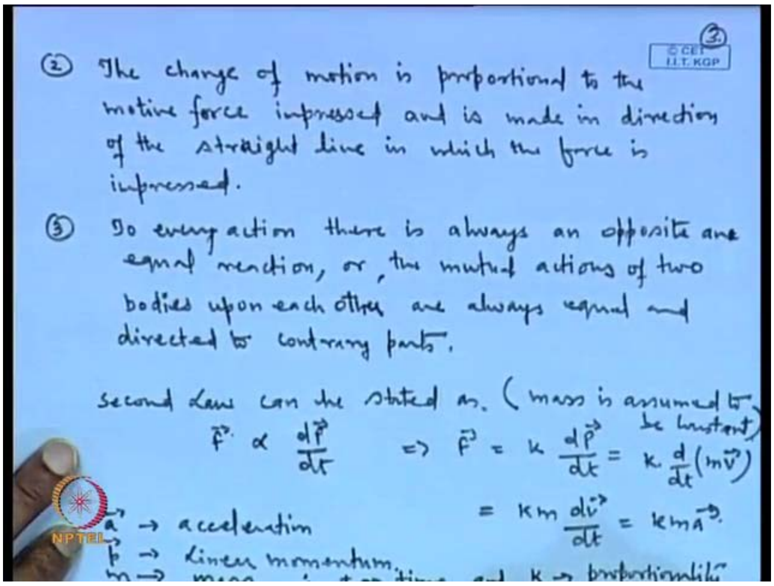

So, second law can be stated as, second law can be stated as the force impressed, the rate of change of momentumwill be proportional to the force (( )). Another way we can write, F is equal to K times d P by d t. So, here, mass is assumed to be constant. So, for the second law, mass is assumed to be constant. So, this can be written as K times d by d t m v. K m times a, where a is acceleration, p is linear momentum, m is mass and t is time, and K is proportionality constant. In S I system, for 1 Newton of force imposed, and for a unit mass, unit mass, that is 1 kg of mass, if the resultant acceleration is taken as a is equal to 1 meter per second square, then, this implies K is equal to 1. So, the second law can be written in this format, F is equal to m a. Now, suppose, we put a equal to 0, so, this implies, F equal to 0. In other words, if a particle is moving along x coordinate axis, coordinate axis, then, we can write m times, but d square x by d t square, F is equal to m times d square x by d t square, this is equal to 0. So, in turn, this implies that, d square x by d t square, this is equal to 0, which implies, d x by d t, this is a constant, a constant.

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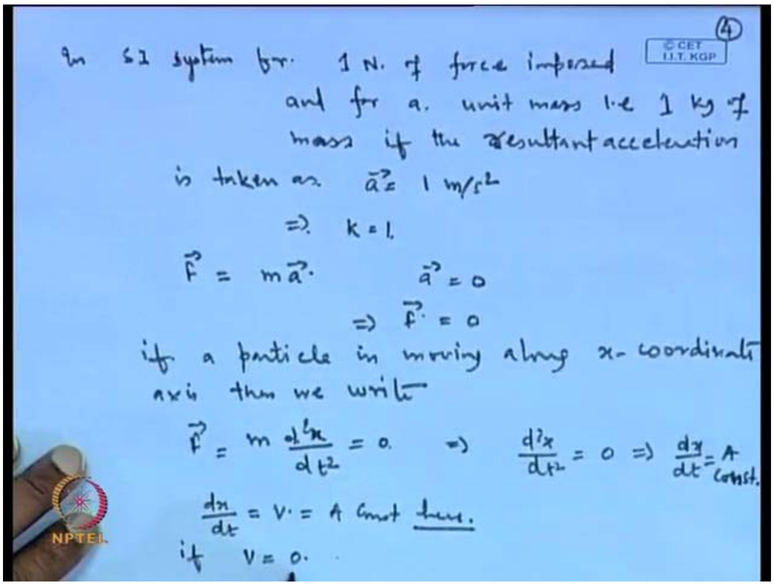

So, this d x by d t, we can write it as v; this is a constant here. So, if this constant is 0, if the constant v is equal to 0, then, what we see that, a particle which is moving, which is initially at rest, will remain at rest, until unless a force is applied; because here, the F, we have set to 0, means no force is applied. And therefore, if the particle is initially at rest, so, it will continue in the state of rest. Now, if v is not equal to 0 initially, so, this implies that, particle will keep moving with the initial velocity, until unless an external force is applied. So, in another way, we can say that, from the Newton's law we have obtained the first law, which states that, if a particle is initially at rest, or is moving with constant velocity, so, it will continue in that state, until unless an external force is applied. Then, as earlier I mentioned, then the question arises, whether Newton did some mistake, or it, this first law was not required at all, because it can be derived from the second law itself; then, the what is the need of the first law. The reason is, the second law, it gives only the mathematical aspect that, if the external force is applied, so, how the change will proceed.

It does not say anything, why this, until unless we apply an external force, why the change in state is not taking place; it does not talk about that. So, the first law talks about that, and therefore, it is a most often, it is a, it is called as the law of inertia. So, this is the reason the first law has been enunciated. Without first law, Newton's, the, Newton's fundamental postulates about the mechanics would have been incomplete. For without inertia, the, we cannot describe, why a system is behaving, or it is resisting to the motion, to the externally applied force; or in the absence of the externally applied force, it is not changing its state. So, it is because of this inertia, this can be, we can give an explanation to this.

Suppose, we have two objects; one is heavy and another one light, and both are resting on the floor, on a friction-less floor, and then, we try to move both the objects. So, obviously, once we try to move the, both the objects, we feel, the, we have to apply more force to the heavier object, than to the lighter object. So, the reason is, why we feel, why we have to apply more force to the heavier object, it is because of the inertia. So, inertia is a property of mass. This is not the mass itself. Because of the mass, this inertia exists, which tends to resist the motion of the object, or the motion, which, which results from the external applied force. So, let us write a little bit about this. So, two objects, one light and other one heavy; light one requiresless force, while the heavy one requires large force. So, this is because, due to inertia, inertia, so, this we are discussing about an object on friction-less surface; so, because of, this is because of inertia, and due to inertia, both objects resist motion; originally, both objects are at rest; hence, having no linear momentum; thus, the resistanceis totally due to inertia. So, the first law, we will call as the law of inertia.

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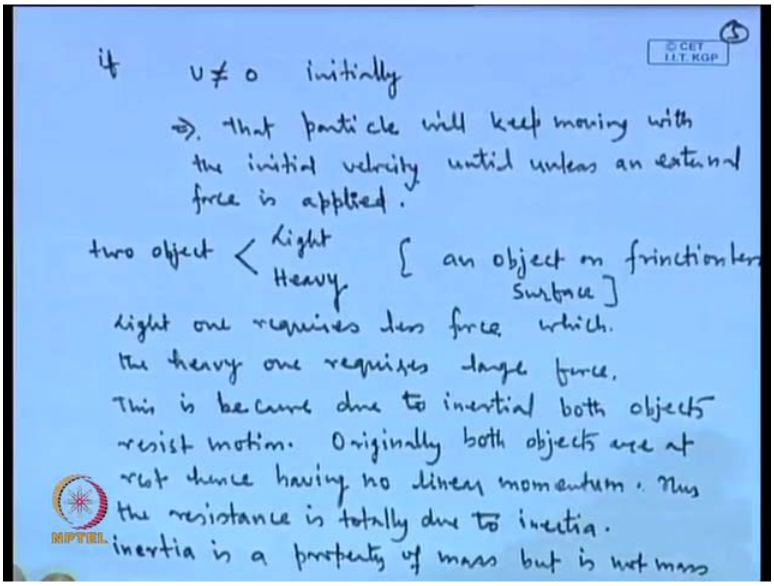

So, inertia is a property of mass, but it is not mass itself. So, starting with this preliminary ideas about the central force motion, and Newton’s law, then, we start about the, we start with those mathematical formulation of the central force motion. Before going into the two body problem, let us consider that, there is a center of attraction and a particle is moving under the field of that attraction. So, we have here, the central force motion. So, what are the characteristics of central force motion? We try to derive them mathematically here, in this place. So, the first characteristic, and we have stated as, the central force motionoccurs in a plane. Let C be the center of attraction and some particle of mass m is moving under this attraction. So, r is the radius vector to the particle m and F is the force acting on the particle m. Then, the moment acting on the particle can be written as, M is equal to r cross F. We can see from this figure that, F and r are collinear; that is angle between F and r is 180 degree. Therefore, the cross product will vanish. This we can write as, magnitude of M, we can write as, magnitude of r times magnitude of F and sin theta.

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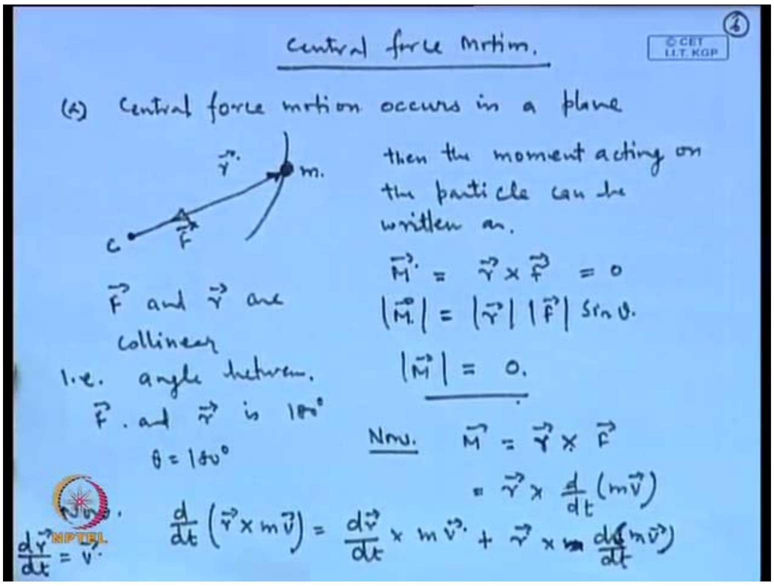

So, here, theta is the angle between r and F. So, theta in this case is 180 degree; therefore, the magnitude of the moment, it becomes 0. Now, M can be also reduced to a different format. So, we can write this as, d by d t…Now, d by d t r cross m v can be written as, d r by d t cross m v plus r cross m d v by d t; or, we can write m inside the bracket itself. Now, d r by d t here, this is nothing, but v. So, this is the cross product, v cross m cross v, m is a constant; therefore, this quantity becomes equal to 0. Therefore, we can write M is equal to r cross d by d t m v, is equal tod by d t r cross m v. Now, r cross m v is nothing, but the angular momentum, and this, we can write as, say H.

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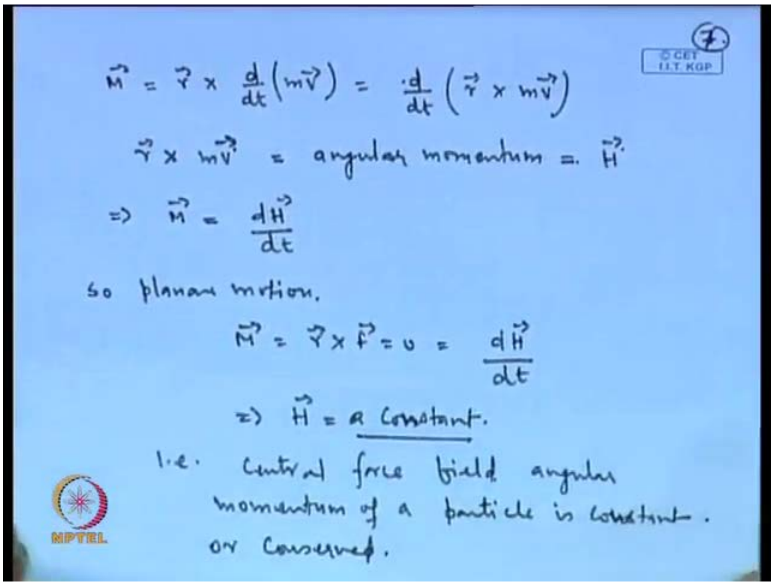

Therefore, M can be written as d H by d t. So, for planar motion, we can write M is equal to r cross F equal to 0, is equal to d H by d t. So, this implies,H is equal to a constant. So, that is,in central force field, angular momentum of a particle is constant or conserved. So, this is the first property that, the angular momentum will remain conserved. And moreover, because we have seen earlier, in the last slide, this is M equals to r cross F. So, here, this is coming to be a 0 and ultimately, the H vector, we are getting this as a constant, means, the particle is moving in a plane; if the angular momentum vector is changing, then, the motion of the plane it will continuously change. If angular vector, if the angular momentum vector is a constant, means, angular momentum will, as it is defined here, this is r cross m v. So, here, the angular momentum vector will be perpendicular to the both r vector and the velocity vector; both to the radius vector and to the velocity vector. Therefore, and this turns out to be a constant; therefore, the r and v vector, they will always lie in a plane. So, the second characteristic of the central force motion is that, the energy is conserved; the total energy is conserved. In central force motion, or the central force motion, you can write, central force motion is conservative. So, total energy remains constant in central force motion. So, let us consider the center of attraction C and a particle, which starts from point A and moves to point B, along certain path. So, the radius vector to A can be written as r A and the radius vector to B can be written as r B. So, work done, work done in taking the particle from A to B can be written as W AB equal to r A r B in the limit F r…So, force is a vector and the work is defined as the dot product of the force and the displacement. Now, we can write F r is equal to…

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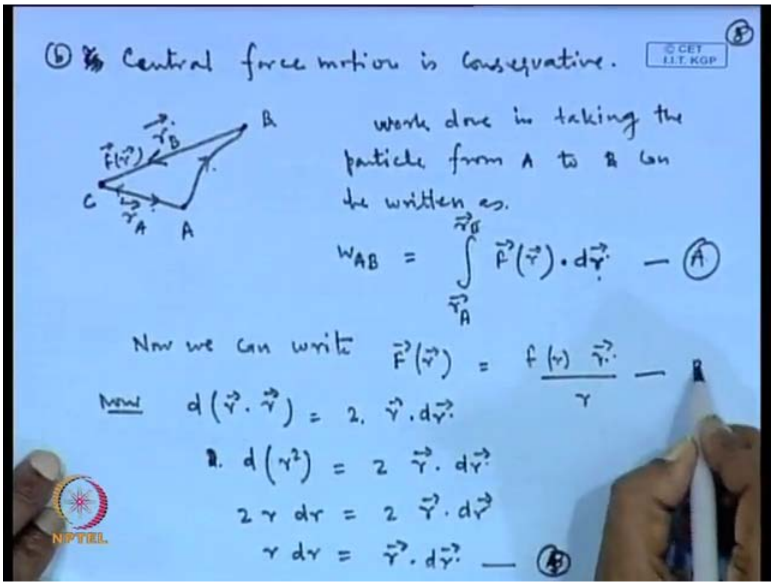

So, the force is acting, because it is a central force motion, the force is directed towards the center. So, this is the F r. Now, we can write d r dot r to r dot d r. So, differentiating this,this results in to…So, the left hand side, we can write as, d r square is equals to 2 r dot d r; or, other way we can write, 2 r time d r is equal to 2 r dot d r. So, this gives us r d r is equal to r dot d r. Now, we can use this and put it into the equation 1, in equation A, and let us say, this is equation B and term this as equation C. Now, using equations A, B and C, write WAB is equal to…Now, this can be written further as,F r by r times r dot d r. So, using C, using equation C, we can write this as r A r BF r by r times r d r.

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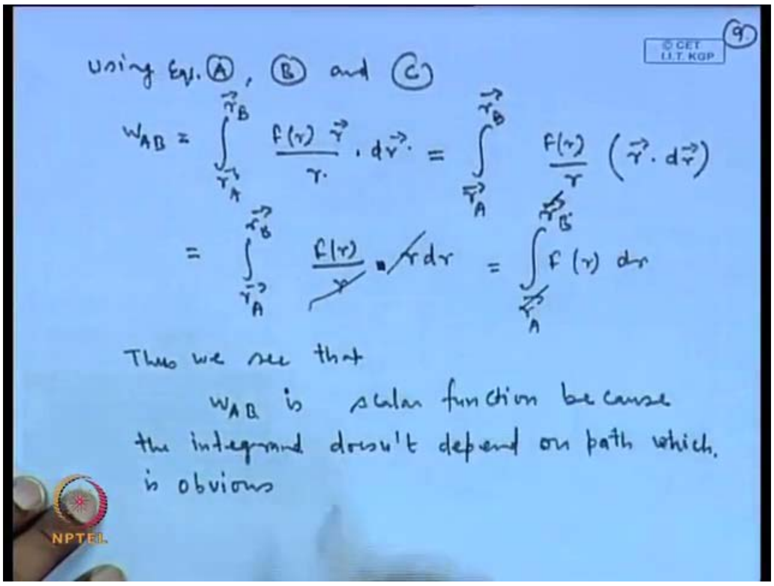

So, this two cancel out and this gives F r d r. Thus, we see that,WAB is a scalar function, because the integrand does not depend on path, which is obvious here. So, we can drop thevector notice on here and just keep, that it ismoving for some point A to point B. So, the integrand being the scalar function, we can define W AB as r A to r B and we can define a function, what is called the potential function, which we can write in this way; that is d u r, minus d u r is equal to F r d r.

So, this implies, d u by d r is equal to minus F r, because we have seen that, the integrand, it is a just a function of r. So, it is a scalar function. So, once we integrate it... So, the, after the, after the integration, the function that we get, that will be just, that will just depend on the value of r. Therefore, it is a, we can replace in terms of a potential function, which is, we have written here as u. So, here, u is the potential function, is a potential function. So, we have F r d r,WAB, which is equal to F r d r and between r A and r B,minus d u r, r A r B, and this can be written as, u r B minus u r A, with a negative sign. So, this is equal to minus u r B plus u r A. Further, we can write F r d r, we can express this as, m times d v by d t. So, by using Newton’s law, we can write this F r d r, or directly in the scalar notation also, we can write m d v by d t and then…So, let us put this as…So, here,F r, we have put as this quantity.

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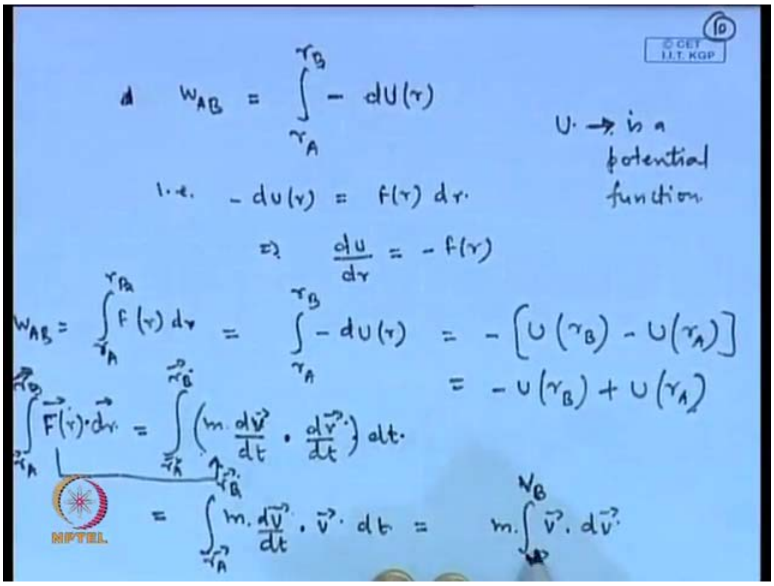

If we try to put here, the vector notation,so, this will be F r d r in this format, and d v by d t and then, here, we will have a dot product; and this we can write in this fashion. So, then, this becomes equal to…Now, d r by d t, this is, d r by d t is nothing, but v. So, here we have m d v by d t; then, dot product v d t and integrating it. So, this, we can further write as, because m is a constant, we can take it outside the integration sign, and this becomes v dot d v. And then, putting the limits of integration from r A to r B…So, instead, here, the limit of integration, we can change and write this as v A to v B; and, this is very simple to integrate. So, F dot d r, r A to r B is nothing, but vA…v dot d v. And, integrating this, this can be written as, m v square by 2. So, this is easy to see here…Let us write this as d v dot v. So, this is equal to d v dot v plus v dot d v, or 2 v dot d v. Therefore, this implies, v dot d v this equals to 1 by 2 v dot v; so, 1 by 2 v square. So, after writing it in this way, we can expand it and write it, v square times m v A square by 2. This is our equation number D, or we write this as the equation number E and we have equation the number D in this place.

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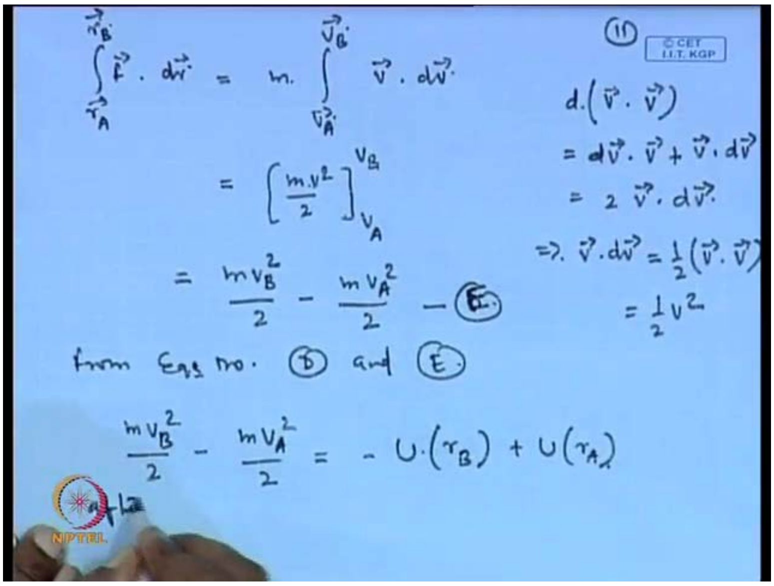

So, from equation number D and E, from equation number D, equation number D and E, we can write m v B square divided by 2 minus m v A square by 2, this equals to minus u r B plus u r A. After rearranging, we can write m v A square divided by 2 plus u r A, this equals to m v B square divided by 2 plus u r B. Now, this is our equation number F. Thus, we see that, from equation number, we can conclude that, the kinetic energy, kinetic energy plus potential energy at point A is equal to that at point B. Therefore, this implies that, the total energy which we can, total energy here is nothing, but the potential energy plus the kinetic energy.

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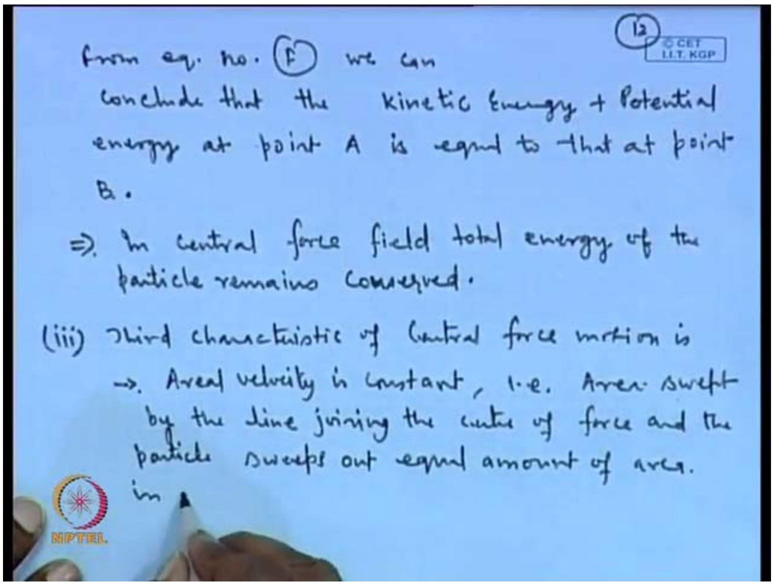

So, it remains as a constant. Therefore, this implies, in central force field, total energy of the particle remains conserved. First windup now, the third law of, the third characteristic, the, here, the third characteristic of the central force motion, motion is areal velocity is constant; that is, areal velocity is constant; that is, area swept, area swept by the line joining the center of force and the particle, sweeps out equal amount of area in equal time. And, this is very easy to deduct. Suppose, again, C is the center of attraction and here, some particle is moving in a path. So, we have the radius vector at point A as r A, and r, let us write this as r, and in time delta t, then, it moves by amount some delta r. So, the quantity, the radius vector of the particle then becomes, so, once it comes to the point B, so, it becomes r plus delta r. Now, if we have this kind of diagram, so, we can find out the area of, contained within this figure. So, let us indicate this angle is delta theta. So, we have this vector, another vector is this one. So, this is r and this is r plus delta r; this angle is delta theta. So, we complete a parallelogram. So, this is the area we are concerned with; this is our delta r. So, area of the parallelogram can be written as…So, the total area of the parallelogram, area of this will be, r cross r plus delta r.

So, the shaded area, the shaded area, then becomes equal to 1 by 2 times r cross r plus delta r; and this is nothing, but the delta A area, which is swept in time delta t. So, the rate of swept of area in time delta t, this can be written as 1 by 2 r cross r plus delta r delta t; or, 1 by 2 delta t times r cross r plus…So, this quantity equal to 0, because it is a cross product and therefore, delta A by delta t becomes equal to 1 by 2 r cross delta r by delta t. So, this is nothing, but 1 by 2 r cross, delta r by delta t as the, in the limit delta t tends to 0…

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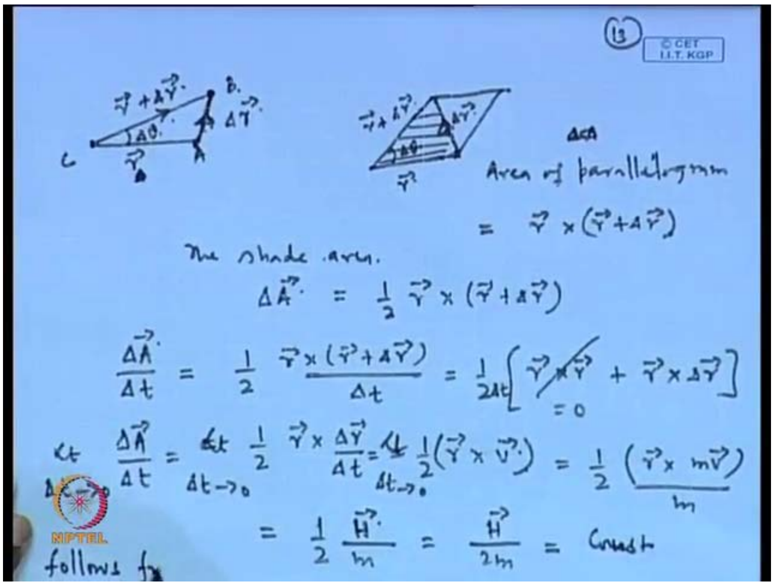

So, we can write here, delta t tends to 0; this can be written as r cross v. So, this, we can write as r cross m v divided by m. So, this is 1 by 2, and r cross m v is nothing, but the angular momentum of the particle; or, this is H by 2 m. So, already, we have seen in the, in the first characteristic that, H is a constant; therefore, d A by d t, this becomes constant; this follows from, follows from first characteristic.